Contents

3 Velocity and Acceleration Analysis

3.1 Introduction

The motion of a rigid body (RB) is defined when the position vector, velocity and acceleration of all points of the rigid body are defined as functions of time with respect to a fixed reference frame with the origin at O_0.

Let $\mathbf{\imath}_0$, $\mathbf{\jmath}_0$ and $\mathbf{k}_0$, be the constant unit vectors of a fixed orthogonal Cartesian reference frame $O_0x_0y_0z_0$ and $\mathbf{\imath}, \mathbf{\jmath}$ and $\mathbf{k}$ be the unit vectors of a mobile (body fixed) orthogonal Cartesian reference frame $Oxyz$ (Fig. 3.1). The unit vectors $\mathbf{\imath}_0, \mathbf{\jmath}_0$, and $\mathbf{k}_0$ of the primary reference frame are constant with respect to time.

A reference frame that moves with the rigid body is a *body fixed* (or mobile) reference frame. The unit vectors $\mathbf{\imath}, \mathbf{\jmath}$, and $\mathbf{k}$ of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame. The location of the point O is arbitrary.

The position vector of a point M $(M \in(RB))$, with respect to the fixed reference frame $O_0x_0y_0z_0$ is denoted by $\mathbf{r}_1 = \mathbf{r}_{O_0M}$ and with respect to the mobile reference frame $Oxyz$ is denoted by $\mathbf{r} = \mathbf{r}_{OM}$. The location of the origin O of the mobile reference frame with respect to the fixed point O_0 is defined by the position vector $\mathbf{r}_O = \mathbf{r}_{O_0O}$.
Then the relation between the vectors $\mathbf{r}_1$, $\mathbf{r}$ and $\mathbf{r}_0$ is given by

$$\mathbf{r}_1 = \mathbf{r}_O + \mathbf{r} = \mathbf{r}_O + x\mathbf{\imath} + y\mathbf{\jmath} + z\mathbf{k}, \tag{3.1}$$

where x, y and z represent the projections of the vector $\mathbf{r}$ on the mobile reference frame.

The magnitude of the vector $\mathbf{r} = \mathbf{r}_{OM}$ is a constant as the distance between the points O and M is constant $(O \in(RB)$ and $M \in(RB))$. Thus, the x, y and z components of the vector $\mathbf{r}$ with respect to the mobile reference frame are constant. The unit vectors $\mathbf{\imath}, \mathbf{\jmath}$ and $\mathbf{k}$ are time-dependent vector functions.
The vectors $\mathbf{\imath}, \mathbf{\jmath}$ and $\mathbf{k}$ are the unit vector of an orthogonal Cartesian reference frame, thus one can write

$$\mathbf{\imath} \cdot \mathbf{\imath} = 1, \quad \mathbf{\jmath} \cdot \mathbf{\jmath} = 1, \quad \mathbf{k} \cdot \mathbf{k} = 1, \tag{3.2}$$

$$\mathbf{\imath} \cdot \mathbf{\jmath} = 0, \quad \mathbf{\jmath} \cdot \mathbf{k} = 0, \quad \mathbf{k} \cdot \mathbf{\imath} = 0. \tag{3.3}$$

3.2 Velocity Field for a Rigid Body

The velocity of an arbitrary point M of the rigid body with respect to the fixed reference frame $Ox_0y_0z_0$, is the derivative with respect to time of the position vector $\mathbf{r}_1$. One can write

$$\mathbf{v} = \frac{d\mathbf{r}_1}{dt} = \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_O + \dot{\mathbf{r}} = \mathbf{v}_O + x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}} + \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k}, \tag{3.4}$$

where $\mathbf{v}_O = \dot{\mathbf{r}}_O$ represent the velocity of the origin of the mobile reference frame $O_1x_1y_1z_1$ with respect to the fixed reference frame $Oxyz$. Because all the points in the rigid body maintain their relative position, their velocity relative to the mobile reference frame $Oxyz$ is zero, i.e., $\dot{x} = \dot{y} = \dot{z} = 0$.
The velocity of point M is

$$\mathbf{v} = \mathbf{v}_O + x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}}.$$

The derivative of the Eqs.(3.2) and (3.3) with respect to time gives

$$\dot{\mathbf{i}} \cdot \boldsymbol{\imath} = 0, \quad \dot{\mathbf{j}} \cdot \boldsymbol{\jmath} = 0, \quad \dot{\mathbf{k}} \cdot \mathbf{k} = 0, \tag{3.5}$$

and

$$\dot{\mathbf{i}} \cdot \boldsymbol{\jmath} + \dot{\mathbf{j}} \cdot \boldsymbol{\imath} = 0, \quad \dot{\mathbf{j}} \cdot \mathbf{k} + \dot{\mathbf{k}} \cdot \boldsymbol{\jmath} = 0, \quad \dot{\mathbf{k}} \cdot \boldsymbol{\imath} + \dot{\mathbf{i}} \cdot \mathbf{k} = 0. \tag{3.6}$$

For Eq.(3.6) one can introduce the convention

$$\begin{aligned} \dot{\mathbf{i}} \cdot \boldsymbol{\jmath} &= -\dot{\mathbf{j}} \cdot \boldsymbol{\imath} = \omega_z, \\ \dot{\mathbf{j}} \cdot \mathbf{k} &= -\dot{\mathbf{k}} \cdot \boldsymbol{\jmath} = \omega_x, \\ \dot{\mathbf{k}} \cdot \boldsymbol{\imath} &= -\dot{\mathbf{i}} \cdot \mathbf{k} = \omega_y, \end{aligned} \tag{3.7}$$

where ω_x, ω_y and ω_z may be considered as the projections of a vector $\boldsymbol{\omega}$, $\boldsymbol{\omega} = \omega_x\boldsymbol{\imath} + \omega_y\boldsymbol{\jmath} + \omega_z\mathbf{k}$.
To calculate $\dot{\mathbf{i}}$, $\dot{\mathbf{j}}$, $\dot{\mathbf{k}}$ one can use the relation for an arbitrary vector $\mathbf{v}$

$$\mathbf{v} = v_x\boldsymbol{\imath} + v_y\boldsymbol{\jmath} + v_z\mathbf{k} = (\mathbf{v} \cdot \boldsymbol{\imath})\,\boldsymbol{\imath} + (\mathbf{v} \cdot \boldsymbol{\jmath})\,\boldsymbol{\jmath} + (\mathbf{v} \cdot \mathbf{k})\,\mathbf{k}. \tag{3.8}$$

Using Eq.(3.8) and the results from Eqs.(3.5) and (3.6) one can write

$$\begin{aligned} \dot{\mathbf{i}} &= (\dot{\mathbf{i}} \cdot \boldsymbol{\imath})\,\boldsymbol{\imath} + (\dot{\mathbf{i}} \cdot \boldsymbol{\jmath})\,\boldsymbol{\jmath} + (\dot{\mathbf{i}} \cdot \mathbf{k})\,\mathbf{k} \\ &= (0)\,\boldsymbol{\imath} + (\omega_z)\,\boldsymbol{\jmath} - (\omega_y)\,\mathbf{k} \\ &= \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix} = \boldsymbol{\omega} \times \boldsymbol{\imath}, \end{aligned}$$

$$
\begin{aligned}
\dot{\boldsymbol{\jmath}} &= \left(\dot{\boldsymbol{\jmath}} \cdot \boldsymbol{\imath}\right) \boldsymbol{\imath} + \left(\dot{\boldsymbol{\jmath}} \cdot \boldsymbol{\jmath}\right) \boldsymbol{\jmath} + \left(\dot{\boldsymbol{\jmath}} \cdot \mathbf{k}\right) \mathbf{k} \\
&= \left(-\omega_z\right) \boldsymbol{\imath} + (0)\, \boldsymbol{\jmath} + \left(\omega_x\right) \mathbf{k} \\
&= \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 1 & 0 \end{vmatrix} = \boldsymbol{\omega} \times \boldsymbol{\jmath},
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{k}} &= \left(\dot{\mathbf{k}} \cdot \boldsymbol{\imath}\right) \boldsymbol{\imath} + \left(\dot{\mathbf{k}} \cdot \boldsymbol{\jmath}\right) \boldsymbol{\jmath} + \left(\dot{\mathbf{k}} \cdot \mathbf{k}\right) \mathbf{k} \\
&= \left(\omega_y\right) \boldsymbol{\imath} - \left(\omega_x\right) \boldsymbol{\jmath} + (0)\, \mathbf{k} \\
&= \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 0 & 1 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{k}.
\end{aligned}
\tag{3.9}
$$

The relations

$$
\dot{\boldsymbol{\imath}} = \boldsymbol{\omega} \times \boldsymbol{\imath}, \quad \dot{\boldsymbol{\jmath}} = \boldsymbol{\omega} \times \boldsymbol{\jmath}, \quad \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k}. \tag{3.10}
$$

are known as *Poisson formulas.*
Using Eqs.(3.4) and (3.10) one can obtain

$$
\mathbf{v} = \mathbf{v}_O + x\boldsymbol{\omega} \times \boldsymbol{\imath} + y\boldsymbol{\omega} \times \boldsymbol{\jmath} + z\boldsymbol{\omega} \times \mathbf{k} = \mathbf{v}_O + \boldsymbol{\omega} \times \left(x\boldsymbol{\imath} + y\boldsymbol{\jmath} + z\mathbf{k}\right),
$$

or

$$
\mathbf{v} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}. \tag{3.11}
$$

Combining Eqs.(3.4) and (3.11) it results

$$
\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}. \tag{3.12}
$$

Using Eq.(3.11) one can write the components of the velocity as

$$
\begin{aligned}
v_x &= v_{Ox} + z\omega_y - y\omega_z, \\
v_y &= v_{Oy} + x\omega_z - z\omega_x, \\
v_z &= v_{Oz} + y\omega_x - x\omega_y.
\end{aligned}
$$

3.3 Acceleration Field for a Rigid Body

The acceleration of an arbitrary point $M \in (RB)$ with respect to a fixed reference frame $O_0x_0y_0z_0$, represents the double derivative with respect to time of the position vector $\mathbf{r}_1$

$$\mathbf{a} = \ddot{\mathbf{r}}_1 = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r}) = \frac{d}{dt}\mathbf{v}_O + \frac{d}{dt}\boldsymbol{\omega}\times\mathbf{r} + \boldsymbol{\omega}\times\frac{d}{dt}\mathbf{r} = \dot{\mathbf{v}}_O + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times\dot{\mathbf{r}}. \tag{3.13}$$

The acceleration of the point O with respect to the fixed reference frame $O_0x_0y_0z_0$ is

$$\mathbf{a}_O = \dot{\mathbf{v}}_O = \ddot{\mathbf{r}}_O. \tag{3.14}$$

The derivative of the vector $\boldsymbol{\omega}$ with respect to the time is the vector $\boldsymbol{\alpha}$ given by

$$\begin{aligned} \boldsymbol{\alpha} &= \dot{\boldsymbol{\omega}} = \dot{\omega}_x \boldsymbol{\imath} + \dot{\omega}_y \boldsymbol{\jmath} + \dot{\omega}_z \mathbf{k} + \omega_x \dot{\mathbf{i}} + \omega_y \dot{\mathbf{j}} + \omega_z \dot{\mathbf{k}} \qquad (3.15) \\ &= \alpha_x \boldsymbol{\imath} + \alpha_y \boldsymbol{\jmath} + \alpha_z \mathbf{k} + \omega_x \boldsymbol{\omega} \times \boldsymbol{\imath} + \omega_y \boldsymbol{\omega} \times \boldsymbol{\jmath} + \omega_z \boldsymbol{\omega} \times \mathbf{k} \\ &= \alpha_x \boldsymbol{\imath} + \alpha_y \boldsymbol{\jmath} + \alpha_z \mathbf{k} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \alpha_x \boldsymbol{\imath} + \alpha_y \boldsymbol{\jmath} + \alpha_z \mathbf{k}. \end{aligned}$$

where $\alpha_x = \dot{\omega}_x, \alpha_y = \dot{\omega}_y$, and $\alpha_z = \dot{\omega}_z$.

In the previous exprerssion the Poisson formulas

$$\dot{\mathbf{i}} = \boldsymbol{\omega} \times \boldsymbol{\imath}, \quad \dot{\mathbf{j}} = \boldsymbol{\omega} \times \boldsymbol{\jmath}, \quad \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k},$$

have been used.

Using Eqs.(3.13), (3.14) and (3.15) one can write the acceleration of the point M as

$$\mathbf{a} = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{3.16}$$

Using Eq.(3.16) one can write the components of the acceleration as

$$\begin{aligned} a_x &= a_{Ox} + (z\alpha_y - y\alpha_z) + \omega_y (y\omega_x - x\omega_y) + \omega_z (x\omega_x - x\omega_z), \\ a_y &= a_{Oy} + (x\alpha_z - z\alpha_x) + \omega_z (z\omega_y - y\omega_z) + \omega_x (x\omega_y - y\omega_z), \\ a_z &= a_{Oz} + (y\alpha_x - x\alpha_y) + \omega_x (x\omega_z - z\omega_x) + \omega_y (y\omega_z - z\omega_y). \end{aligned}$$

REMARKS

1. If the orientation of a rigid body RB in a reference frame RF_0 depends on only a single scalar variable $\boldsymbol{\zeta}$, there exists for each value of $\boldsymbol{\zeta}$ a vector $\boldsymbol{\omega}$ such that the derivative with respect to $\boldsymbol{\zeta}$ in RF_0 of every vector $\mathbf{c}$ fixed in rigid body RB is given by

$$\frac{d\,\mathbf{c}}{d\boldsymbol{\zeta}} = \boldsymbol{\omega} \times \mathbf{c}, \tag{3.17}$$

where the vector $\boldsymbol{\omega}$ is the rate of change of orientation of the rigid body RB in the reference frame RF_0 with respect to $\boldsymbol{\zeta}$. The vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}}, \tag{3.18}$$

where $\mathbf{a}$ and $\mathbf{b}$ are any two nonparallel vectors fixed in the rigid body RB.

The vector $\boldsymbol{\omega}$ is a free vector, i.e. is not associated with any particular point. With the help of $\boldsymbol{\omega}$ one can replace the process of differentiation with that of cross multiplication.

The vector $\boldsymbol{\omega}$ may be expressed in a symmetrical relation in $\mathbf{a}$ and $\mathbf{b}$

$$\boldsymbol{\omega} = \frac{1}{2}\left(\frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} + \frac{\dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} \cdot \mathbf{a}} \right). \tag{3.19}$$

2. The first derivatives of a vector $\mathbf{p}$ with respect to a scalar variable $\boldsymbol{\zeta}$ in two reference frames RF_i and RF_j are related as follows

$$\frac{{}^{(j)}d\,\mathbf{p}}{d\boldsymbol{\zeta}} = \frac{{}^{(i)}d\,\mathbf{p}}{d\boldsymbol{\zeta}} + \boldsymbol{\omega}_{ij} \times \mathbf{p}, \tag{3.20}$$

where $\boldsymbol{\omega}_{ij}$ is the rate of change of orientation of RF_i in RF_j with respect to $\boldsymbol{\zeta}$ and $\dfrac{{}^{(j)}d\,\mathbf{p}}{d\boldsymbol{\zeta}}$ is the total derivative of $\mathbf{p}$ with respect to $\boldsymbol{\zeta}$ in RF_j.

Proof

The vector $\mathbf{p}$ can be expressed as

$$\mathbf{p} = p_1\boldsymbol{\imath}_1 + p_2\boldsymbol{\imath}_2 + p_3\boldsymbol{\imath}_3,$$

where $\boldsymbol{\imath}_1$, $\boldsymbol{\imath}_2$, $\boldsymbol{\imath}_3$ are three units vectors not parallel to the same plane fixed in RF_i, and p_x, p_y, p_z are the scalar measure numbers of $\mathbf{p}$. Differentiating in RF_j

$$\begin{aligned}
\frac{{}^{(j)}d\,\mathbf{p}}{d\boldsymbol{\zeta}} &= \frac{{}^{(j)}d}{d\boldsymbol{\zeta}}\left(p_1\boldsymbol{\imath}_1 + p_2\boldsymbol{\imath}_2 + p_3\boldsymbol{\imath}_3\right)\\
&= \frac{{}^{(j)}d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_1 + \frac{{}^{(j)}d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_2 + \frac{{}^{(j)}d\,p_3}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_3 + p_1\frac{{}^{(j)}d\,\boldsymbol{\imath}_1}{d\boldsymbol{\zeta}} + p_2\frac{{}^{(j)}d\,\boldsymbol{\imath}_2}{d\boldsymbol{\zeta}} + p_3\frac{{}^{(j)}d\,\boldsymbol{\imath}_3}{d\boldsymbol{\zeta}}\\
&= \frac{d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_1 + \frac{d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_2 + \frac{d\,p_3}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_3 + p_1\boldsymbol{\omega}_{ij}\times\boldsymbol{\imath}_1 + p_2\boldsymbol{\omega}_{ij}\times\boldsymbol{\imath}_2 + p_3\boldsymbol{\omega}_{ij}\times\boldsymbol{\imath}_3\\
&= \frac{{}^{(i)}d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_1 + \frac{{}^{(i)}d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_2 + \frac{{}^{(i)}d\,p_3}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_3 + \boldsymbol{\omega}_{ij}\times\left(p_1\boldsymbol{\imath}_1 + p_2\boldsymbol{\imath}_2 + p_3\boldsymbol{\imath}_3\right)\\
&= \frac{{}^{(i)}d\,\mathbf{p}}{d\boldsymbol{\zeta}} + \boldsymbol{\omega}_{ij}\times\mathbf{p}. \qquad (3.21)
\end{aligned}$$

3. The angular velocity of a rigid body RB in a reference frame RF_0 is the rate of change of orientation with respect to the time t

$$\boldsymbol{\omega} = \frac{1}{2}\left(\frac{\dfrac{d\,\mathbf{a}}{dt}\times\dfrac{d\,\mathbf{b}}{dt}}{\dfrac{d\,\mathbf{a}}{dt}\cdot\mathbf{b}} + \frac{\dfrac{d\,\mathbf{b}}{dt}\times\dfrac{d\,\mathbf{a}}{dt}}{\dfrac{d\,\mathbf{b}}{dt}\cdot\mathbf{a}}\right) = \frac{1}{2}\left(\frac{\dot{\mathbf{a}}\times\dot{\mathbf{b}}}{\dot{\mathbf{a}}\cdot\mathbf{b}} + \frac{\dot{\mathbf{b}}\times\dot{\mathbf{a}}}{\dot{\mathbf{b}}\cdot\mathbf{a}}\right).$$

The direction of $\boldsymbol{\omega}$ is related to the direction of the rotation of the rigid body through a right-hand rule.

4. Let RF_i, $i = 1, 2, ..., n$ be n reference frames. The angular velocity of a rigid body r in the reference frame RF_n, can be expressed as

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r1} + \boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{23} + ... + \boldsymbol{\omega}_{r,n-1}.$$

Proof

Let $\mathbf{p}$ be any vector fixed in the rigid body. Then

$$\frac{{}^{(i)}d\,\mathbf{p}}{dt} = \boldsymbol{\omega}_{ri} \times \mathbf{p}$$
$$\frac{{}^{(i-1)}d\,\mathbf{p}}{dt} = \boldsymbol{\omega}_{r,i-1} \times \mathbf{p}.$$

On the other hand

$$\frac{{}^{(i)}d\,\mathbf{p}}{dt} = \frac{{}^{(i-1)}d\,\mathbf{p}}{dt} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p}.$$

Hence

$$\boldsymbol{\omega}_{ri} \times \mathbf{p} = \boldsymbol{\omega}_{r,i-1} \times \mathbf{p} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p},$$

as this equation is satisfied for all $\mathbf{p}$ fixed in the rigid body

$$\boldsymbol{\omega}_{ri} = \boldsymbol{\omega}_{r,i-1} + \boldsymbol{\omega}_{i,i-1}. \tag{3.22}$$

With $i = n$, Eq. (3.22) gives

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-1} + \boldsymbol{\omega}_{n,n-1}. \tag{3.23}$$

With $i = n - 1$, Eq. (3.22) gives

$$\boldsymbol{\omega}_{r,n-1} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2}. \tag{3.24}$$

Substitute Eq. (3.24) into Eq. (3.23)

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2} + \boldsymbol{\omega}_{n,n-1}.$$

Next use Eq. (3.22) with $i = n - 2$, then with $i = n - 3$, and so forth.

3.4 Motion of a Point that Moves Relative to a Rigid Body

A reference frame that moves with the rigid body is a body fixed reference frame. Figure 3.2 shows a rigid body (RB), in motion relative to a primary reference frame with its origin at point O_0, $O_0x_0y_0z_0$. The primary reference frame is a fixed reference frame or an earth fixed reference frame. The unit vectors $\mathbf{\imath}_0, \mathbf{\jmath}_0$, and $\mathbf{k}_0$ of the primary reference frame are constant.

The body fixed reference frame, $Oxyz$, has its origin at a point O of the rigid body ($O \in (RB)$), and is a moving reference frame relative to the primary reference. The unit vectors $\mathbf{\imath}, \mathbf{\jmath}$, and $\mathbf{k}$ of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame.

The position vector of a point P of the rigid body ($P \in (RB)$) relative to the origin, O, of the body fixed reference frame is the vector $\mathbf{r}_{OP}$. The velocity of P relative to O is

$$\frac{d\mathbf{r}_{OP}}{dt} = \mathbf{v}_P^O = \boldsymbol{\omega} \times \mathbf{r}_{OP},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body. The position vector of a point A (the point A is not assumed to be a point of the rigid body), relative to the origin O_0 of the primary reference frame is, Fig. 3.3

$$\mathbf{r}_A = \mathbf{r}_O + \mathbf{r},$$

where

$$\mathbf{r} = \mathbf{r}_{OA} = x\mathbf{\imath} + y\mathbf{\jmath} + z\mathbf{k}$$

is the position vector of A relative to the origin O, of the body fixed reference frame, and x, y, and z are the coordinates of A in terms of the body fixed reference frame. The velocity of the point A is the time derivative of the position vector $\mathbf{r}_A$

$$\begin{aligned} \mathbf{v}_A \; = \; & \frac{d\mathbf{r}_O}{dt} + \frac{d\mathbf{r}}{dt} = \mathbf{v}_O + \mathbf{v}_{OA} = \\ & \mathbf{v}_O + \frac{dx}{dt}\mathbf{\imath} + x\frac{d\mathbf{\imath}}{dt} + \frac{dy}{dt}\mathbf{\jmath} + y\frac{d\mathbf{\jmath}}{dt} + \frac{dz}{dt}\mathbf{k} + z\frac{d\mathbf{k}}{dt}. \end{aligned}$$

Using Poisson formula, the total derivative of the the position vector $\mathbf{r}$ is

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{\imath} + \dot{y}\mathbf{\jmath} + \dot{z}\mathbf{k} + \boldsymbol{\omega} \times \mathbf{r}.$$

The velocity of A relative to the body fixed reference frame is a derivative in the body fixed reference frame

$$\mathbf{v}_{Arel} = \frac{^{(RB)}d\,\mathbf{r}}{dt} = \frac{dx}{dt}\boldsymbol{\imath} + \frac{dy}{dt}\boldsymbol{\jmath} + \frac{dz}{dt}\mathbf{k} = \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k}, \tag{3.25}$$

A general formula for the total derivative of a moving vector $\mathbf{r}$ may be written as

$$\frac{d\mathbf{r}}{dt} = \frac{^{(RB)}d\,\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}, \tag{3.26}$$

where $\dfrac{d\mathbf{r}}{dt} = \dfrac{^{(0)}d\,\mathbf{r}}{dt}$ is the derivative in the fixed reference frame (0) $(O_0x_0y_0z_0)$, and $\dfrac{^{(RB)}d\,\mathbf{r}}{dt}$ is the derivative in the mobile reference frame.

The velocity of the point A relative to the primary reference frame is

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}, \tag{3.27}$$

Equation (3.27) expresses the velocity of a point A as the sum of three terms:
- the velocity of a point O of the rigid body,
- the velocity $\mathbf{v}_{Arel}$ of A relative to the rigid body, and
- the velocity $\boldsymbol{\omega} \times \mathbf{r}$ of A relative to O due to the rotation of the rigid body.

The acceleration of the point A relative to the primary reference frame is obtained by taking the time derivative of Eq. (3.27)

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO}, \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \end{aligned} \tag{3.28}$$

where

$$\mathbf{a}_{Arel} = \frac{^{(RB)}d^2\,\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\boldsymbol{\imath} + \frac{d^2y}{dt^2}\boldsymbol{\jmath} + \frac{d^2z}{dt^2}\mathbf{k}, \tag{3.29}$$

is the acceleration of A relative to the body fixed reference frame or relative to the rigid body. The term

$$\mathbf{a}_{Cor} = 2\boldsymbol{\omega} \times \mathbf{v}_{Arel}.$$

is called the Coriolis acceleration.

In the case of planar motion, Eq. (3.28) becomes

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{OA}, \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} - \boldsymbol{\omega}^2\mathbf{r}, \end{aligned} \tag{3.30}$$

The motion of the rigid body (RB) is described relative to the primary reference frame. The velocity $\mathbf{v}_A$ and the acceleration $\mathbf{a}_A$ of point a point A are relative to the primary reference frame. The terms $\mathbf{v}_{Arel}$ and $\mathbf{a}_{Arel}$ are the velocity and acceleration of point A relative to the body fixed reference frame i.e., they are the velocity and acceleration measured by an observer moving with the rigid body, Fig. 3.4.

If A is a point of the rigid body, $A \in RB$, $\mathbf{v}_{Arel} = \mathbf{0}$ and $\mathbf{a}_{Arel} = \mathbf{0}$.

Motion of a point relative to a moving reference frame

The velocity and acceleration of an arbitrary point A relative to a point O of a rigid body, in terms of the body fixed reference frame, are given by Eqs. (3.27) and (3.28)

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}_{OA}, \tag{3.31}$$

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r}_{OA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{OA}). \tag{3.32}$$

These results apply to any reference frame having a moving origin O and rotating with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ relative to a primary reference frame (Fig. 3.5). The terms $\mathbf{v}_A$ and $\mathbf{a}_A$ are the velocity and acceleration of an arbitrary point A relative to the primary reference frame. The terms $\mathbf{v}_{Arel}$ and $\mathbf{a}_{Arel}$ are the velocity and acceleration of A relative to the secondary moving reference frame i.e., they are the velocity and acceleration measured by an observer moving with the secondary reference frame.

3.5 Slider-Crank (R-RRT) Mechanism

The R-RRT (slider-crank) mechanism shown in Fig. 2.2(a) has the dimensions: $AB = 1$ m and $BC = 1$ m. The driver link 1 rotates with a constant speed of $n = 30/\pi$ rpm. Find the velocities and the accelerations of the joints B and C and the angular velocity and acceleration of the link 2 at the moment when the driver link 1 makes an angle $\phi = \phi_1 = \pi/4$ rad with the horizontal axis.

Solution

The point A is selected as the origin of the xyz reference frame. The position vectors of the joints B and C are:

$$\mathbf{r}_B = x_B\mathbf{\imath} + y_B\mathbf{\jmath} = \frac{\sqrt{2}}{2}\mathbf{\imath} + \frac{\sqrt{2}}{2}\mathbf{\jmath} \text{ m} \quad \text{and} \quad \mathbf{r}_C = x_C\mathbf{\imath} + y_C\mathbf{\jmath} = \sqrt{2}\mathbf{\imath} + 0\mathbf{\jmath} \text{ m}.$$

The angular velocity of link 1 is

$$\boldsymbol{\omega}_1 = \boldsymbol{\omega} = \omega_1\ \mathbf{k} = \frac{\pi n}{30}\ \mathbf{k} = \frac{\pi(30/\pi)}{30}\ \mathbf{k} = 1\ \mathbf{k}\ \text{ rad/s}.$$

The angular acceleration of link 1 is $\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\omega}}_1 = \mathbf{0}$.
The MATLAB statements for the angular velocity and acceleration of link 1 are

```
n = 30/pi;   % (rpm) driver link
omega1 = [ 0 0 pi*n/30 ];  % (rad/s) angular velocity driver link
alpha1 = [0 0 0 ];  % (rad/s^2) angular acceleration driver link
fprintf('omega1 = [ %g, %g, %g] (rad/s)\n', omega1);
fprintf('alpha1 = [ %g, %g, %g] (rad/s^2)\n', alpha1 );
```

In the MATLAB environment, a three-dimensional vector `v` is written as a list of variables `v = [ x y z ]`, where `x`, `y`, and `z` are the spatial coordinates of the vector `v`. The first component of the vector `v` is `x=v(1)`, the second component is `y=v(2)`, and the third component is `z=v(3)`.

The velocity and acceleration of the origin $A \equiv O$ are $\mathbf{v}_A = \mathbf{a}_A = \mathbf{0}$ or in MATLAB

```
vA = [0 0 0 ];  % (m/s) velocity of A (fixed)
```

```
aA = [0 0 0 ];  % (m/s^2) acceleration of A
```

Velocity and acceleration of joint B

The points A and $B = B_1$ are on the link 1 ($A,\ B_1 \in$ link 1). The velocity of the point $B = B_1$ is calculated in terms of the known $\mathbf{v}_A$ using Eq. (3.11)

$$\mathbf{v}_B = \mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} = \mathbf{0} + \boldsymbol{\omega}_1 \times \mathbf{r}_B = \boldsymbol{\omega}_1 \times \mathbf{r}_B.$$

The velocity of point B_2 on the link 2 is $\mathbf{v}_{B_2} = \mathbf{v}_{B_1}$ because the links 1 and 2 are connected at a rotational joint. The velocity of $B_1 = B_2$ is

$$\mathbf{v}_B = \mathbf{v}_{B_1} = \mathbf{v}_{B_2} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ x_B & y_B & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{vmatrix} = -\frac{\sqrt{2}}{2}\boldsymbol{\imath} + \frac{\sqrt{2}}{2}\boldsymbol{\jmath} \ \text{ m/s}.$$

The acceleration of the point $B = B_1$ on the link 1 is calculated in term of known acceleration $\mathbf{a}_A$ using Eq. (3.16)

$$\begin{aligned} \mathbf{a}_B &= \mathbf{a}_{B_1} = \mathbf{a}_{B_2} = \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_B) = \boldsymbol{\alpha}_1 \times \mathbf{r}_B - \boldsymbol{\omega}_1^2 \mathbf{r}_B \\ &= -\boldsymbol{\omega}_1^2 \mathbf{r}_B = -1^2(\frac{\sqrt{2}}{2}\boldsymbol{\imath} + \frac{\sqrt{2}}{2}\boldsymbol{\jmath}) = -\frac{\sqrt{2}}{2}\boldsymbol{\imath} - \frac{\sqrt{2}}{2}\boldsymbol{\jmath} \ \text{ m/s}^2. \end{aligned}$$

The MATLAB commands for the velocity and acceleration of the B are

```
vB1 = vA + cross(omega1,rB);  % velocity of B
vB2 = vB1;  % between 1 & 2 there is a rotational joint B_R
aB1 = aA + cross(alpha1,rB) - dot(omega1,omega1)*rB;
aB2 = aB1;
fprintf('vB = vB1 = vB2 = [ %g, %g, %g] (m/s)\n', vB1);
fprintf('aB = aB1 = aB2 = [ %g, %g, %g] (m/s^2)\n', aB1);
```

The command `dot(u,v)` calculates the scalar product (or vector dot product) of the vectors `u` and `v`. The command `cross(u,v)` performs the cross product of the vectors `u` and `v`.

Velocity of joint C

The points B_2 and C_2 are on the link 2 and

$$\mathbf{v}_C = \mathbf{v}_{C_2} = \mathbf{v}_{B_2} + \boldsymbol{\omega}_2 \times \mathbf{r}_{BC} = \mathbf{v}_B + \boldsymbol{\omega}_2 \times (\mathbf{r}_C - \mathbf{r}_B), \tag{3.33}$$

where the angular velocity of link 2 is $\boldsymbol{\omega}_2 = \omega_2\mathbf{k}$ (ω_2 is unknown).

On the other hand the velocity of C is along the horizontal axis (x-axis) because slider 2 translates along x-axis

$$\mathbf{v}_C = \mathbf{v}_{C_3} = v_C\mathbf{\imath}. \tag{3.34}$$

Equations (3.43) and (3.44) give

$$\mathbf{v}_B + \boldsymbol{\omega}_2 \times (\mathbf{r}_C - \mathbf{r}_B) = v_C\mathbf{\imath},$$

or

$$\mathbf{v}_B + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_2 \\ x_C - x_B & y_C - y_B & 0 \end{vmatrix} = v_C\mathbf{\imath}. \tag{3.35}$$

Equation (3.45) represents a vectorial equation with two scalar components on x-axis and y-axis and with two unknowns ω_2 and v_C

$$v_{Bx} - \omega_2(y_C - y_B) = v_C, \tag{3.36}$$
$$v_{Bx} + \omega_2(x_C - x_B) = 0, \tag{3.37}$$

or

$$-\frac{\sqrt{2}}{2} - \omega_2(0 - \frac{\sqrt{2}}{2}) = v_C,$$
$$\frac{\sqrt{2}}{2} + \omega_2(\sqrt{2} - \frac{\sqrt{2}}{2}) = 0.$$

It results

$$\omega_2 = -1 \text{ rad/s} \quad \text{and} \quad v_C = -\sqrt{2} \text{ m/s}.$$

In MATLAB the `sym` command constructs symbolic variables and expressions. The commands

```
omega2z = sym('omega2z','real');
vCx = sym('vCx','real');
```

create a symbolic variables `omega2z` and `vCx` for the unknowns ω_2 and v_C. The commands `sym('omega2z','real')` and `sym('vCx','real')` also assume that `omega2z` and `vCx` real numbers. The vectors $\boldsymbol{\omega}_2 = \omega_2 \, \mathbf{k}$ and $\mathbf{v}_C = v_C \, \boldsymbol{\imath}$ are expressed in MATLAB with

```
omega2 = [ 0 0 omega2z ];
vC = [ vCx 0 0 ];
```

Equation (3.45) or $\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_2 \times (\mathbf{r}_C - \mathbf{r}_B)$ in MATLAB is

```
eqvC = vC - (vB2 + cross(omega2,rC-rB));
```

This vectorial equation has a component on
- x-axis given by Eq. (3.36), or in MATLAB, `eqvC(1)` and
- y-axis given by Eq. (3.37), or in MATLAB, `eqvC(2)`.

The two algebraic equations can be solve using the command `solve`

```
eqvCx = eqvC(1);   % equation component on x-axis
eqvCy = eqvC(2);   % equation component on y-axis
solvC = solve(eqvCx,eqvCy);
```

with the solutions

```
omega2zs = eval(solvC.omega2z);
vCxs = eval(solvC.vCx);
```

The angular velocity of the link 2 and the velocity of C in vectorial form are

```
Omega2 = [0 0 omega2zs];
VC = [vCxs 0 0];
```

To display the correct expression for the equations `eqvCx` and `eqvCy` the following MATLAB statements can be used

```
qvCx = vpa(eqvCx,6);
fprintf('x-axis:  %s = 0 \n', char(qvCx));
qvCy = vpa(eqvCy,6);
```

```
fprintf('y-axis:  %s = 0 \n', char(qvCy));
```

The command `vpa(S,D)` uses variable-precision arithmetic (vpa) to compute each element of `S` to `D` decimal digits of accuracy and the command `char()` creates character array (string).
The equations will be displayed as

```
x-axis:  vCx+.707105-.707105*omega2z = 0
y-axis:  -.707105-.707105*omega2z = 0
```

Acceleration of joint C
The points C_2 and B_2 are on the link 2 and

$$\begin{aligned}\mathbf{a}_C = \mathbf{a}_{C_2} &= \mathbf{a}_{B_2} + \boldsymbol{\alpha}_2 \times \mathbf{r}_{BC} - \omega_2^2\,\mathbf{r}_{BC} = \\ &\mathbf{a}_B + \boldsymbol{\alpha}_2 \times (\mathbf{r}_C - \mathbf{r}_B) - \omega_2^2\,(\mathbf{r}_C - \mathbf{r}_B), \end{aligned} \tag{3.38}$$

where the angular acceleration of link 2 is $\boldsymbol{\alpha}_2 = \alpha_2\mathbf{k}$ (α_2 is unknown).
The slider C has a translational motion along x-axis and

$$\mathbf{a}_C = \mathbf{a}_{C_3} = a_C\boldsymbol{\imath}. \tag{3.39}$$

Equations (3.49) and (3.50) give

$$\mathbf{a}_B + \boldsymbol{\alpha}_2 \times (\mathbf{r}_C - \mathbf{r}_B) - \omega_2^2\,(\mathbf{r}_C - \mathbf{r}_B) = a_C\boldsymbol{\imath},$$

or

$$\mathbf{a}_B + \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha_2 \\ x_C - x_B & y_C - y_B & 0 \end{vmatrix} - \omega_2^2\,[(x_C - x_B)\boldsymbol{\imath} + (y_C - y_B)\boldsymbol{\jmath}] = a_C\boldsymbol{\imath}. \tag{3.40}$$

Equation (3.51) represents a vectorial equation with two scalar components on x-axis and y-axis and with two unknowns α_2 and α_3

$$a_{Bx} - \alpha_2(y_C - y_B) - \omega_2^2(x_C - x_B) = a_C, \tag{3.41}$$

$$a_{By} + \alpha_2(x_C - x_B) - \omega_2^2(y_C - y_B) = 0, \tag{3.42}$$

or

$$\begin{aligned} -\frac{\sqrt{2}}{2} - \alpha_2(0 - \frac{\sqrt{2}}{2}) - (-1)^2(\sqrt{2} - \frac{\sqrt{2}}{2}) = a_C, \\ -\frac{\sqrt{2}}{2} + \alpha_2(\sqrt{2} - \frac{\sqrt{2}}{2}) - (-1)^2(0 - \frac{\sqrt{2}}{2}) = 0. \end{aligned}$$

It results

$$\alpha_2 = 0 \text{ rad/s}^2 \quad \text{and} \quad a_C = -\sqrt{2} \text{ m/s}^2.$$

To calculate $\boldsymbol{\alpha}_2$ and $\mathbf{a}_C$ the following commands are used with MATLAB

```
alpha2z=sym('alpha2z','real');
aCx=sym('aCx','real');
alpha2 = [ 0 0 alpha2z ];    % alpha3z unknown
aC = [aCx 0 0 ];             % aCx unknown
eqaC = aC - (aB1 + cross(alpha2,rC-rB) - dot(Omega2,Omega2)*(rC-rB));
eqaCx = eqaC(1);    % equation component on x-axis
eqaCy = eqaC(2);    % equation component on y-axis
solaC = solve(eqaCx,eqaCy);
alpha2zs=eval(solaC.alpha2z);
aCxs=eval(solaC.aCx);
Alpha2 = [0 0 alpha2zs];
aCs = [aCxs 0 0];
```

The MATLAB program for the velocities and accelerations is given in Program 3.1. The results are shown at the end of the program.

3.6 R-RRR-RRT Mechanism

The planar R-RRR-RRT mechanism is shown in Fig. 2.4. The following data are given: AB=0.150 m, BC=0.400 m, CD=0.370 m, CE=0.230 m, EF=CE, L_a=0.300 m, L_b=0.450 m, and L_c=CD. The constant angular speed of the driver link 1 is $n = n_1$=60 rpm.
The joints have the the coordinates [m]:
$x_A = y_A = 0;\ x_D = 0.3,\ y_D = 0.45;\ x_B = 0.129904,\ y_B = 0.075;$
$x_C = -0.0689445,\ y_C = 0.422073;\ x_E = -0.298288,\ y_E = 0.404712;$
$x_F = -0.37,\ y_F = 0.186177.$

Find the velocities and the accelerations of the mechanism at the moment when the driver link 1 makes an angle $\phi = \phi_1$ =30° with the horizontal axis.

Solution

The angular velocity of link 1 is

$$\boldsymbol{\omega}_1 = \omega_1\,\mathbf{k} = \frac{\pi n}{30}\,\mathbf{k} = \frac{\pi(60)}{30}\,\mathbf{k} = 6.28319\,\mathbf{k}\ \text{ rad/s.}$$

The angular acceleration of link 1 is $\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\omega}}_1 = \mathbf{0}$.
The MATLAB statements for the angular velocity and acceleration of link 1 are

```
n = 60; omega1 = [ 0 0 pi*n/30 ]; alpha1 = [0 0 0 ];
```

Velocity and acceleration of joint B

The velocity of the point $B = B_1$ on the link 1 is

$$\mathbf{v}_B = \mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} = \boldsymbol{\omega}_1 \times \mathbf{r}_B,$$

where $\mathbf{v}_A \equiv \mathbf{0}$ is the velocity of the origin $A \equiv O$. The velocity of point B_2 on the link 2 is $\mathbf{v}_{B_2} = \mathbf{v}_{B_1}$ because between the links 1 and 2 there is a rotational joint. The velocity of $B = B_1 = B_2$ is

$$\mathbf{v}_B = \mathbf{v}_{B_1} = \mathbf{v}_{B_2} \ = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega \\ x_B & y_B & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & 6.28319 \\ 0.129904 & 0.075 & 0 \end{vmatrix} =$$
$$-0.471239\,\boldsymbol{\imath} + 0.81621\,\boldsymbol{\jmath}\ \text{ m/s.}$$

The acceleration of the point $B = B_1 = B_2$ is

$$\begin{aligned} \mathbf{a}_B &= \mathbf{a}_{B_1} = \mathbf{a}_{B_2} = \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_B) = \boldsymbol{\alpha}_1 \times \mathbf{r}_B - \boldsymbol{\omega}_1^2 \mathbf{r}_B \\ &= -(6.28319)^2(0.129904\,\boldsymbol{\imath} + 0.075\,\boldsymbol{\jmath}) = -5.1284\,\boldsymbol{\imath} - 2.96088\,\boldsymbol{\jmath}\ \text{ m/s}^2. \end{aligned}$$

The MATLAB statements for the velocity and acceleration of the driver link 1 are

```
vA = [0 0 0 ]; aA = [0 0 0 ];
vB1 = vA + cross(omega1,rB); vB2 = vB1;
aB1 = aA + cross(alpha1,rB) - dot(omega1,omega1)*rB; aB2 = aB1;
```

Velocity of joint C

The points B_2 and C_2 are on the link 2 and

$$\mathbf{v}_{C_2} = \mathbf{v}_{B_2} + \boldsymbol{\omega}_2 \times \mathbf{r}_{BC} = \mathbf{v}_B + \boldsymbol{\omega}_2 \times (\mathbf{r}_C - \mathbf{r}_B), \tag{3.43}$$

where the angular velocity of link 2 is $\boldsymbol{\omega}_2 = \omega_2\,\mathbf{k}$ (ω_2 is unknown).

The points D_3 and C_3 are on the link 3 and

$$\mathbf{v}_{C_3} = \mathbf{v}_{D_3} + \boldsymbol{\omega}_3 \times \mathbf{r}_{DC} = \boldsymbol{\omega}_3 \times (\mathbf{r}_C - \mathbf{r}_D), \tag{3.44}$$

where $\mathbf{v}_D = \mathbf{v}_{D_3} \equiv \mathbf{0}$ and the angular velocity of link 3 is $\boldsymbol{\omega}_3 = \omega_3\,\mathbf{k}$ (ω_3 is unknown).

Equations (3.43) and (3.44) give ($\mathbf{v}_{C_2} = \mathbf{v}_{C_3}$)

$$\mathbf{v}_B + \boldsymbol{\omega}_2 \times (\mathbf{r}_C - \mathbf{r}_B) = \boldsymbol{\omega}_3 \times (\mathbf{r}_C - \mathbf{r}_D),$$

or

$$\mathbf{v}_B + \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_2 \\ x_C - x_B & y_C - y_B & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_C - x_D & y_C - y_D & 0 \end{vmatrix} \tag{3.45}$$

Equation (3.45) represents a vectorial equation with two scalar components on x-axis and y-axis and with two unknowns ω_2 and ω_3

$$\begin{aligned} v_{Bx} - \omega_2(y_C - y_B) &= -\omega_3(y_C - y_D), \\ v_{By} + \omega_2(x_C - x_B) &= \omega_3(x_C - x_D), \end{aligned}$$

or

$$\begin{aligned} -0.471239 - \omega_2(0.422073 - 0.075) &= -\omega_3(0.422073 - 0.45), \\ 0.81621 + \omega_2(-0.0689445 - 0.129904) &= \omega_3(-0.0689445 - 0.3). \end{aligned}$$

It results

$$\omega_2 = -1.1307 \text{ rad/s} \quad \text{and} \quad \omega_3 = -2.82169 \text{ rad/s}.$$

The velocity of C is

$$\begin{aligned} & \mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\omega}_3 \times (\mathbf{r}_C - \mathbf{r}_D) = -\omega_3(y_C - y_D)\,\imath + \omega_3(x_C - x_D)\,\jmath \\ = \; & -(-2.82169)(0.422073 - 0.45)\,\imath + (-2.82169)(-0.0689445 - 0.3)\,\jmath \\ = \; & -0.0788027\,\imath + 1.04105\,\jmath \;\; \text{m/s}. \end{aligned}$$

The MATLAB commands for the angular velocities of links 2 and 3, and the velocity of C are

```
omega2z = sym('omega2z','real'); omega3z = sym('omega3z','real');
omega2 = [ 0 0 omega2z ]; omega3 = [ 0 0 omega3z ];
eqvC = vB2 + cross(omega2,rC-rB) - ( vD + cross(omega3,rC-rD) );
eqvCx = eqvC(1); eqvCy = eqvC(2);
solvC = solve(eqvCx,eqvCy);
omega2zs=eval(solvC.omega2z); omega3zs=eval(solvC.omega3z);
Omega2 = [0 0 omega2zs]; Omega3 = [0 0 omega3zs];
vC = vB2 + cross(Omega2,rC-rB);
```

Velocity of joint E

The points E_3 and D_3 are on the link 3 and

$$\begin{aligned} \mathbf{v}_E &= \mathbf{v}_{E_3} = \mathbf{v}_{D_3} + \boldsymbol{\omega}_3 \times \mathbf{r}_{DE} = \boldsymbol{\omega}_3 \times (\mathbf{r}_E - \mathbf{r}_D) \\ &= -\omega_3(y_E - y_D)\,\imath + \omega_3(x_E - x_D)\,\jmath \\ &= -(-2.82169)(0.404712 - 0.45)\,\imath + (-2.82169)(-0.298288 - 0.3)\,\jmath \\ &= -0.127788\,\imath + 1.68819\,\jmath \;\; \text{m/s}, \end{aligned}$$

or in MATLAB

```
vE = vD + cross(Omega3,rE-rD);
```

Velocity of joint F

The points F_4 and E_4 are on the link 4 and

$$\mathbf{v}_F = \mathbf{v}_{F_4} = \mathbf{v}_{E_4} + \boldsymbol{\omega}_4 \times \mathbf{r}_{EF} = \mathbf{v}_E + \boldsymbol{\omega}_4 \times (\mathbf{r}_F - \mathbf{r}_E), \tag{3.46}$$

where the angular velocity of link 4 is $\boldsymbol{\omega}_4 = \omega_4\, st\mathbf{k}$ (ω_4 is unknown).

On the other hand the velocity of F is along the vertical axis (y-axis) because slider 5 translates along y- axis

$$\mathbf{v}_F = \mathbf{v}_{F_5} = \mathbf{v}_{F_4} = v_F\,\mathbf{j}. \tag{3.47}$$

Equations (3.46) and (3.47) give

$$\mathbf{v}_E + \boldsymbol{\omega}_4 \times (\mathbf{r}_F - \mathbf{r}_E) = v_F\,\mathbf{j},$$

or

$$\mathbf{v}_E + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_4 \\ x_F - x_E & y_F - y_E & 0 \end{vmatrix} = v_F\mathbf{j} \tag{3.48}$$

Equation (3.48) represents a vectorial equation with two scalar components on x-axis and y-axis and with two unknowns ω_4 and v_F

$$\begin{aligned} v_{Ex} - \omega_4(y_F - y_E) &= 0, \\ v_{Ey} + \omega_4(x_F - x_E) &= v_F, \end{aligned}$$

or

$$\begin{aligned} -0.127788 - \omega_4(0.186177 - 0.404712) &= 0, \\ 1.68819 + \omega_4(-0.37 + 0.298288) &= v_F. \end{aligned}$$

It results

$$\omega_4 = 0.58475 \text{ rad/s} \quad \text{and} \quad v_F = 1.64625 \text{ m/s}.$$

The MATLAB commands for the angular velocities of links 4 and the velocity of F are

```
omega4z = sym('omega4z','real'); vFy = sym('vFy','real');
omega4 = [ 0 0 omega4z ]; vF = [ 0 vFy 0 ];
eqvF = vF - (vE + cross(omega4,rF-rE));
eqvFx = eqvF(1); eqvFy = eqvF(2);
solvF = solve(eqvFx,eqvFy);
omega4zs=eval(solvF.omega4z); vFys=eval(solvF.vFy);
Omega4 = [0 0 omega4zs]; VF = [0 vFys 0];
```

Acceleration of joint C

The points C_2 and B_2 are on the link 2 and

$$\mathbf{a}_{C_2} = \mathbf{a}_{B_2} + \boldsymbol{\alpha}_2 \times \mathbf{r}_{BC} - \omega_2^2\, \mathbf{r}_{BC} = \mathbf{a}_B + \boldsymbol{\alpha}_2 \times (\mathbf{r}_C - \mathbf{r}_B) - \omega_2^2\, (\mathbf{r}_C - \mathbf{r}_B), \quad (3.49)$$

where the angular acceleration of link 2 is $\boldsymbol{\alpha}_2 = \alpha_2\, \mathbf{k}$ (α_2 is unknown).

The points C_3 and D_3 are on the link 3 and

$$\mathbf{a}_{C_3} = \mathbf{a}_{D_3} + \boldsymbol{\alpha}_3 \times \mathbf{r}_{DC} - \omega_3^2\, \mathbf{r}_{DC} = \boldsymbol{\alpha}_3 \times (\mathbf{r}_C - \mathbf{r}_D) - \omega_3^2\, (\mathbf{r}_C - \mathbf{r}_D), \quad (3.50)$$

where $\mathbf{a}_D = \mathbf{a}_{D_3} \equiv \mathbf{0}$ and the angular velocity of link 3 is $\boldsymbol{\alpha}_3 = \alpha_3\, \mathbf{k}$ (α_3 is unknown).

Equations (3.49) and (3.50) give

$$\mathbf{a}_B + \boldsymbol{\alpha}_2 \times (\mathbf{r}_C - \mathbf{r}_B) - \omega_2^2\, (\mathbf{r}_C - \mathbf{r}_B) = \boldsymbol{\alpha}_3 \times (\mathbf{r}_C - \mathbf{r}_D) - \omega_3^2\, (\mathbf{r}_C - \mathbf{r}_D),$$

or

$$\mathbf{a}_B + \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha_2 \\ x_C - x_B & y_C - y_B & 0 \end{vmatrix} - \omega_2^2\, [(x_C - x_B)\, \boldsymbol{\imath} + (y_C - y_B)\, \boldsymbol{\jmath}] =$$

$$\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha_3 \\ x_C - x_D & y_C - y_D & 0 \end{vmatrix} - \omega_3^2\, [(x_C - x_D)\, \boldsymbol{\imath} + (y_C - y_D)\, \boldsymbol{\jmath}]\,. \quad (3.51)$$

Equation (3.51) represents a vectorial equation with two scalar components on x-axis and y-axis and with two unknowns α_2 and α_3

$$a_{Bx} - \alpha_2(y_C - y_B) - \omega_2^2(x_C - x_B) = -\alpha_3(y_C - y_D) - \omega_3^2(x_C - x_D),$$
$$a_{By} + \alpha_2(x_C - x_B) - \omega_2^2(y_C - y_B) = \alpha_3(x_C - x_D) - \omega_3^2(y_C - y_D),$$

or

$$-5.1284 - \alpha_2(0.422073 - 0.075) - (-1.1307)^2(-0.0689445 - 0.129904)$$
$$= -\alpha_3(0.422073 - 0.45) - (-2.82169)^2(-0.0689445 - 0.3),$$
$$-2.96088 + \alpha_2(-0.0689445 - 0.129904) - (-1.1307)^2(0.422073 - 0.075)$$
$$= \alpha_3(-0.0689445 - 0.3) - (-2.82169)^2(0.422073 - 0.45).$$

It results

$$\alpha_2 = -22.33 \text{ rad/s}^2 \quad \text{and} \quad \alpha_3 = -2.20443 \text{ rad/s}^2.$$

The acceleration of C is

$$\begin{aligned}
\mathbf{a}_C &= \boldsymbol{\alpha}_3 \times (\mathbf{r}_C - \mathbf{r}_D) - \omega_3^2\ (\mathbf{r}_C - \mathbf{r}_D) \\
&= [-\alpha_3(y_C - y_D) - \omega_3^2(x_C - x_D)]\boldsymbol{\imath} + [\alpha_3(x_C - x_D) - \omega_3^2(y_C - y_D)]\boldsymbol{\jmath} \\
&= [-(-2.20443)(0.422073 - 0.45) - (-2.82169)^2(-0.0689445 - 0.3)]\,\boldsymbol{\imath} \\
&+ [(-2.20443)(-0.0689445 - 0.3) - (-2.82169)^2(0.422073 - 0.45)]\,\boldsymbol{\jmath} \\
&= 2.87595\,\boldsymbol{\imath} + 1.03567\,\boldsymbol{\jmath}\ \ \text{m/s}^2.
\end{aligned}$$

The MATLAB commands for the angular accelerations of links 2 and 3, and the acceleration of C are

```
alpha2z = sym('alpha2z','real'); alpha3z = sym('alpha3z','real');
alpha2 = [ 0 0 alpha2z ]; alpha3 = [ 0 0 alpha3z ];
eqaC2= aB2 + cross(alpha2,rC-rB) - dot(Omega2,Omega2)*(rC-rB);
eqaC3= aD + cross(alpha3,rC-rD) - dot(Omega3,Omega3)*(rC-rD);
eqaC = eqaC2 - eqaC3;
eqaCx = eqaC(1); eqaCy = eqaC(2);
solaC = solve(eqaCx,eqaCy);
alpha2zs=eval(solaC.alpha2z); alpha3zs=eval(solaC.alpha3z);
Alpha2 = [0 0 alpha2zs]; Alpha3 = [0 0 alpha3zs];
aC = aB2 + cross(Alpha2,rC-rB) - dot(Omega2,Omega2)*(rC-rB);
```

Acceleration of joint E

The points E and D are on the link 3 and the acceleration of E is

$$\begin{aligned}
\mathbf{a}_E &= \mathbf{a}_D + \boldsymbol{\alpha}_3 \times \mathbf{r}_{DE} - \omega_2^2\ \mathbf{r}_{DE} = \boldsymbol{\alpha}_3 \times (\mathbf{r}_E - \mathbf{r}_D) - \omega_3^2\ (\mathbf{r}_E - \mathbf{r}_D) \\
&= [-\alpha_3(y_E - y_D) - \omega_3^2(x_E - x_D)]\,\boldsymbol{\imath} + [\alpha_3(x_E - x_D) - \omega_3^2(y_E - y_D)]\,\boldsymbol{\jmath} \\
&= [-(-2.20443)(0.404712 - 0.45) - (-2.82169)^2(-0.298288 - 0.3)]\,\boldsymbol{\imath} \\
&+ [(-2.20443)(-0.298288 - 0.3) - (-2.82169)^2(0.404712 - 0.45)]\,\boldsymbol{\jmath} \\
&= 4.66371\,\boldsymbol{\imath} + 1.67947\,\boldsymbol{\jmath}\ \ \text{m/s}^2.
\end{aligned}$$

The MATLAB command for the acceleration of E is

```
aE = aD + cross(Alpha3,rE-rD) - dot(Omega3,Omega3)*(rE-rD);
```

Acceleration of joint F

The points F and E are on the link 4 and

$$\mathbf{a}_F = \mathbf{a}_E + \boldsymbol{\alpha}_4 \times \mathbf{r}_{EF} - \omega_4^2\, \mathbf{r}_{EF} = \mathbf{a}_E + \boldsymbol{\alpha}_4 \times (\mathbf{r}_F - \mathbf{r}_E) - \omega_4^2\, (\mathbf{r}_F - \mathbf{r}_E), \quad (3.52)$$

where $\boldsymbol{\alpha}_4 = \alpha_4\, \mathbf{k}$ (α_4 is unknown) is the angular acceleration of link 4.

The slider 5 moves along the y-axis and the acceleration of F is

$$\mathbf{a}_F = \mathbf{a}_{F_5} = a_F\, \boldsymbol{\jmath}. \quad (3.53)$$

Equations (3.52) and (3.53) yield

$$\mathbf{a}_E + \boldsymbol{\alpha}_4 \times (\mathbf{r}_F - \mathbf{r}_E) - \omega_4^2\, (\mathbf{r}_F - \mathbf{r}_E) = a_F\, \boldsymbol{\jmath},$$

or

$$\mathbf{a}_F + \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha_2 \\ x_F - x_E & y_F - y_E & 0 \end{vmatrix} - \omega_4^2\, [(x_F - x_E)\, \boldsymbol{\imath} + (y_F - y_E)\, \boldsymbol{\jmath}] = a_F\, \boldsymbol{\jmath}. \quad (3.54)$$

Equation (3.54) represents a vectorial equation with two scalar components on x-axis and y-axis and with two unknowns α_4 and a_F

$$a_{Ex} - \alpha_4(y_F - y_E) - \omega_4^2(x_F - x_E) = 0,$$
$$a_{Ey} + \alpha_4(x_F - x_E) - \omega_4^2(y_F - y_E) = a_F,$$

or

$$4.66371 - \alpha_4(0.186177 - 0.404712) - (0.58475)^2(-0.37 + 0.298288) = 0,$$
$$1.67947 + \alpha_4(-0.37 + 0.298288) - (0.58475)^2(0.186177 - 0.404712) = a_F.$$

The following results are obtained

$$\alpha_4 = -21.453 \text{ rad/s}^2 \quad \text{and} \quad a_F = 3.29262 \text{ m/s}^2.$$

The MATLAB commands for the angular accelerations of links 4 and the acceleration of F are

```
alpha4z = sym('alpha4z','real'); aFy = sym('aFy','real');
alpha4 = [ 0 0 alpha4z ]; aF = [ 0 aFy 0 ];
eqaF = aF - (aE + cross(alpha4,rF-rE) - dot(Omega4,Omega4)*(rF-rE));
```

```
eqaFx = eqaF(1); eqaFy = eqaF(2);
solaF = solve(eqaFx,eqaFy);
alpha4zs=eval(solaF.alpha4z); aFys=eval(solaF.aFy);
Alpha4 = [0 0 alpha4zs]; AF = [0 aFys 0];
```

The MATLAB program with the results for the velocities and accelerations is given in Program 3.2.

3.7 R-TRR Mechanism

The following dimensions are given for the mechanism shown in the Fig. 3.6 $AC = 0.100$ m and $BC = 0.300$ m. The length AD is selected as $1.5\,BC$ ($AD = 1.5\,BC$). The driver link 1 rotates with a constant speed of $n = n_1 =$ 30 rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 45°$.

Solution

A cartesian reference frame with the origin at A is selected and the coordinates of joint A are $x_A = y_A = 0$. The coordinates of the joint C are $x_C = AC = 0.100$ m and $y_C = 0$. The coordinates of joint B for $\phi_1 = 45°$ are $x_B = 0.256$ m and $y_B = 0.256$ m, and the position vector of point B is $\mathbf{r}_B = x_B\,\mathbf{\imath} + y_B\,\mathbf{\jmath} = 0.256\,\mathbf{\imath} + 0.256\,\mathbf{\jmath}$ m.

The position of joint B was calculated from the equations

$$\tan\phi = \frac{y_B}{x_B} \quad \text{and} \quad (x_B - x_C)^2 + (y_B - y_C)^2 = BC^2.$$

The MATLAB commands for the position vector of B are

```
eqB1 = 'xBsol*sin(phi) = yBsol*cos(phi)';
eqB2 = 'yBsol^2+(xC-xBsol)^2-BC^2 = 0';
solB = solve(eqB1, eqB2, 'xBsol, yBsol');
xBpositions = eval(solB.xBsol); yBpositions = eval(solB.yBsol);
xB1 = xBpositions(1); xB2 = xBpositions(2);
yB1 = yBpositions(1); yB2 = yBpositions(2);
if (phi>=0 && phi<= pi)
    if yB1 >= 0 xB = xB1; yB=yB1; else xB = xB2; yB=yB2; end
end
if (phi>pi && phi<=2*pi)
    if yB1 < 0 xB = xB1; yB=yB1; else xB = xB2; yB=yB2; end
end
rB = [ xB, yB, 0 ];
```

The magnitude of the angular velocity of the driver link 1 is

$$\omega = \omega_1 = \dot{\phi}(t) = \frac{\pi\, n_1}{30} = \frac{\pi\,(30 \text{ rpm})}{30} = 3.141 \text{ rad/s}. \tag{3.55}$$

The angular velocity of link 1 is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 = \omega\,\mathbf{k} = 3.141\,\mathbf{k} \quad \text{rad/s}.$$

The link 2 and the driver link 1 have the same angular velocity $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$.

The angular acceleration of link 1 is $\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\omega}}_1 = \mathbf{0}$.

Velocity and acceleration of B_1

The velocity of the point B_1 on the link 1 is

$$\mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_B = \boldsymbol{\omega}_1 \times \mathbf{r}_B,$$

where $\mathbf{v}_A \equiv \mathbf{0}$ is the velocity of the origin $A \equiv O$.
The velocity of B_1 is

$$\mathbf{v}_{B_1} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega \\ x_B & y_B & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & 3.141 \\ 0.256 & 0.256 & 0 \end{vmatrix} = -0.804\,\boldsymbol{\imath} + 0.804\,\boldsymbol{\jmath} \ \ \text{m/s}.$$

The acceleration of the point B_1 on the link 1 is

$$\begin{aligned} \mathbf{a}_{B_1} &= \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_B) = \boldsymbol{\alpha}_1 \times \mathbf{r}_B - \boldsymbol{\omega}_1^2 \mathbf{r}_B \\ &= -\boldsymbol{\omega}_1^2 \mathbf{r}_B = -3.141^2(0.256\,\boldsymbol{\imath} + 0.256\,\boldsymbol{\jmath}) = -2.528\,\boldsymbol{\imath} - 2.528\,\boldsymbol{\jmath} \ \ \text{m/s}^2. \end{aligned}$$

Angular velocity of link 3

The velocity of the point B_2 on the link 2 is equal to the velocity of the point B_3 on the link 3 (link 2 and link 3 are connected with a rotational joint).

The points B_3 and C are on the link 3 and

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_3} = \mathbf{v}_C + \boldsymbol{\omega}_3 \times \mathbf{r}_{CB} = \boldsymbol{\omega}_3 \times (\mathbf{r}_B - \mathbf{r}_C), \tag{3.56}$$

where $\mathbf{v}_C \equiv \mathbf{0}$ and the angular velocity of link 3 is $\boldsymbol{\omega}_3 = \omega_3\,\mathbf{k}$. The angular velocity of link 3 ω_3 to be calculated.

The velocity of the point B_2 on the link 2 is calculated in terms of the velocity of the point B_1 on the link 1

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_1} + \mathbf{v}^{rel}_{B_2B_1} = \mathbf{v}_{B_1} + \mathbf{v}_{B_{21}}, \tag{3.57}$$

where $\mathbf{v}^{rel}_{B_2B_1} = \mathbf{v}_{B_{21}}$ is the relative acceleration of B_2 with respect to B_1 on link 1. This relative velocity is parallel to the sliding direction AB, $\mathbf{v}_{B_{21}} || AB$, or

$$\mathbf{v}_{B_{21}} = v_{B_{21}} \cos\phi_1\,\boldsymbol{\imath} + v_{B_{21}} \sin\phi_1\,\boldsymbol{\jmath}, \tag{3.58}$$

where $\phi_1 = 45°$. Equations (3.67), (3.65), and (3.66) give

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_B - x_C & y_B - y_C & 0 \end{vmatrix} = \mathbf{v}_{B_1} + v_{B_{21}} \cos\phi_1 \, \mathbf{\imath} + v_{B_{21}} \sin\phi_1 \, \mathbf{\jmath}. \tag{3.59}$$

Equation (3.68) represents a vectorial equations with two scalar components on x-axis and y-axis and with two unknowns ω_3 and $v_{B_{21}}$

$$-\omega_3(y_B - y_C) = v_{B_{1x}} + v_{B_{21}} \cos\phi_1,$$
$$\omega_3(x_B - x_C) = v_{B_{1y}} + v_{B_{21}} \sin\phi_1,$$

or

$$-\omega_3(0.256 - 0) = -0.804 + v_{B_{21}} \cos 45°,$$
$$\omega_3(0.256 - 0.1) = 0.804 + v_{B_{21}} \sin 45°.$$

It results

$$\omega_3 = 3.903 \text{ rad/s} \ \text{ and } \ v_{B_{21}} = -0.276 \ \text{ m/s},$$

or in vectorial form

$$\boldsymbol{\omega}_3 = 3.903\,\mathbf{k} \text{ rad/s} \ \text{ and } \ \mathbf{v}_{B_{21}} = -0.195\,\mathbf{\imath} - 0.195\,\mathbf{\jmath} \ \text{ m/s}.$$

The velocity of B_3 (or B_2) is

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_3} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & 3.903 \\ 0.256 - 0.1 & 0.256 & 0 \end{vmatrix} = -0.999\,\mathbf{\imath} + 0.609\,\mathbf{\jmath} \ \text{ m/s}.$$

The MATLAB commands for the $\boldsymbol{\omega}_3$, $\mathbf{v}_{B_{21}}$, and $\mathbf{v}_{B_3}$ are

```
omega3z = sym('omega3z','real');   % omega3z unknown
omega3 = [ 0 0 omega3z ];
vB21 = sym('vB21','real');   % vB21 unknown
vB2B1 = [ vB21*cos(phi1) vB21*sin(phi1) 0 ];
vC = [0 0 0 ];
vB3 = vC + cross(omega3,rB-rC); % vB2 = vB3 = vC + omega3 x (rB-rC)
vB2 = vB3;
eqvB = vB2 - ( vB1 + vB2B1 ); % vB2 = vB1 + vB2B1
```

```
eqvBx = eqvB(1); eqvBy = eqvB(2);
solvB = solve(eqvBx,eqvBy);
omega3zs = eval(solvB.omega3z);
vB21s = eval(solvB.vB21);
Omega3 = [0 0 omega3zs];
VB21 = vB21s*[cos(phi1) sin(phi1) 0];
VB3 = vC + cross(Omega3,rB-rC);
```

Angular acceleration of link 3

The points B_3 and C are on the link 3 and

$$\mathbf{a}_{B_2} = \mathbf{a}_{B_3} = \mathbf{a}_C + \boldsymbol{\alpha}_3 \times \mathbf{r}_{CB} - \omega_3^2 \mathbf{r}_{CB} = \boldsymbol{\alpha}_3 \times \mathbf{r}_{CB} - \omega_3^2 \mathbf{r}_{CB}, \tag{3.60}$$

where $\mathbf{a}_C \equiv \mathbf{0}$ and the angular acceleration of link 3 is $\boldsymbol{\alpha}_3 = \alpha_3 \mathbf{k}$. The angular acceleration of link 3 α_3 to be calculated.

The acceleration of the point B_2 on the link 2 is calculated in terms of the acceleration of the point B_1 on the link 1

$$\mathbf{a}_{B_2} = \mathbf{a}_{B_1} + \mathbf{a}^{rel}_{B_2B_1} + \mathbf{a}^{cor}_{B_2B_1} = \mathbf{a}_{B_1} + \mathbf{a}_{B_{21}} + \mathbf{a}^{cor}_{B_{21}}, \tag{3.61}$$

where $\mathbf{a}^{rel}_{B_2B_1} = \mathbf{a}_{B_{21}}$ is the relative acceleration of B_2 with respect to B_1 on link 1. This relative acceleration is parallel to the sliding direction AB, $\mathbf{a}_{B_{21}} || AB$, or

$$\mathbf{a}_{B_{21}} = a_{B_{21}} \cos\phi_1 \,\imath + a_{B_{21}} \sin\phi_1 \,\jmath. \tag{3.62}$$

The Coriolis acceleration of B_2 realative to B_1 is

$$\mathbf{a}^{cor}_{B_{21}} = 2\,\boldsymbol{\omega}_1 \times \mathbf{v}_{B_{21}} = 2\,\boldsymbol{\omega}_2 \times \mathbf{v}_{B_{21}} = 2 \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \omega_1 \\ v_{B_{21}} \cos\phi_1 & v_{B_{21}} \sin\phi_1 & 0 \end{vmatrix} =$$

$$2(-\omega_1 v_{B_{21}} \sin\phi_1 \,\imath + \omega_1 v_{B_{21}} \cos\phi_1 \,\jmath) =$$

$$2[-3.141(-0.276)\sin 45^\circ \,\imath + 3.141(-0.276)\cos 45^\circ \,\jmath] =$$

$$1.226\,\imath - 1.226\,\jmath \ \ \text{m/s}^2. \tag{3.63}$$

Equations (3.72), (3.69), (3.70), and (3.71) give

$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \alpha_3 \\ x_B - x_C & y_B - y_C & 0 \end{vmatrix} - \omega_3^2(\mathbf{r}_B - \mathbf{r}_C) =$$

$$\mathbf{a}_{B_1} + a_{B_{21}}(\cos\phi_1 \imath + \sin\phi_1 \jmath) + 2\,\boldsymbol{\omega}_1 \times \mathbf{v}_{B_{21}}. \tag{3.64}$$

Equation (3.73) represents a vectorial equations with two scalar components on x-axis and y-axis and with two unknowns α_3 and $a_{B_{21}}$

$$-\alpha_3(y_B - y_C) - \omega_3^2(x_B - x_C) = a_{B_{1x}} + a_{B_{21}}\cos\phi_1 - 2\omega_1 v_{B_{21}}\sin\phi_1,$$
$$\alpha_3(x_B - x_C) - \omega_3^2(y_B - y_C) = a_{B_{1y}} + a_{B_{21}}\sin\phi_1 + 2\omega_1 v_{B_{21}}\cos\phi_1,$$

or

$$-\alpha_3(0.256 - 0) - 3.903^2(0.256 - 0.1) =$$
$$-2.528 + a_{B_{21}}\cos 45^\circ + 1.226,$$
$$\alpha_3(0.256 - 0.1) - 3.903^2(0.256 - 0) =$$
$$-2.528 + a_{B_{21}}\sin 45^\circ - 1.226.$$

It results

$$\alpha_3 = -2.252 \text{ rad/s}^2 \quad \text{and} \quad a_{B_{21}} = -0.707 \ \text{ m/s}^2.$$

The relative acceleration of B_2 with respect to B_1 is

$$\mathbf{a}_{B_{21}} = -0.707\cos 45^\circ\boldsymbol{\imath} - 0.707\sin 45^\circ\boldsymbol{\jmath} = -0.500\,\boldsymbol{\imath} - 0.500\,\boldsymbol{\jmath},$$

and the acceleration of B_3 is

$$\mathbf{a}_{B_2} = \mathbf{a}_{B_3} = \boldsymbol{\alpha}_3 \times \mathbf{r}_{CB} - \omega_3^2\mathbf{r}_{CB} =$$
$$-2.252\mathbf{k} \times [(0.256 - 0.1)\boldsymbol{\imath} + (0.256 - 0)\boldsymbol{\jmath}] - 3.903^2[(0.256 - 0.1)\boldsymbol{\imath} + (0.256 - 0)\boldsymbol{\jmath}] =$$
$$-1.802\,\boldsymbol{\imath} - 4.25\,\boldsymbol{\jmath} \ \text{ m/s}^2.$$

The MATLAB commands for the $\boldsymbol{\alpha}_3$, $\mathbf{a}_{B_{21}}$, and $\mathbf{a}_{B_3}$ are

```
alpha3z = sym(’alpha3z’,’real’);   % alpha3z unknown
alpha3 = [ 0 0 alpha3z ];
aB21 = sym(’aB21’,’real’);   % aB21 unknown
aB2B1 = [ aB21*cos(phi1) aB21*sin(phi1) 0 ];
aC = [ 0 0 0 ];
aB3 = aC + cross(alpha3,rB-rC) - dot(Omega3,Omega3)*(rB-rC);
aB2 = aB3;
aB2B1cor = 2*cross(omega1,VB21);   % aB2B1cor = 2 omega1 x vB2B1
eqaB = aB2 - ( aB1 + aB2B1 + aB2B1cor ); % aB2=aB1+aB2B1+aB2B1cor
eqaBx = eqaB(1); eqaBy = eqaB(2);
solaB = solve(eqaBx,eqaBy);
```

```
alpha3zs = eval(solaB.alpha3z);
aB21s = eval(solaB.aB21);
Alpha3 = [0 0 alpha3zs];
AB21 = aB21s*[cos(phi1) sin(phi1) 0];
AB3 = aC + cross(Alpha3,rB-rC) - dot(Omega3,Omega3)*(rB-rC);
```

The relation between the angular velocities of link 2 and link 3 is

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_3 + \boldsymbol{\omega}_{23},$$

and the relative angular velocity of link 2 with respect to link 3 is

$$\boldsymbol{\omega}_{23} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_3 = 3.141\,\mathbf{k} - 3.903\,\mathbf{k} = -0.762\,\mathbf{k} \quad \text{rad/s}.$$

The relative angular acceleration of link 2 with respect to link 3 is

$$\boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3 = -\boldsymbol{\alpha}_3 = 2.252\,\mathbf{k} \quad \text{rad/s}^2,$$

where $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_1 = \mathbf{0}$.
The MATLAB program and the results for the velocities and accelerations analysis are given in Program 3.3.

3.8 R-RTR-RTR Mechanism

The planar R-RTR-RTR mechanism considered is shown in Fig. 2.9. The driver link is the rigid link 1 (the link AB). The following numerical data are given: $AB = 0.140$ m, $AC = 0.060$ m, $AE = 0.250$ m, $CD = 0.150$ m. The length of the links DF and EG are selected as $DF = 0.400$ m and $EG = 0.5$ m. The constant angular speed of the driver link 1 is 50 rpm. Find the velocities and accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 30°$.

Solution

A Cartesian reference frame xOy is selected. The joint A is the origin of the reference frame, that is, $A \equiv O$, and $x_A = 0,\ y_A = 0$. The coordinates of the joint C are: $x_C = 0,\ y_C = AC = 0.060$ m. The coordinates of the joint E are $x_E = 0,\ y_E = -AE = -0.250$ m. The coordinates of the joint B are computed for the givev angle $\phi_1 = 30°$:

$$x_B = AB\cos\phi_1 = 0.121\,\text{m} \quad \text{and} \quad y_B = AB\sin\phi_1 = 0.070\,\text{m}.$$

The coordinates of the joint D are $x_D = -0.149$ m and $y_D = 0.047$ m.
The angle of links 2 (or link 3) and 5 (or link 4) with the horizontal axis are

$$\phi_2 = \phi_3 = 0.082 \text{ rad} = 4.715° \quad \text{and} \quad \phi_4 = \phi_5 = -1.105 \text{ rad} = -63.333°.$$

The angular velocity of link 1 is constant and has the value

$$\boldsymbol{\omega}_1 = \omega_1\,\mathbf{k} = \frac{\pi n}{30}\,\mathbf{k} = \frac{\pi(50)}{30}\,\mathbf{k} = 5.235\,\mathbf{k} \ \text{ rad/s}.$$

The angular acceleration of link 1 is $\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\omega}}_1 = \mathbf{0}$.

Velocity and acceleration of $B_1 = B_2$

The velocity of the point B_1 on the link 1 is

$$\mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} = \boldsymbol{\omega}_1 \times \mathbf{r}_B,$$

where $\mathbf{v}_A \equiv \mathbf{0}$ is the velocity of the origin $A \equiv O$ and $\mathbf{r}_B = x_B\,\imath + y_B\,\jmath = 0.121\,\imath + 0.070\,\jmath$ m.

The velocity of point B_2 on the link 2 is $\mathbf{v}_{B_2} = \mathbf{v}_{B_1}$ because between the links 1 and 2 there is a rotational joint. The velocity of $B_1 = B_2$ is

$$\mathbf{v}_{B_1} = \mathbf{v}_{B_2} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \omega \\ x_B & y_B & 0 \end{vmatrix} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & 5.235 \\ 0.121 & 0.070 & 0 \end{vmatrix} = -0.366\imath + 0.634\jmath \ \text{ m/s}.$$

The acceleration of $B_1 = B_2$ is

$$\begin{aligned}\mathbf{a}_{B_1} &= \mathbf{a}_{B_2} = \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_B) = \boldsymbol{\alpha}_1 \times \mathbf{r}_B - \boldsymbol{\omega}_1^2 \mathbf{r}_B \\ &= -\boldsymbol{\omega}_1^2 \mathbf{r}_B = -5.235^2(0.121\,\boldsymbol{\imath} + 0.070\,\boldsymbol{\jmath}) = -3.323\,\boldsymbol{\imath} - 1.919\,\boldsymbol{\jmath} \ \ \text{m/s}^2.\end{aligned}$$

Angular velocity of link 3

The velocity of the point B_3 on the link 3 is calculated in terms of the velocity of the point B_2 on the link 2

$$\mathbf{v}_{B_3} = \mathbf{v}_{B_2} + \mathbf{v}^{rel}_{B_3B_2} = \mathbf{v}_{B_2} + \mathbf{v}_{B_{32}}, \tag{3.65}$$

where $\mathbf{v}^{rel}_{B_3B_2} = \mathbf{v}_{B_{32}}$ is the relative acceleration of B_3 with respect to B_2 on link 3. This relative velocity is parallel to the sliding direction BC, $\mathbf{v}_{B_{32}} || BC$, or

$$\mathbf{v}_{B_{32}} = v_{B_{32}} \cos\phi_2\,\boldsymbol{\imath} + v_{B_{32}} \sin\phi_2\,\boldsymbol{\jmath}, \tag{3.66}$$

where $\phi_2 = 4.715°$ is known from position analysis. The points B_3 and C are on the link 3 and

$$\mathbf{v}_{B_3} = \mathbf{v}_C + \boldsymbol{\omega}_3 \times \mathbf{r}_{CB} = \boldsymbol{\omega}_3 \times (\mathbf{r}_B - \mathbf{r}_C), \tag{3.67}$$

where $\mathbf{v}_C \equiv \mathbf{0}$ and the angular velocity of link 3 is

$$\boldsymbol{\omega}_3 = \omega_3\,\mathbf{k}.$$

Equations (3.65), (3.66), and (3.67) give

$$\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_B - x_C & y_B - y_C & 0 \end{vmatrix} = \mathbf{v}_{B_2} + v_{B_{32}} \cos\phi_2\,\boldsymbol{\imath} + v_{B_{32}} \sin\phi_2\,\boldsymbol{\jmath}. \tag{3.68}$$

Equation (3.68) represents a vectorial equations with two scalar components on x-axis and y-axis and with two unknowns ω_3 and $v_{B_{32}}$

$$\begin{aligned}-\omega_3(y_B - y_C) &= v_{Bx} + v_{B_{32}} \cos\phi_2, \\ \omega_3(x_B - x_C) &= v_{By} + v_{B_{32}} \sin\phi_2,\end{aligned}$$

or

$$\begin{aligned}-\omega_3(0.070 - 0.060) &= -0.366 + v_{B_{32}} \cos 4.715°, \\ \omega_3(0.121 - 0) &= 0.634 + v_{B_{32}} \sin 4.715°.\end{aligned}$$

It results

$$\omega_3 = \omega_2 = 5.448 \text{ rad/s} \quad \text{and} \quad v_{B_{32}} = 0.313 \text{ m/s},$$

or in vectorial form

$$\boldsymbol{\omega}_3 = \boldsymbol{\omega}_2 = 5.448\,\mathbf{k} \text{ rad/s} \quad \text{and} \quad \mathbf{v}_{B_{32}} = 0.312\,\mathbf{\imath} + 0.025\,\mathbf{\jmath}. \text{ m/s}.$$

The MATLAB commands for the $\boldsymbol{\omega}_3$ and $\mathbf{v}_{B_{32}}$ are

```
omega3z=sym('omega3z','real'); vB32=sym('vB32','real');
omega3 = [ 0 0 omega3z ];   % omega3z unknown (to be calculated)
vC = [0 0 0 ];  % C is fixed
vB3 = vC + cross(omega3,rB-rC);   % vB3 = vC + omega3 x rCB
vB3B2 = vB32*[ cos(phi2) sin(phi2) 0];   % vB32 unknown
eqvB = vB3 - vB2 - vB3B2;   % vB3 = vB2 + vB3B2 (vectorial equation)
eqvBx = eqvB(1); eqvBy = eqvB(2);
% two equations eqvBx & eqvBy with two unknowns omega3z & vB32
solvB = solve(eqvBx,eqvBy);   % solve for omega3z and vB32
omega3zs=eval(solvB.omega3z); vB32s=eval(solvB.vB32);
Omega3 = [0 0 omega3zs]; Omega2 = Omega3;
VB32 = vB32s*[cos(phi2) sin(phi2) 0];
```

Angular acceleration of link 3

The acceleration of the point B_3 on the link 3 is calculated in terms of the acceleration of the point B_2 on the link 2

$$\mathbf{a}_{B_3} = \mathbf{a}_{B_2} + \mathbf{a}^{rel}_{B_3B_2} + \mathbf{a}^{cor}_{B_3B_2} = \mathbf{a}_{B_2} + \mathbf{a}_{B_{32}} + \mathbf{a}^{cor}_{B_{32}}, \tag{3.69}$$

where $\mathbf{a}^{rel}_{B_3B_2} = \mathbf{a}_{B_{32}}$ is the relative acceleration of B_3 with respect to B_2 on link 3. This relative acceleration is parallel to the sliding direction BC, $\mathbf{a}_{B_{32}}||BC$, or

$$\mathbf{a}_{B_{32}} = a_{B_{32}} \cos\phi_2\,\mathbf{\imath} + a_{B_{32}} \sin\phi_2\,\mathbf{\jmath}. \tag{3.70}$$

The Coriolis acceleration of B_3 realative to B_2 is

$$\mathbf{a}^{cor}_{B_{32}} = 2\,\boldsymbol{\omega}_3 \times \mathbf{v}_{B_{32}} = 2\,\boldsymbol{\omega}_2 \times \mathbf{v}_{B_{32}} = 2\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ v_{B_{32}}\cos\phi_2 & v_{B_{32}}\sin\phi_2 & 0 \end{vmatrix} =$$

$$2(-\omega_3 v_{B_{32}} \sin\phi_2\,\mathbf{\imath} + \omega_3 v_{B_{32}} \cos\phi_2\,\mathbf{\jmath}) =$$

$$2[-5.448(0.313)\sin 4.715^\circ\,\mathbf{\imath} + 5.448(0.313)\cos 4.715^\circ\,\mathbf{\jmath}] =$$

$$-0.280\,\mathbf{\imath} + 3.400\,\mathbf{\jmath} \text{ m/s}^2. \tag{3.71}$$

The points B_3 and C are on the link 3 and

$$\mathbf{a}_{B_3} = \mathbf{a}_C + \boldsymbol{\alpha}_3 \times \mathbf{r}_{CB} - \omega_3^2 \mathbf{r}_{CB}, \tag{3.72}$$

where $\mathbf{a}_C \equiv \mathbf{0}$ and the angular acceleration of link 3 is

$$\boldsymbol{\alpha}_3 = \alpha_3 \mathbf{k}.$$

Equations (3.69), (3.70), (3.71), and (3.72) give

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha_3 \\ x_B - x_C & y_B - y_C & 0 \end{vmatrix} - \omega_3^2(\mathbf{r}_B - \mathbf{r}_C) =$$
$$\mathbf{a}_{B_2} + a_{B_{32}}(\cos\phi_2 \mathbf{\imath} + \sin\phi_2 \mathbf{\jmath}) + 2\,\boldsymbol{\omega}_3 \times \mathbf{v}_{B_{32}}. \tag{3.73}$$

Equation (3.73) represents a vectorial equations with two scalar components on x-axis and y-axis and with two unknowns α_3 and $a_{B_{32}}$

$$-\alpha_3(y_B - y_C) - \omega_3^2(x_B - x_C) = a_{Bx} + a_{B_{32}}\cos\phi_2 - 2\omega_3 v_{B_{32}}\sin\phi_2,$$
$$\alpha_3(x_B - x_C) - \omega_3^2(y_B - y_C) = a_{By} + a_{B_{32}}\sin\phi_2 + 2\omega_3 v_{B_{32}}\cos\phi_2,$$

or

$$-\alpha_3(0.070 - 0.060) - 5.448^2(0.121 - 0) =$$
$$-3.323 + a_{B_{32}}\cos 4.715^\circ - 2(5.448)(0.313)\sin 4.715^\circ,$$
$$\alpha_3(0.121 - 0) - 5.448^2(0.070 - 0.060) =$$
$$-1.919 + a_{B_{32}}\sin 4.715^\circ + 2(5.448)(0.313)\cos 4.715^\circ.$$

It results

$$\alpha_3 = \alpha_2 = 14.568 \text{ rad/s}^2 \quad \text{and} \quad a_{B_{32}} = -0.140 \text{ m/s}^2.$$

The MATLAB commands for the $\boldsymbol{\alpha}_3$ and $\mathbf{a}_{B_{32}}$ are

```
aB3B2cor = 2*cross(Omega3,VB32);   % Coriolis acceleration
alpha3z=sym('alpha3z','real'); aB32=sym('aB32','real');
alpha3 = [ 0 0 alpha3z ];   % alpha3z unknown
aC = [0 0 0 ];   % C is fixed
aB3 = aC + cross(alpha3,rB-rC) - dot(Omega3,Omega3)*(rB-rC);
aB3B2 = aB32*[ cos(phi2) sin(phi2) 0];   % aB32 unknown
```

```
eqaB = aB3 - aB2 - aB3B2 - aB3B2cor;   % aB3 = aB2 + aB3B2 + aB3B2cor
eqaBx = eqaB(1); eqaBy = eqaB(2);
solaB = solve(eqaBx,eqaBy);
alpha3zs=eval(solaB.alpha3z);
aB32s=eval(solaB.aB32);
Alpha3 = [0 0 alpha3zs];
Alpha2 = Alpha3;
AB32 = aB32s*[cos(phi2) sin(phi2) 0];
```

Velocity and acceleration of $D_3 = D_4$
The velocity of $D_3 = D_4$ is

$$\mathbf{v}_{D_3} = \mathbf{v}_{D_4} = \mathbf{v}_C + \boldsymbol{\omega}_3 \times \mathbf{r}_{CD} = \boldsymbol{\omega}_3 \times (\mathbf{r}_D - \mathbf{r}_C) =$$

$$\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_D - x_C & y_D - y_C & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & 5.448 \\ -0.149 - 0 & 0.047 - 0.060 & 0 \end{vmatrix} =$$

$$0.067\,\boldsymbol{\imath} - 0.814\,\boldsymbol{\jmath} \text{ m/s}.$$

The acceleration of $D_3 = D_4$ is

$$\mathbf{a}_{D_3} = \mathbf{a}_{D_4} = \mathbf{a}_C + \boldsymbol{\alpha}_3 \times \mathbf{r}_{CD} - \omega_3^2 \mathbf{r}_{CD} = \boldsymbol{\alpha}_3 \times (\mathbf{r}_D - \mathbf{r}_C) - \omega_3^2(\mathbf{r}_D - \mathbf{r}_C) =$$

$$\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha_3 \\ x_D - x_C & y_D - y_C & 0 \end{vmatrix} - \omega_3^2\left[(x_D - x_C)\boldsymbol{\imath} + (y_D - y_C)\boldsymbol{\jmath}\right] =$$

$$\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & 14.568 \\ -0.149 - 0 & 0.047 - 0.060 & 0 \end{vmatrix} -$$

$$5.448^2\left[(-0.149 - 0)\,\boldsymbol{\imath} + (0.047 - 0.060)\,\boldsymbol{\jmath}\right] =$$

$$4.617\,\boldsymbol{\imath} - 1.811\,\boldsymbol{\jmath} \text{ m/s}^2.$$

The MATLAB commands for the velocity and acceleration of $D_3 = D_4$ are

```
vD3 = vC + cross(Omega3,rD-rC); % D3 & C points on link 3
vD4 = vD3;
aD3 = aC + cross(Alpha3,rD-rC) - dot(Omega3,Omega3)*(rD-rC);
aD4 = aD3;
```

Angular velocity of link 5

The velocity of the point D_5 on the link 5 is calculated in terms of the velocity of the point D_4 on the link 4

$$\mathbf{v}_{D_5} = \mathbf{v}_{D_4} + \mathbf{v}_{D_{54}}. \tag{3.74}$$

This relative velocity of D_5 with respect to D_4 is parallel to the sliding direction DE, $\mathbf{v}_{D_{54}}||DE$, or

$$\mathbf{v}_{D_{54}} = v_{D_{54}} \cos\phi_5\, \mathbf{\imath} + v_{D_{54}} \sin\phi_5\, \mathbf{\jmath}. \tag{3.75}$$

The points D_5 and E are on the link 5 and

$$\mathbf{v}_{D_5} = \mathbf{v}_E + \boldsymbol{\omega}_5 \times \mathbf{r}_{ED} = \boldsymbol{\omega}_5 \times (\mathbf{r}_D - \mathbf{r}_E), \tag{3.76}$$

where $\mathbf{v}_E \equiv \mathbf{0}$ and the angular velocity of link 5 is

$$\boldsymbol{\omega}_5 = \omega_5 \mathbf{k}.$$

Equations (3.74), (3.75), and (3.76) give

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_5 \\ x_D - x_E & y_D - y_E & 0 \end{vmatrix} = \mathbf{v}_{D_4} + v_{D_{54}}(\cos\phi_5\, \mathbf{\imath} + \sin\phi_5\, \mathbf{\jmath}). \tag{3.77}$$

Equation (3.77) represents a vectorial equations with two scalar components on x-axis and y-axis and with two unknowns ω_5 and $v_{D_{54}}$

$$\begin{aligned} -\omega_5(y_D - y_E) &= v_{D4x} + v_{D_{54}} \cos\phi_5, \\ \omega_5(x_D - x_E) &= v_{D4y} + v_{D_{54}} \sin\phi_5, \end{aligned}$$

or

$$\begin{aligned} -\omega_5(0.047 - 0.250) &= 0.067 + v_{D_{54}} \cos(-63.333°), \\ \omega_5(-0.149 - 0) &= -0.814 + v_{D_{54}} \sin(-63.333°). \end{aligned}$$

It results

$$\omega_5 = \omega_4 = 0.917 \text{ rad/s} \text{ and } v_{D_{54}} = -0.757 \text{ m/s}.$$

The MATLAB commands for the $\boldsymbol{\omega}_5$ and $\mathbf{v}_{D_{54}}$ are

```
omega5z=sym('omega5z','real');
vD54=sym('vD54','real');
omega5 = [ 0 0 omega5z ];   % omega5z unknown
vE = [0 0 0 ];
vD5 = vE + cross(omega5,rD-rE);   % D5 & E points on link 5
vD5D4 = vD54*[ cos(phi5) sin(phi5) 0];   % vD54 unknown
eqvD = vD5 - vD4 - vD5D4;   % vD5 = vD4 + vD5D4
eqvDx = eqvD(1); eqvDy = eqvD(2);
solvD = solve(eqvDx,eqvDy);
omega5zs=eval(solvD.omega5z);
vD54s=eval(solvD.vD54);
Omega5 = [0 0 omega5zs];
Omega4 = Omega5;
VD54 = vD54s*[cos(phi5) sin(phi5) 0];
```

Angular acceleration of link 5

The acceleration of the point D_5 on the link 5 is calculated in terms of the acceleration of the point D_4 on the link 4

$$\mathbf{a}_{D_5} = \mathbf{a}_{D_4} + \mathbf{a}_{D_{54}} + \mathbf{a}^{cor}_{D_{54}}, \tag{3.78}$$

This relative acceleration $\mathbf{a}_{B_{32}}$ is parallel to the sliding direction DE, $\mathbf{a}_{D_{54}}||DE$, or

$$\mathbf{a}_{D_{54}} = a_{D_{54}} \cos\phi_5 \mathbf{\imath} + a_{D_{54}} \sin\phi_5 \mathbf{\jmath}. \tag{3.79}$$

The Coriolis acceleration of D_5 realative to D_4 is

$$\mathbf{a}^{cor}_{D_{54}} = 2\,\boldsymbol{\omega}_4 \times \mathbf{v}_{D_{54}} = 2\,\boldsymbol{\omega}_5 \times \mathbf{v}_{D_{54}} = 2\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_5 \\ v_{D_{54}}\cos\phi_5 & v_{D_{54}}\sin\phi_5 & 0 \end{vmatrix} =$$

$$2(-\omega_5 v_{D_{54}} \sin\phi_5\,\mathbf{\imath} + \omega_5 v_{D_{54}} \cos\phi_5\,\mathbf{\jmath}) =$$

$$2[-0.917(-0.757)\sin(-63.333^\circ)\,\mathbf{\imath} + 0.917(-0.757)\cos(-63.333^\circ)\,\mathbf{\jmath}] =$$

$$-0.280\,\mathbf{\imath} + 3.400\,\mathbf{\jmath} \ \ \text{m/s}^2. \tag{3.80}$$

The points D_5 and E are on the link 5 and

$$\mathbf{a}_{D_5} = \mathbf{a}_E + \boldsymbol{\alpha}_5 \times \mathbf{r}_{ED} - \omega_5^2 \mathbf{r}_{ED}, \tag{3.81}$$

where $\mathbf{a}_E \equiv \mathbf{0}$ and the angular acceleration of link 5 is

$$\boldsymbol{\alpha}_5 = \alpha_5\,\mathbf{k}.$$

Equations (3.78), (3.79), (3.80), and (3.81) give

$$\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha_5 \\ x_D - x_E & y_D - y_E & 0 \end{vmatrix} - \omega_5^2(\mathbf{r}_D - \mathbf{r}_E) =$$
$$\mathbf{a}_{D_4} + a_{D_{54}}(\cos\phi_5\,\boldsymbol{\imath} + \sin\phi_5\,\boldsymbol{\jmath}) + 2\ \boldsymbol{\omega}_5 \times \mathbf{v}_{D_{54}}. \tag{3.82}$$

Equation (3.82) represents a vectorial equations with two scalar components on x-axis and y-axis and with two unknowns α_5 and $a_{D_{54}}$

$$-\alpha_5(y_D - y_E) - \omega_5^2(x_D - x_E) = a_{D4x} + a_{D_{54}}\cos\phi_5 - 2\omega_5 v_{D_{54}}\sin\phi_5,$$
$$\alpha_5(x_D - x_E) - 2\omega_5^2(y_D - y_E) = a_{D4y} + a_{D_{54}}\sin\phi_5 + 2\omega_5 v_{D_{54}}\cos\phi_5,$$

or

$$-\alpha_5(0.047 - 0.250) - 0.917^2(-0.149 - 0) =$$
$$4.617 + a_{D_{54}}\cos(-63.333^\circ) - 2(0.917)(-0.757)\sin(-63.333^\circ),$$
$$\alpha_5(-0.149 - 0) - 0.917^2(0.047 - 0.250) =$$
$$-1.811 + a_{D_{54}}\sin(-63.333^\circ) + 2(0.917)(-0.757)\cos(-63.333^\circ).$$

It results

$$\alpha_5 = \alpha_4 = -5.771 \text{ rad/s}^2 \text{ and } a_{D_{54}} = 3.411 \text{ m/s}^2.$$

The MATLAB commands for the $\boldsymbol{\alpha}_5$ and $\mathbf{a}_{D_{54}}$ are

```
aD5D4cor = 2*cross(Omega5,VD54);   % Coriolis acceleration
alpha5z=sym('alpha5z','real');
aD54=sym('aD54','real');
alpha5 = [ 0 0 alpha5z ];   % alpha5z unknown
aE = [0 0 0 ];
aD5 = aE + cross(alpha5,rD-rE) - dot(Omega5,Omega5)*(rD-rE);
aD5D4 = aD54*[ cos(phi5) sin(phi5) 0];   % aD54 unknown
eqaD = aD5 - aD4 - aD5D4 - aD5D4cor;
eqaDx = eqaD(1); eqaDy = eqaD(2);
```

```
solaD = solve(eqaDx,eqaDy);
alpha5zs=eval(solaD.alpha5z);
aD54s=eval(solaD.aD54);
Alpha5 = [0 0 alpha5zs];
Alpha4 = Alpha5;
AD54 = aD54s*[cos(phi5) sin(phi5) 0];
```

The MATLAB program and the results for the velocities and accelerations analysis are given in Program 3.4.

3.9 Derivative Method

Another method for obtaining the velocities and/or accelerations of links and joints is the compute the derivatives of the positions and/or velocities with respect to time.

R-RTR-RTR Mechanism

The derivative method will be explained using the planar R-RTR-RTR mechanism considered in subsection 3.8 and shown in Fig. 2.9.

The angular velocity of link 1 is constant and has the value

$$n = 50 \text{ rpm and } \omega = \dot{\phi} = \frac{\pi n}{30} = 5.235 \text{ rad/s},$$

or in MATLAB

```
n = 50 ;    % rpm of the driver link (constant)
omega = n*pi/30;    % rad/s (omega=constant)
```

The velocity is obtained taking the derivative of the position with respect to time, t. The symbolic variable t is introduced in MATLAB with the statement `sym`

```
t = sym('t','real');
```

The coordinates of the joint B are

$$x_B(t) = AB\cos\phi(t) \quad \text{and} \quad y_B(t) = AB\sin\phi(t),$$

and the position vector of B is $\mathbf{r}_B = x_B\,\mathbf{\imath} + x_B\,\mathbf{\jmath}$.
To calculate symbolically the position of the joint B, the following MATLAB commands are used

```
xB = AB*cos(sym('phi(t)'));
yB = AB*sin(sym('phi(t)'));
rB = [ xB yB 0 ];   % position vector of B in terms of phi(t)
```

The statement `sym('phi(t)')` represents the mathematical function $\phi(t)$ and is introduced with the command `sym` that constructs symbolic numbers, variables and objects. The function `phi` has one argument, the time `t`.
To calculate numerically the position of the joint B, the symbolic variables

need to be substituted with the input data. To apply a transformation rule to a particular expression `expr`, type `subs(expr,lhs,rhs)`. The statement `subs(expr,lhs,rhs)` replaces `lhs` with `rhs` in the symbolic expression `expr`.

For the mechanism, the numerical values for the joint B are

```
xBn = subs(xB, 'phi(t)', pi/6);    % xB for phi(t)=pi/6
yBn = subs(yB, 'phi(t)', pi/6);    % yB for phi(t)=pi/6
rBn = subs(rB, 'phi(t)', pi/6);    % rB for phi(t)=pi/6
```

The numerical values of the vector `rBn` are printed with

```
fprintf('rB = [ %g, %g, %g ] (m) \n', rBn);
```

The linear velocity vector of $B_1 = B_2$ is

$$\mathbf{v}_{B_1} = \mathbf{v}_{B_2} = \dot{x}_B \mathbf{\imath} + \dot{y}_B \mathbf{\jmath},$$

where

$$\dot{x}_B = \frac{dx_B}{dt} = -AB\dot{\phi}\sin\phi \quad \text{and} \quad \dot{y}_B = \frac{dy_B}{dt} = AB\dot{\phi}\cos\phi,$$

are the components of the velocity vector of $B_1 = B_2$.
To calculate symbolically the components of the velocity vector using the MATLAB the command `diff(f,t)` is used, which gives the derivative of `f` with respect to `t`.
The symbolical expression of the velocity vector of $B_1 = B_2$ is obtain with the statement

```
vB = diff(rB,t); % vB1=vB2 in terms of phi(t) and diff(phi(t),t)
```

The vector `vB` is a symbolic expression in terms of `phi(t)` and `diff(phi(t),t)`

```
vB = [-7/50*sin(phi(t))*diff(phi(t),t), 7/50*cos(phi(t))*diff(phi(t),t), 0]
```

The numerical values for the components of the velocity of $B_1 = B_2$ are

$$\dot{x}_B = -0.140\ (5.235)\sin 30^\circ = -0.366\,\text{m/s},$$
$$\dot{y}_B = 0.140\ (5.235)\cos 30^\circ = 0.634\,\text{m/s}.$$

To obtain the numerical values in MATLAB first `diff('phi(t)',t)` is replaced with `omega` and then `phi(t)` is replaced with `pi/6`

```
vBnn = subs(vB,diff('phi(t)',t),omega);
% replaces diff('phi(t)',t) with omega ( phi'(t)=omega ) in vB
vBn = subs(vBnn,'phi(t)',pi/6);
% replaces phi(t) with pi/6 in vBnn
```

Instead of replacing `diff('phi(t)',t)` with `omega` and then replacing `'phi(t)'` with `pi/6`, a list with the symbolical variable `'phi(t)'`, `diff('phi(t)',t)`, and `diff('phi(t)',t,2)` is created

```
slist = {'phi(t)', diff('phi(t)',t), diff('phi(t)',t,2)};
```

Next a list with the numerical values for `slist` is introduced

```
nlist = {pi/6, omega, 0};  % numerical values for slist
% 'phi(t)' -> pi/6
% diff('phi(t)',t) -> omega
% diff('phi(t)',t,2) -> 0
```

The velocities and accelerations need to be calculated at the moment when driver link makes and angle $\phi(t) = \pi/6$ with the horizontal and $\dot{\phi}(t) = \omega$ and $\ddot{\phi}(t) = \dot{\omega} = 0$. To obtain the numerical value for the symbolic vector `rB` the following statements are introduced

```
vBn = subs(vB,slist,nlist);  % replaces slist with nlist in vB
VB = double(vBn); %converts the symbolic vBn to a numeric object
fprintf('vB1 = vB2 = [ %g, %g, %g ] (m/s) \n', VB);
```

The statement `double(S)` converts the symbolic object `S` to a numeric object. The linear acceleration vector of $B_1 = B_2$ is

$$\mathbf{a}_{B_1} = \mathbf{a}_{B_2} = \ddot{x}_B \mathbf{\imath} + \ddot{y}_B \mathbf{\jmath},$$

where

$$\ddot{x}_B = \frac{d\dot{x}_B}{dt} = -AB\dot{\phi}^2 \cos\phi - AB\ddot{\phi}\sin\phi,$$

$$\ddot{y}_B = \frac{d\dot{y}_B}{dt} = -AB\dot{\phi}^2 \sin\phi + AB\ddot{\phi}\cos\phi.$$

For the considered mechanism the angular acceleration of the link 1 is $\ddot{\phi} = \dot{\omega} = 0$. The numerical values of the acceleration of B are

$$\ddot{x}_B = -0.140\ (5.235)^2 \cos 30^\circ = -3.323\,\mathrm{m/s^2},$$
$$\ddot{y}_B = -0.140\ (5.235)^2 \sin 30^\circ = -1.919\,\mathrm{m/s^2}.$$

The MATLAB command used to calculate symbolically the acceleration vector is

```
aB = diff(vB,t);  % acceleration of B1=B2
```

The numerical value for the vector `aB` is obtained with

```
aBn = double(subs(aB,slist,nlist));  % numerical value for aB
fprintf('aB1 = aB2 = [ %g, %g, %g ] (m/s^2) \n', aBn);
```

The coordinates of the joint D are x_D and y_D. The MATLAB commands used to calculate the position of D are

```
eqnD1 = '( xDsol - xC )^2 + ( yDsol - yC )^2 = CD^2 ';
eqnD2 = '(yB - yC) / (xB - xC) = (yDsol - yC) / (xDsol - xC)';
solD = solve(eqnD1, eqnD2, 'xDsol, yDsol');
```

Two sets of solutions are found for the position of the joint D that are functions of the angle $\phi(t)$ (i.e., functions of time):

```
xDpositions = eval(solD.xDsol); yDpositions = eval(solD.yDsol);
xD1 = xDpositions(1); xD2 = xDpositions(2);
yD1 = yDpositions(1); yD2 = yDpositions(2);
```

To determine the correct position of the joint D for the mechanism, an additional condition is needed. For the first quadrant, $0 \leq \phi \leq 90^\circ$, the condition is $x_D \leq x_C$.
This condition using the MATLAB command is:

```
xD1n = subs(xD1,'phi(t)',pi/6); % xD1 for phi(t)=pi/6
```

```
if xD1n < xC
   xD = xD1; yD = yD1;
else
   xD = xD2; yD = yD2;
end
rD = [ xD yD 0 ]; % position vector of D in term of phi(t)
```

The numerical solutions are printed using MATLAB

```
xDn = subs(xD,'phi(t)',pi/6); % xD for phi(t)=pi/6
yDn = subs(yD,'phi(t)',pi/6); % yD for phi(t)=pi/6
rDn = [ xDn yDn 0 ]; % rD for phi(t)=pi/6
fprintf('rD = [ %g, %g, %g ] (m) \n', rDn);
```

The linear velocity vector of the joint $D_3 = D_4$ (on link 3 or link 4) is

$$\mathbf{v}_{D_3} = \mathbf{v}_{D_4} = \dot{x}_D \mathbf{\imath} + \dot{y}_D \mathbf{\jmath},$$

where

$$\dot{x}_D = \frac{dx_D}{dt} \quad \text{and} \quad \dot{y}_D = \frac{dy_D}{dt},$$

are the components of the velocity vector of the joint D, respectively, on the x-axis and the y-axis.
To calculate symbolically the components of this velocity vector the following MATLAB commands are used

```
vD = diff(rD,t); % vD in terms of phi(t) and diff('phi(t)',t)
```

The numerical solutions are printed using MATLAB

```
vDn = double(subs(vD,slist,nlist)); % numerical value for vD
fprintf('vD3 = vD4 = [ %g, %g, %g ] (m/s) \n', vDn);
```

For the considered mechanism the numerical values are

$$\dot{x}_D = 0.067\,\text{m/s and } \dot{y}_D = -0.814\,\text{m/s}.$$

The linear acceleration vector of $D_3 = D_4$ is

$$\mathbf{a}_{D_3} = \mathbf{a}_{D_4} = \ddot{x}_D \mathbf{\imath} + \ddot{y}_D \mathbf{\jmath},$$

where

$$\ddot{x}_D = \frac{d\dot{x}_D}{dt} \quad \text{and} \quad \ddot{y}_D = \frac{d\dot{y}_D}{dt}.$$

To calculate symbolically the components of the acceleration vector the following MATLAB commands are used:

```
aD = diff(vD,t);
```

The numerical values of the acceleration of $D_3 = D_4$ are

$$\ddot{x}_D = 4.617\,\mathrm{m/s^2} \quad \text{and} \quad \ddot{y}_D = -1.811\,\mathrm{m/s^2},$$

and can be printed using MATLAB

```
aDn = double(subs(aD,slist,nlist)); % numerical value for aD
fprintf('aD3 = aD4 = [ %g, %g, %g ] (m/s^2) \n', aDn);
```

The angle $\phi_2(t) = \phi_3(t)$ is determined as a function of time t from the equation of the slope of the line BC:

$$\tan\phi_2(t) = \tan\phi_3(t) = \frac{y_B(t) - y_C}{x_B(t) - x_C}.$$

The MATLAB function `atan(z)` gives the arc tangent of the number `z` and the angle ϕ_2 is calculated symbolically

```
phi2 = atan((yB-yC)/(xB-xC));
```

The numerical value is given by

```
phi2n = subs(phi2,'phi(t)',pi/6);  % phi2 for phi(t)=pi/6
```

The numerical solutions are printed using MATLAB

```
fprintf('phi2 = phi3 = %g (degrees) \n', phi2n*180/pi);
```

The angular velocity $\omega_2(t) = \omega_3(t)$ is the derivative with respect to time of the angle $\phi_2(t)$

$$\omega_2 = \frac{d\phi_2(t)}{dt}.$$

Symbolically, the angular velocity $\omega_2 = \omega_3$ is calculated using MATLAB

```
dphi2 = diff(phi2,t);  % omega2 in terms of phi(t) and diff('phi(t)',t)
```

and is the numerical values are printed using the MATLAB statements

```
dphi2nn = subs(dphi2,diff('phi(t)',t),omega);
dphi2n = subs(dphi2nn,'phi(t)',pi/6); % numerical value for omega2
fprintf('omega2 = omega3 = %g (rad/s) \n', dphi2n);
```

The angular acceleration $\alpha_2(t) = \alpha_3(t)$ is the derivative with respect to time of the angular velocity $\omega_2(t)$:

$$\alpha_2(t) = \frac{d\omega_2(t)}{dt}.$$

Symbolically, using MATLAB, the angular acceleration α_2 is

```
ddphi2 = diff(dphi2,t);
```

The numerical values of the angles, angular velocities, and angular accelerations for the links 2 and 3 are

```
ddphi2n = double(subs(ddphi2,slist,nlist)); % numerical value
```

The numerical solutions are printed using MATLAB

```
fprintf('alpha2 = alpha3 = %g (rad/s ^2) \n', ddphi2n);
```

The results are

$$\phi_3 = \phi_2 = 0.082 \text{ rad}, \quad \omega_3 = \omega_2 = 5.448 \text{ rad/s}, \quad \alpha_3 = \alpha_2 = 14.568 \text{ rad/s}^2.$$

The angle $\phi_4(t) = \phi_5(t)$ is determined as a function of time t from the following equation:

$$\tan\phi_4(t) = \tan\phi_5(t) = \frac{y_D(t) - y_E}{x_D(t) - x_E},$$

and symbolically using MATLAB

```
ddphi4 = diff(dphi4,t);
```

The angular velocity $\omega_4(t) = \omega_5(t)$ is the derivative with respect to time of the angle $\phi_4(t)$

$$\omega_4 = \frac{d\phi_4(t)}{dt}.$$

To calculate symbolically the angular velocity ω_4 using MATLAB, the following command is used

```
dphi4 = diff(phi4,t);
```

The angular acceleration $\alpha_4(t) = \alpha_5(t)$ is the derivative with respect to time of the angular velocity $\omega_4(t)$:

$$\alpha_4(t) = \frac{d\omega_4(t)}{dt},$$

and it is calculated symbolically with MATLAB

```
ddphi4 = diff(dphi4,t);
```

The numerical values of the angles, angular velocities, and angular accelerations for the links 5 and 4 are

$$\phi_5 = \phi_4 = 2.036 \text{ rad}, \quad \omega_5 = \omega_4 = 0.917 \text{ rad/s}, \quad \alpha_5 = \alpha_4 = -5.771 \text{ rad/s}^2.$$

The numerical solutions printed with MATLAB are

```
dphi4n = double(subs(dphi4,slist,nlist)); % numeric omega4
fprintf('omega4 = omega5 = %g (rad/s) \n', dphi4n);
ddphi4n = double(subs(ddphi4,slist,nlist)); % numeric alpha4
fprintf('alpha4 = alpha5 = %g (rad/s^2) \n', ddphi4n);
```

The MATLAB program for velocity and acceleration analysis and the results are given in Program 3.5.

R-TRR Mechanism

The mechanism considered at subsection 3.7 [Fig. 3.6(a)] will be analyzed using the derivative method. The following dimensions are given $AC = 0.100$ m, $BC = 0.300$ m, and $\phi = \phi_1 = 45°$. The coordinates of joint B are $x_B = y_B = 0.256$ m and the driver link 1 rotates with a constant speed of $n_1 = 30$ rpm. Find the velocities and the accelerations of the mechanism using the derivative method.

Solution

A cartesian reference frame with the origin at A is selected. The coordinates of joint A are $x_A = y_A = 0$. The coordinates of the joint C are $x_C = AC = 0.100$ m and $y_C = 0$.

The position of joint B is calculated from the equations

$$\tan\phi(t) = \frac{y_B(t)}{x_B(t)} \quad \text{and} \quad [x_B(t) - x_C]^2 + [y_B(t) - y_C]^2 = BC^2,$$

or

$$\begin{aligned} x_B(t)\,\sin\phi(t) &= y_B(t)\,\cos\phi(t),\\ [x_B(t) - x_C]^2 + [y_B(t) - y_C]^2 &= BC^2. \end{aligned} \tag{3.83}$$

The coordinates of joint B are $x_B = y_B = 0.256$ m.

The MATLAB statements for the positions are

```
AC = 0.1 ; BC = 0.30 ; xA = 0 ; yA = 0 ; xC = AC ; yC = 0 ;
n = 30 ; omega = n*pi/30;
t = sym('t','real') ;
phi = sym('phi(t)') ;
xB = sym('xB(t)') ;
yB = sym('yB(t)') ;

eqB1 = tan(phi) - yB/xB ;
eqB2 = ( xB - xC )^2 + ( yB - yC )^2 - BC^2 ;

sp = {'phi(t)','xB(t)','yB(t)'} ;
np = {pi/4,'xBn','yBn'} ;
eqB1p = subs(eqB1,sp,np) ;
```

```
eqB2p = subs(eqB2,sp,np) ;
solBp = solve(eqB1p, eqB2p) ;
xBpositions = eval(solBp.xBn) ;
yBpositions = eval(solBp.yBn) ;
xB1 = xBpositions(1); xB2 = xBpositions(2) ;
yB1 = yBpositions(1); yB2 = yBpositions(2);
if yB1 > 0 xBp = xB1; yBp = yB1; else xBp = xB2; yBp = yB2; end
rB = [xBp yBp 0] ;
fp = {pi/4,xBp,yBp} ;
```

The linear velocity of point B on link 3 or 2 is

$$\mathbf{v}_{B_3} = \mathbf{v}_{B_2} = \dot{x}_B\boldsymbol{\imath} + \dot{y}_B\boldsymbol{\jmath},$$

where

$$\dot{x}_B = \frac{dx_B}{dt} \quad \text{and} \quad \dot{y}_B = \frac{dy_B}{dt}.$$

The velocity analysis is carried out differentiating Eq. (3.83):

$$\begin{gathered}\dot{x}_B \sin\phi + x_B\,\dot{\phi}\,\cos\phi = \dot{y}_B\,\cos\phi - y_B\,\dot{\phi}\,\sin\phi,\\ \dot{x}_B(x_B - x_C) + \dot{y}_B(y_B - y_C) = 0,\end{gathered}$$

or

$$\begin{gathered}\dot{x}_B \sin\phi + x_B\,\omega\,\cos\phi = \dot{y}_B\,\cos\phi - y_B\,\omega\,\sin\phi,\\ \dot{x}_B(x_B - x_C) + \dot{y}_B(y_B - y_C) = 0.\end{gathered} \tag{3.84}$$

The magnitude of the angular velocity of the driver link 1 is

$$\omega = \omega_1 = \dot{\phi} = \frac{\pi\, n_1}{30} = \frac{\pi\,(30 \text{ rpm})}{30} = 3.141 \text{ rad/s}. \tag{3.85}$$

The link 2 and the driver link 1 have the same angular velocity $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$. For the given numerical data Eq. (3.84) becomes

$$\begin{gathered}\dot{x}_B \sin 45^\circ + 0.256\,(3.141)\,\cos 45^\circ = \dot{y}_B\,\cos 45^\circ - 0.256\,(3.141)\,\sin 45^\circ,\\ \dot{x}_B(0.256 - 0.1) + \dot{y}_B(0.256 - 0) = 0.\end{gathered} \tag{3.86}$$

The solution of Eq. (3.86) gives

$$\dot{x}_B = -0.999 \text{ m/s} \quad \text{and} \quad \dot{y}_B = 0.609 \text{ m/s}.$$

The velocity of B is

$$\mathbf{v}_{B_3} = \mathbf{v}_{B_2} = -0.999\,\boldsymbol{\imath} + 0.609\,\boldsymbol{\jmath} \quad \text{m/s},$$

$$|\mathbf{v}_{B_3}| = |\mathbf{v}_{B_2}| = \sqrt{(-0.999)^2 + (0.609)^2} = 1.171 \quad \text{m/s}.$$

The MATLAB statements for the velocity of $B_2 = B_3$ are

```
deqB1 = diff(eqB1,t) ;
deqB2 = diff(eqB2,t) ;
sv = {diff('phi(t)',t),diff('xB(t)',t),diff('yB(t)',t)} ;
nv = {omega,'vxB','vyB'} ;
deqB1p=subs(deqB1,sv,nv) ;
deqB1n=subs(deqB1p,sp,fp) ;
deqB2p=subs(deqB2,sv,nv) ;
deqB2n=subs(deqB2p,sp,fp) ;
solvB = solve(deqB1n, deqB2n) ;
vBx = eval(solvB.vxB) ;
vBy = eval(solvB.vyB) ;
fv = {omega,vBx,vBy} ;
```

The acceleration analysis is obtained using the derivative of the velocities given by Eq. (3.84):

$$\begin{gathered}\ddot{x}_B \sin\phi + \dot{x}_B\,\omega\,\cos\phi + \dot{x}_B\,\omega\,\cos\phi - x_B\,\omega^2\,\sin\phi = \\ \ddot{y}_B\,\cos\phi - \dot{y}_B\,\omega\,\sin\phi - \dot{y}_B\,\omega\,\sin\phi + y_B\,\omega^2\,\cos\phi, \\ \ddot{x}_B(x_B - x_C) + \dot{x}_B^2 + \ddot{y}_B(y_B - y_C) + \dot{y}_B^2 = 0. \end{gathered} \tag{3.87}$$

The magnitude of the angular acceleration of the driver link 1 is

$$\alpha = \dot{\omega} = \ddot{\phi} = 0.$$

Numerically, Eq. (3.87) gives

$$\begin{gathered}\ddot{x}_B\,\sin 45^\circ + 2\,(-0.999)\,(3.141)\,\cos 45^\circ - 0.256\,(3.141)^2\,\sin 45^\circ = \\ \ddot{y}_B\,\cos 45^\circ - 2\,(0.609)\,(3.141)\,\sin 45^\circ + 0.256\,(3.141)^2\,\cos 45^\circ, \\ \ddot{x}_B(0.256 - 0.1) + (-0.999)^2 + \ddot{y}_B(0.256) + 0.609^2 = 0. \end{gathered} \tag{3.88}$$

The solution of Eq. (3.88) is

$$\ddot{x}_B = -1.802 \text{ m/s}^2 \quad \text{and} \quad \ddot{y}_B = -4.255 \text{ m/s}^2.$$

The acceleration of B on link 3 or 2 is

$$\mathbf{a}_{B_3} = \mathbf{a}_{B_2} = \ddot{x}_B\,\imath + \ddot{y}_B\,\jmath = -1.802\,\imath - 4.255\,\jmath \quad \text{m/s}^2,$$

$$|\mathbf{a}_{B_3}| = |\mathbf{a}_{B_2}| = \sqrt{(-1.802)^2 + (-4.255)^2} = 4.620 \quad \text{m/s}^2.$$

The MATLAB statements for the acceleration of $B_2 = B_3$ are

```
ddeqB1 = diff(deqB1,t) ;
ddeqB2 = diff(deqB2,t) ;
sa = {diff('phi(t)',t,2),diff('xB(t)',t,2),diff('yB(t)',t,2)};
na = {0,'axB','ayB'} ;
ddeqB1p=subs(ddeqB1,sa,na) ;
ddeqB1n=subs(ddeqB1p,sv,fv) ;
ddeqB1f=subs(ddeqB1n,sp,fp) ;
ddeqB2p=subs(ddeqB2,sa,na) ;
ddeqB2n=subs(ddeqB2p,sv,fv) ;
ddeqB2f=subs(ddeqB2n,sp,fp) ;
solaB = solve(ddeqB1f, ddeqB2f) ;
aBx = eval(solaB.axB) ;
aBy = eval(solaB.ayB) ;
fa = {0,aBx,aBy};
```

The slope of the link 3 (the points B and C are on the straight line BC) is

$$\tan\phi_3(t) = \frac{y_B(t) - y_C}{x_B(t) - x_C},$$

or

$$[x_B(t) - x_C]\,\sin\phi_3(t) = [y_B(t) - y_C]\,\cos\phi_3(t). \tag{3.89}$$

The angle ϕ_3 is computed as follows:

$$\phi_3 = \arctan\frac{y_B - y_C}{x_B - x_C} = \arctan\frac{0.256}{0.256 - 0.1} = 1.023 \text{ rad} = 58.633°.$$

The derivative of Eq. (3.89) yields

$$\dot{x}_B\,\sin\phi_3 + (x_B - x_C)\,\dot{\phi}_3\,\cos\phi_3 = \dot{y}_B\,\cos\phi_3 - (y_B - y_C)\,\dot{\phi}_3\,\sin\phi_3,$$

or

$$\dot{x}_B \sin\phi_3 + (x_B - x_C)\,\omega_3 \cos\phi_3 = \dot{y}_B \cos\phi_3 - (y_B - y_C)\,\omega_3 \sin\phi_3, \quad (3.90)$$

where $\omega_3 = \dot{\phi}_3$.
Numerically Eq. (3.90) gives

$$\begin{gathered} -0.999 \sin 58.633^\circ + (0.256 - 0.1)\,\omega_3 \cos 58.633^\circ = \\ 0.609 \cos 58.633^\circ - 0.256\,\omega_3 \sin 58.633^\circ, \end{gathered}$$

with the solution $\omega_3 = 3.903$ rad/s.
The angular velocity of link 3 is

$$\boldsymbol{\omega}_3 = \omega_3\,\mathbf{k} = 3.903\,\mathbf{k} \quad \text{rad/s}.$$

The MATLAB statements for the angular velocity of link 3 are

```
phi3 = atan((yB-yC)/(xB-xC)) ;
phi3n = subs(phi3,sp,fp) ;
dphi3 = diff(phi3,t) ;
dphi3nn = subs(dphi3,sv,fv) ;
dphi3n = subs(dphi3nn,sp,fp) ;
fprintf('omega3 = %g (rad/s) \n', double(dphi3n)) ;
```

The angular acceleration of link 3, $\alpha_3 = \dot{\omega}_3 = \ddot{\phi}_3$, is obtained using the derivative of the Eq. (3.90):

$$\begin{gathered} \ddot{x}_B \sin\phi_3 + \dot{x}_B\,\omega_3 \cos\phi_3 + \\ \dot{x}_B\,\omega_3 \cos\phi_3 + (x_B - x_C)\,\dot{\omega}_3 \cos\phi_3 - (x_B - x_C)\,\omega_3^2 \sin\phi_3 = \\ \ddot{y}_B \cos\phi_3 - \dot{y}_B\,\omega_3 \sin\phi_3 - \\ \dot{y}_B\,\omega_3 \sin\phi_3 - (y_B - y_C)\,\dot{\omega}_3 \sin\phi_3 - (y_B - y_C)\,\omega_3^2 \cos\phi_3, \end{gathered}$$

or

$$\begin{gathered} \ddot{x}_B \sin\phi_3 + 2\,\dot{x}_B\,\omega_3 \cos\phi_3 + (x_B - x_C)\,\alpha_3 \cos\phi_3 - (x_B - x_C)\,\omega_3^2 \sin\phi_3 = \\ \ddot{y}_B \cos\phi_3 - 2\,\dot{y}_B\,\omega_3 \sin\phi_3 - (y_B - y_C)\,\alpha_3 \sin\phi_3 - (y_B - y_C)\,\omega_3^2 \cos\phi_3. \end{gathered}$$

Numerically, the previous equation becomes

$$\begin{gathered} -1.802 \sin 58.633^\circ + 2\,(-0.999)\,(3.903) \cos 58.633^\circ + \\ (0.256 - 0.1)\,\alpha_3 \cos 58.633^\circ - (0.256 - 0.1)\,(3.903)^2 \sin 58.633^\circ = \\ -4.255 \cos 58.633^\circ - 2\,(0.609)\,(3.903) \sin 58.633^\circ - \\ 0.256\,\alpha_3 \sin 58.633^\circ - 0.256\,(3.903)^2 \cos 58.633^\circ, \end{gathered}$$

with the solution $\alpha_3 = -2.252$ rad/s^2. The angular acceleration of link 3 is

$$\boldsymbol{\alpha}_3 = \alpha_3\,\mathbf{k} = -2.252\,\mathbf{k} \quad \text{rad/s}^2.$$

The MATLAB statements for the angular acceleration of link 3 are

```
ddphi3 = diff(dphi3,t) ;
ddphi3nnn = subs(ddphi3,sa,fa) ;
ddphi3nn = subs(ddphi3nnn,sv,fv) ;
ddphi3n = subs(ddphi3nn,sp,fp) ;
fprintf('alpha3 = %g (rad/s^2 ) \n', double(ddphi3n)) ;
```

The MATLAB program for velocity and acceleration analysis and the results using the derivative method are given in Program 3.6.

R-RTR-RRT Mechanism

The mechanism shown in Fig. 3.6(b) and has the dimensions: $AB = 0.100$ m, $AC = 0.150$ m, $CD = 0.075$ m, and $DE = 0.200$ m. The angular speed of the driver link 1 is $\omega = \omega_1 = 4.712$ rad/s. Find the velocities and the accelerations of the mechanism using the derivative method for the case when the angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 45°$.

Solution

The origin of the fixed reference frame is at $C \equiv 0$. The position of the fixed joint A is $x_A = 0$ and $y_A = AC = 0.150$ m. The position of joint B is

$$x_B(t) = x_A + AB\cos\phi(t), \quad y_B(t) = y_A + AB\sin\phi(t),$$

and for $\phi = 45°$, the position is

$$x_B = 0 + 0.100\ \cos 45° = 0.070 \text{ m}, \ \ y_B = 0.150 + 0.100\ \sin 45° = 0.220 \text{ m}.$$

The linear velocity vector of $B_1 = B_2$ is

$$\mathbf{v}_{B_1} = \mathbf{v}_{B_2} = \dot{x}_B \mathbf{\imath} + \dot{y}_B \mathbf{\jmath},$$

where

$$\dot{x}_B = \frac{dx_B}{dt} = -AB\,\dot{\phi}\ \sin\phi,\ \ \dot{y}_B = \frac{dy_B}{dt} = AB\,\dot{\phi}\ \cos\phi.$$

With $\phi = 45°$ and $\dot{\phi} = \omega = 4.712$ rad/s:

$$\begin{aligned}&\dot{x}_B = -0.100\,(4.712)\ \sin 45° = -0.333 \ \text{ m/s},\\ &\dot{y}_B = 0.100\,(4.712)\ \cos 45° = 0.333 \ \text{ m/s},\\ &v_{B_1} = v_{B_2} = |\mathbf{v}_{B_1}| = |\mathbf{v}_{B_2}| = \sqrt{\dot{x}_B^2 + \dot{y}_B^2} = \sqrt{(-0.333)^2 + 0.333^2} = 0.471 \ \text{ m/s}.\end{aligned}$$

The linear acceleration vector of $B_1 = B_2$ is

$$\mathbf{a}_{B_1} = \mathbf{a}_{B_2} = \ddot{x}_B \mathbf{\imath} + \ddot{y}_B \mathbf{\jmath},$$

where

$$\begin{aligned}\ddot{x}_B &= \frac{d\dot{x}_B}{dt} = -AB\dot{\phi}^2\cos\phi - AB\ddot{\phi}\sin\phi,\\ \ddot{y}_B &= \frac{d\dot{y}_B}{dt} = -AB\dot{\phi}^2\sin\phi + AB\ddot{\phi}\cos\phi.\end{aligned}$$

The angular acceleration of link 1 is $\ddot{\phi} = \dot{\omega} = 0$. The numerical values for the acceleration of B are

$$\begin{aligned}
&\ddot{x}_B = -0.100\,(4.712)^2 \cos 45^\circ = -1.569 \ \text{ m/s}^2,\\
&\ddot{y}_B = -0.100\,(4.712)^2 \sin 45^\circ = -1.569 \ \text{ m/s}^2,\\
&a_{B_1} = a_{B_2} = |\mathbf{a}_{B_1}| = |\mathbf{a}_{B_2}| = \sqrt{\ddot{x}_B^2 + \ddot{y}_B^2} = \sqrt{(-1.569)^2 + (-1.569)^2} = 2.220 \ \text{ m/s}^2.
\end{aligned}$$

The MATLAB statements for the velocity and acceleration of $B_1 = B_2$ are

```
AB = 0.1 ; CD = 0.075 ; DE = 0.2 ; AC = 0.15 ; % (m)
xC = 0 ; yC = 0 ; xA = 0 ; yA = AC ;
phi1 = pi/4 ; omega = 4.712 ; alpha = 0;
t = sym('t','real');
xB1 = xA + AB*cos(sym('phi(t)'));
yB1 = yA + AB*sin(sym('phi(t)'));
rB = [ xB1 yB1 0 ]; % symbolic in function of phi(t)
xBn = subs(xB1,'phi(t)',pi/4); % xB for phi(t)=pi/4
yBn = subs(yB1,'phi(t)',pi/4); % yB for phi(t)=pi/4
rBn = subs(rB,'phi(t)',pi/4); % rB for phi(t)=pi/4
fprintf('rB = [ %g, %g, %g ] (m)\n', rBn);
vB = diff(rB,t); % differentiates rB with respect to t
% list for the symbolical variables phi''(t), phi'(t), phi(t)
slist = {diff('phi(t)',t,2), diff('phi(t)',t), 'phi(t)'};
% list with the numerical values for phi''(t), phi'(t), phi(t)
nlist = {alpha, omega, phi1}; % numerical values for slist
vBn = double(subs(vB,slist,nlist));
fprintf('vB1 = vB2 = [ %g, %g, %g ] (m/s)\n', vBn);
% acceleration of B1=B2
aB = diff(vB,t); % differentiates vB with respect to t
aBn = double(subs(aB,slist,nlist));
fprintf('aB1 = aB2 = [ %g, %g, %g ] (m/s^2)\n', aBn);
```

The position of the joint D is calculated from the following equations

$$\begin{aligned}
&[x_D(t) - x_C]^2 + [y_D(t) - y_C]^2 = CD^2,\\
&\frac{y_D(t) - y_C}{x_D(t) - x_C} = \frac{y_B(t) - y_C}{x_B(t) - x_C}.
\end{aligned}$$

The previous equations are rewritten as follows

$$\begin{aligned}&x_D^2(t) + y_D^2(t) = CD^2,\\ &y_D(t)\; x_B(t) = x_D(t)\; y_B(t).\end{aligned} \tag{3.91}$$

For $\phi = 45°$, the coordinates of joint D are $x_D = -0.023$ m and $y_D = -0.071$ m. The negative value for y_D was selected for this position of the mechanism.
The MATLAB statements for the position of D are

```
xB = sym('xB(t)'); % xB(t) symbolic
yB = sym('yB(t)'); % yB(t) symbolic
% list for the symbolical variables of B
% xB''(t), yB''(t), xB'(t), yB'(t), xB(t), yB(t)
sB={diff('xB(t)',t,2),diff('yB(t)',t,2),...
diff('xB(t)',t),diff('yB(t)',t),'xB(t)','yB(t)'};
% three dots (...)  are used whenever a line break is needed
% list with the numerical values for the sB list
nB={aBn(1),aBn(2),vBn(1),vBn(2),xBn,yBn};
xD = sym('xD(t)'); % xB(t) symbolic
yD = sym('yD(t)'); % xB(t) symbolic
% equations for xD(t) and yD(t) function of xB(t) and yB(t)
eqD1 = (xD - xC)^2 + (yD - yC)^2 - CD^2;
eqD2 = (yD - yC)/(xD - xC) - (yB - yC)/(xB - xC);
sDp = {'xD(t)','yD(t)'};
nDp = {'xDn','yDn'};
% replaces {'xD(t)','yD(t)'} with {'xDn','yDn'}
eqD1n = subs(eqD1,sDp,nDp);
eqD2n = subs(eqD2,sDp,nDp);
eqD1p=subs(eqD1n,sB,nB); %equation function of only 'xDn' and 'yDn'
eqD2p=subs(eqD2n,sB,nB); %equation function of only 'xDn' and 'yDn'
solDp = solve(eqD1p, eqD2p);
xDpositions = eval(solDp.xDn);
yDpositions = eval(solDp.yDn);
xD1 = xDpositions(1); xD2 = xDpositions(2);
yD1 = yDpositions(1); yD2 = yDpositions(2);
if yD1 < 0
      xDp = xD1; yDp = yD1;
```

```
else
      xDp = xD2; yDp = yD2;
end
rD = [xDp yDp 0];
fprintf('rD = [ %g, %g, %g ] (m)\n', rD);
```

The velocity analysis is carried out differentiating Eq. (3.91)

$$\begin{aligned} &x_D\,\dot{x}_D + y_D\,\dot{y}_D = 0,\\ &\dot{y}_D\,x_B + y_D\,\dot{x}_B = \dot{x}_D\,y_B + x_D\,\dot{y}_B. \end{aligned} \tag{3.92}$$

For the given data, Eq. (3.92) becomes

$$\begin{aligned} &-0.023\dot{x}_D - 0.071\dot{y}_D = 0,\\ &0.070\,\dot{y}_D + (-0.071)\,(-0.333) = 0.220\,\dot{x}_D + (-0.023)\,(0.333). \end{aligned}$$

The solution is

$$\dot{x}_D = 0.129\,\text{m/s}, \quad \dot{y}_D = -0.041\,\text{m/s}.$$

The magnitude of the velocity of joint D is

$$v_D = |\mathbf{v}_D| = \sqrt{\dot{x}_D^2 + \dot{y}_D^2} = \sqrt{0.129^2 + (-0.041)^2} = 0.135\,\text{m/s}.$$

The MATLAB statements for the velocity of D are

```
deqD1 = diff(eqD1,t);
deqD2 = diff(eqD2,t);
sDv = {diff('xD(t)',t),diff('yD(t)',t),'xD(t)','yD(t)'};
nDv = {'vxD','vyD',xDp,yDp};
deqD1v = subs(subs(deqD1,sDv,nDv),sB,nB);
deqD2v = subs(subs(deqD2,sDv,nDv),sB,nB);
solvD = solve(deqD1v, deqD2v);
vDx = eval(solvD.vxD);
vDy = eval(solvD.vyD);
fprintf('vD = [ %g, %g, %g ] (m/s)\n', [vDx vDy 0]);
fDv = {vDx, vDy, xDp, yDp};
```

The acceleration analysis is obtained using the derivative of the velocity given by Eq. (3.92):

$$\dot{x}_D^2 + x_D\ddot{x}_D + \dot{y}_D^2 + y_D\ddot{y}_D = 0,$$
$$\ddot{y}_D\, x_B + \dot{y}_D\, \dot{x}_B + \dot{y}_D\, \dot{x}_B + y_D\, \ddot{x}_B = \ddot{x}_D\, y_B + \dot{x}_D\, \dot{y}_B + \dot{x}_D\, \dot{y}_B + x_D\, \ddot{y}_B.$$

or

$$0.129^2 + (-0.022)\ddot{x}_D + (-0.041)^2 + (-0.071)\ddot{y}_D = 0,$$
$$0.070\, \ddot{y}_D + 2(-0.041)\,(-0.333) + (-0.071)\,(-1.569) =$$
$$0.220\, \ddot{x}_D \ + 2(0.129)\,(0.333) + (-0.023)\,(-1.569).$$

The solution of the previous system is

$$\ddot{x}_D = 0.147 \ \ \text{m/s}^2, \quad \ddot{y}_D = 0.210 \ \ \text{m/s}^2.$$

The absolute acceleration of joint D is

$$a_D = |\mathbf{a}_D| = \sqrt{\ddot{x}_D^2 + \ddot{y}_D^2} = \sqrt{(0.150)^2 + (0.212)^2} = 0.256 \quad \text{m/s}^2.$$

The MATLAB statements for the acceleration of D are

```
ddeqD1 = diff(deqD1,t);
ddeqD2 = diff(deqD2,t);
sDa = {diff('xD(t)',t,2),diff('yD(t)',t,2)};
nDa = {'axD','ayD'};
ddeqD1n = subs(ddeqD1,sDa,nDa);
ddeqD2n = subs(ddeqD2,sDa,nDa);
ddeqD1a = subs(subs(ddeqD1n,sDv,fDv),sB,nB);
ddeqD2a = subs(subs(ddeqD2n,sDv,fDv),sB,nB);
solaD = solve(ddeqD1a, ddeqD2a);
aDx = eval(solaD.axD);
aDy = eval(solaD.ayD);
fprintf('aD = [ %g, %g, %g ] (m/s^2)\n', [aDx aDy 0]);
sD={diff('xD(t)',t,2),diff('yD(t)',t,2),...
diff('xD(t)',t),diff('yD(t)',t),'xD(t)','yD(t)'};
nD={aDx,aDy,vDx,vDy,xDp,yDp};
```

The position of joint E is determined from the following equation:

$$[x_E(t) - x_D(t)]^2 + [y_E(t) - y_D(t)]^2 = DE^2,$$

and with the coordinate $y_E = 0$:

$$[x_E(t) - x_D(t)]^2 + y_D^2(t) = DE^2. \tag{3.93}$$

With the given numerical values Eq. (3.93) becomes

$$(x_E + 0.023)^2 + (0.071)^2 = 0.2^2,$$

with the correct solution $x_E = 0.164$ m.
The MATLAB statements for the position of E are

```
xE = sym('xE(t)');
yE = 0;
% equations for xE(t) function of xD(t) and yD(t)
eqE = (xE - xD)^2 + (yE - yD)^2 - DE^2;
eqEp = subs(subs(eqE,'xE(t)','xEn'),sD,nD);
solEp = solve(eqEp);
xE1 = eval(solEp(1)); xE2 = eval(solEp(2));
if xE1 > xDp xEp = xE1; else xEp = xE2; end
rE = [xEp yE 0];
fprintf('rE = [ %g, %g, %g ] (m)\n', rE);
```

The velocity of joint E is determined by differentiating Eq. (3.93) as follows

$$2(\dot{x}_E - \dot{x}_D)(x_E - x_D) + 2y_D\dot{y}_D = 0, \tag{3.94}$$

or

$$\dot{x}_E - \dot{x}_D = -\frac{y_D\dot{y}_D}{x_E - x_D}.$$

The solution of the above equation is

$$\dot{x}_E = 0.129 - \frac{(-0.071)(-0.041)}{0.164 + 0.023} = 0.113\,\text{m/s}.$$

The MATLAB statements for the velocity of E are

```
deqE = diff(eqE,t);
sEv = diff('xE(t)',t),'xE(t)';
nEv = 'vxE',xEp;
deqEv = subs(subs(deqE,sEv,nEv),sD,nD);
solvE = solve(deqEv);
vEx = eval(solvE);
vEy = 0;
fprintf('vE = [ %g, %g, %g ] (m/s)\n', [vEx vEy 0]);
fEv = vEx,xEp;
```

The derivative of Eq. (3.94) yields

$$(\ddot{x}_E - \ddot{x}_D)(x_E - x_D) + (\dot{x}_E - \dot{x}_D)^2 + \dot{y}_D^2 + y_D\ddot{y}_D = 0,$$

with the solution

$$\ddot{x}_E = \ddot{x}_D - \frac{\dot{y}_D^2 + y_D\ddot{y}_D + (\dot{x}_E - \dot{x}_D)^2}{x_E - x_D},$$

or with numerical values

$$\ddot{x}_E = 0.150 - \frac{(-0.041)^2 + (-0.071)(0.21) + (0.112 - 0.129)^2}{0.164 + 0.023} = 0.217 \quad \text{m/s}^2.$$

The MATLAB statements for the acceleration of E are

```
ddeqE = diff(deqE,t);
ddeqEn = subs(ddeqE,diff('xE(t)',t,2),'axD');
ddeqEa = subs(subs(ddeqEn,sEv,fEv),sD,nD);
solaE = solve(ddeqEa);
aEx = eval(solaE);
aEy = 0;
fprintf('aE = [ %g, %g, %g ] (m/s^2)\n', [aEx aEy 0]);
```

The points B and C are located on the same straight line BD:

$$y_B(t) - y_C - [x_B(t) - x_C]\ \tan\phi_3(t) = 0. \tag{3.95}$$

The angle $\phi_3 = \phi_2$ is computed as follows:

$$\phi_3 = \phi_2 = \arctan \frac{y_B - y_C}{x_B - x_C},$$

and for $\phi = 45^\circ$ is obtained by

$$\phi_3 = \arctan \frac{0.22}{0.07} = 72.235^\circ.$$

The derivative of Eq. (3.95) yields

$$\dot{y}_B - \dot{y}_C - (\dot{x}_B - \dot{x}_C)\ \tan\phi_3 - (x_B - x_C)\frac{1}{\cos^2\phi_3}\ \dot{\phi}_3 = 0. \tag{3.96}$$

The angular velocity of link 3, $\omega_3 = \omega_2 = \dot{\phi}_3$, is computed as follows

$$\omega_3 = \omega_2 = \frac{\cos^2\phi_3[\dot{y}_B - \dot{y}_C - (\dot{x}_B - \dot{x}_C)\ \tan\phi_3]}{x_B - x_C},$$

and

$$\omega_3 = \frac{\cos^2 72.235^\circ(0.333 + 0.333\tan 72.235^\circ)}{0.07} = 1.807\,\text{rad/s}.$$

The angular acceleration of link 3, $\alpha_3 = \alpha_2 = \ddot{\phi}_3$, is computed from the time derivative of Eq. (3.96)

$$\begin{gathered}\ddot{y}_B - \ddot{y}_C - (\ddot{x}_B - \ddot{x}_C)\tan\phi_3 - 2(\dot{x}_B - \dot{x}_C)\frac{1}{\cos^2\phi_3}\dot{\phi}_3 - \\ 2(x_B - x_C)\frac{\sin\phi_3}{\cos^3\phi_3}\ \dot{\phi}_3^2 - (x_B - x_C)\frac{1}{\cos^2\phi_3}\ \ddot{\phi}_3 = 0.\end{gathered}$$

The solution of the previous equation is

$$\begin{aligned}\alpha_3 &= \alpha_2 = [\ddot{y}_B - \ddot{y}_C - (\ddot{x}_B - \ddot{x}_C)\tan\phi_3 - 2(\dot{x}_B - \dot{x}_C)\frac{1}{\cos^2\phi_3}\dot{\phi}_3 - \\ &2(x_B - x_C)\frac{\sin\phi_3}{\cos^3\phi_3}\ \dot{\phi}_3^2]\frac{\cos^2\phi_3}{x_B - x_C},\end{aligned}$$

and for the given numerical data:

$$\begin{aligned}\alpha_3 &= \alpha_2 = [-1.569 + 1.569\tan 72.235^\circ + 2(0.333)\frac{1}{\cos^2 72.235^\circ}1.807 - \\ &2(0.07)\frac{\sin 72.235^\circ}{\cos^3 72.235^\circ}(1.807)^2]\frac{\cos^2 72.235^\circ}{0.07} = 1.020\,\text{rad/s}^2.\end{aligned}$$

The MATLAB statements for the angular velocity and acceleration of links 2 and 3 are

```
phi3 = atan((yB-yC)/(xB-xC));
phi3n = subs(phi3,sB,nB);
fprintf('phi2 = phi3 = %g (degrees)\n', double(phi3n*180/pi));
dphi3 = diff(phi3,t);
dphi3n = subs(dphi3,sB,nB) ;
fprintf('omega2 = omega3 = %g (rad/s)\n', double(dphi3n));
ddphi3 = diff(dphi3,t);
ddphi3n = subs(ddphi3,sB,nB);
fprintf('alpha2 = alpha3 = %g (rad/s^2)\n', double(ddphi3n));
```

The angle ϕ_4 is determined from the following equation:

$$y_E - y_D(t) - [x_E(t) - x_D(t)]\ \tan\phi_4(t) = 0, \tag{3.97}$$

where $y_E = 0$. The above equation can be rewritten as

$$-y_D(t) - [x_E(t) - x_D(t)]\ \tan\phi_4(t) = 0, \tag{3.98}$$

and the solution is

$$\phi_4 = \arctan\left(\frac{-y_D}{x_E - x_D}\right) = \arctan\left(\frac{0.071}{0.164 + 0.023}\right) = 20.923^\circ.$$

The derivative of Eq. (3.98) yields

$$-\dot{y}_D - (\dot{x}_E - \dot{x}_D)\ \tan\phi_4 - (x_E - x_D)\frac{1}{\cos^2\phi_4}\dot{\phi}_4 = 0. \tag{3.99}$$

Hence,

$$\begin{aligned}
\omega_4 &= \dot{\phi}_4 = -\frac{\cos^2\phi_4[\dot{y}_D + (\dot{x}_E - \dot{x}_D)\tan\phi_4]}{x_E - x_D} \\
&= -\frac{\cos^2 20.923^\circ[-0.041 + (0.113 - 0.129)\tan 20.923^\circ]}{0.164 - (-0.022)} \\
&= 0.221 \quad \text{rad/s.}
\end{aligned}$$

The angular acceleration of link 4 is determined by differentiating Eq. (3.99) as follows:

$$-\ddot{y}_D - (\ddot{x}_E - \ddot{x}_D)\tan\phi_4 - 2(\dot{x}_E - \dot{x}_D)\frac{1}{\cos^2\phi_4}\dot{\phi}_4 -$$
$$2(x_E - x_D)\frac{\sin\phi_4}{\cos^3\phi_4}\dot{\phi}_4^2 - (x_E - x_D)\frac{1}{\cos^2\phi_4}\ddot{\phi}_4 = 0,$$

or

$$-0.210 - (0.217 - 0.147)\tan 20.923^\circ - 2(0.113 - 0.129)\frac{1}{\cos^2 20.923^\circ}0.221 -$$
$$2(0.164 + 0.022)\frac{\sin 20.923^\circ}{\cos^3 20.923^\circ}0.221^2 - (0.164 + 0.022)\frac{1}{\cos^2 20.923^\circ}\ddot{\phi}_4 = 0,$$

The solution of the previous equation is

$$\alpha_4 = \ddot{\phi}_4 = -1.105 \quad \text{rad/s}^2.$$

The MATLAB statements for the angular velocity and acceleration of link 4 are

```
phi4 = atan((yE-yD)/(xE-xD));
phi4n = subs(subs(phi4,sD,nD),sE,nE);
fprintf('phi4 = %g (degrees)\n', double(phi4n*180/pi));
dphi4 = diff(phi4,t);
dphi4n = subs(subs(dphi4,sD,nD),sE,nE);
fprintf('omega4 = %g (rad/s)\n', double(dphi4n));
ddphi4 = diff(dphi4,t);
ddphi4n = subs(subs(ddphi4,sD,nD),sE,nE);
fprintf('alpha4 = %g (rad/s^2)\n', double(ddphi4n));
```

The MATLAB program for velocity and acceleration analysis using derivative method and the results are given in Program 3.7.

3.10 Independent Contour Equations

This section provides an algebraic method to compute the velocities and accelerations of any closed kinematic chain. The classical method for obtaining the velocities and accelerations involves the computation of the derivative with respect to time of the position vectors. The method of contour equations avoids this task and uses only algebraic equations [Atanasiu,Voinea]. Using this approach, a numerical implementation is much more efficient. The method described here can be applied to planar and spatial mechanisms.

Figure 3.7 shows a monocontour closed kinematic chain with n rigid links. The joint A_i, $i = 0, 1, 2, ..., n$ is the connection between the links (i) and $(i-1)$. The last link n is connected with the first link 0 of the chain. For the closed kinematic chain, a path is chosen from link 0 to link n. At the joint A_i there are two instantaneously coincident points: the point $A_{i,i}$ belonging to link (i), $A_{i,i} \in (i)$, and the point $A_{i,i-1}$ belonging to link $(i-1)$, $A_{i,i-1} \in (i-1)$.

The velocity equations for a simple closed kinematic chain are

$$\sum_{(i)} \boldsymbol{\omega}_{i,i-1} = \mathbf{0} \text{ and } \sum_{(i)} \mathbf{r}_{A_i} \times \boldsymbol{\omega}_{i,i-1} + \sum_{(i)} \mathbf{v}_{A_{i,i-1}} = \mathbf{0}, \tag{3.100}$$

where

$\boldsymbol{\omega}_{i,i-1}$ is the relative angular velocity of link (i) with respect to link $(i-1)$;

$\mathbf{r}_{A_i}$ is the position vector of the joint A_i;

$\mathbf{v}_{A_{i,i-1}} = \mathbf{v}^{rel}_{A_{i,i}A_{i,i-1}}$ is the relative velocity of $A_{i,i}$ on link (i) with respect to $A_{i,i-1}$ on link $(i-1)$.

The acceleration equations for a simple closed kinematic chain are

$$\sum_{(i)} \boldsymbol{\alpha}_{i,i-1} + \sum_{(i)} \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1} = \mathbf{0} \text{ and}$$

$$\sum_{(i)} \mathbf{r}_{A_i} \times (\boldsymbol{\alpha}_{i,i-1} + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1}) + \sum_{(i)} \mathbf{a}_{A_{i,i-1}} + \sum_{(i)} \mathbf{a}^{cor}_{A_{i,i-1}} +$$

$$\sum_{(i)} \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}}) = \mathbf{0}. \tag{3.101}$$

where

$\boldsymbol{\alpha}_{i,i-1}$ is the relative angular acceleration of link (i) with respect to link $(i-1)$;

$\boldsymbol{\omega}_i$ is the absolute angular velocity of the link (i), or the angular velocity of link (i) with respect to the "fixed" reference frame $Oxyz$, $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i,0}$;

$\mathbf{a}_{A_{i,i-1}} = \mathbf{a}^{rel}_{A_{i,i}A_{i,i-1}}$ is the relative acceleration of $A_{i,i}$ on link (i) with respect to $A_{i,i-1}$ on link $(i-1)$;

$\mathbf{a}^{cor}_{A_{i,i-1}} = 2\,\boldsymbol{\omega}_{i-1} \times \mathbf{v}_{A_{i,i-1}}$ is the Coriolis acceleration;

$\mathbf{r}_{A_{i-1}A_i} = \mathbf{r}_{A_i} - \mathbf{r}_{A_{i-1}}$.

For a closed kinematic chain in planar motion the acceleration equations are

$$\sum_{(i)} \boldsymbol{\alpha}_{i,i-1} = \mathbf{0} \ \text{ and}$$

$$\sum_{(i)} \mathbf{r}_{A_i} \times \boldsymbol{\alpha}_{i,i-1} + \sum_{(i)} \mathbf{a}_{A_{i,i-1}} + \sum_{(i)} \mathbf{a}^{cor}_{A_{i,i-1}} - \omega_i^2 \mathbf{r}_{A_iA_{i+1}} = \mathbf{0}. \quad (3.102)$$

For planar motion the following relations exist

$$\boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}}) = -\omega_i^2 \mathbf{r}_{A_iA_{i+1}} \text{ and } \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1} = \mathbf{0}.$$

A systematic procedure, using the contour method, is presented below. The equations for velocities and accelerations are written for any closed contour of the mechanism. However, it is best to write the contour equations only for the independent loops of the diagram representing the mechanism.

Step 1. Determine the position analysis of the mechanism.

Step 2. Draw the contour diagram representing the mechanism and select the independent contours. For the contour diagram the numbered links are the nodes of the diagram and are represented by circles, and the joints are represented by lines which connect the nodes. Determine a path for each contour.

Step 3. For each closed loop write the contour velocity relations, Eq. (3.100), and contour acceleration relations, Eq. (3.101). For a closed kinematic chain in planar motion Eq. (3.100) and Eq. (3.102) will be used.

Step 4. Project on a cartesian reference system the velocity and acceleration equations. Linear algebraic equations are obtained where the unknowns are

- the components of the relative angular velocities $\boldsymbol{\omega}_{j,j-1}$;
- the components of the relative angular accelerations $\boldsymbol{\alpha}_{j,j-1}$;
- the components of the relative linear velocities $\mathbf{v}_{Aj,j-1}$;
- the components of the relative linear accelerations $\mathbf{a}_{Aj,j-1}$.

Solve the algebraic system of equations and determine the unknown kinematic parameters.

Step 5. Determine the absolute angular velocities $\boldsymbol{\omega}_j$ and the absolute angular accelerations $\boldsymbol{\alpha}_j$. Compute the velocities and accelerations of the characteristic points and joints.

Example: R-RTR-RTR mechanism

For the planar R-RTR-RTR mechanism considered at section **3.8** and shown in Fig. 3.8(a) the contour equations method will be applied and a MATLAB program for velocity and acceleration analysis will be presented.

The mechanism has five moving links and seven full joints. The number of independent contours is $n_c = c - n = 7 - 5 = 2$, where c is the number of joints and n is the number of moving links.

The mechanism has two independent contours. The contour diagram of the mechanism is represented in Fig. 3.8(b). The first contour I contains the links 0, 1, 2, and 3, while the second contour II contains the links 0, 3, 4, and 5. Clockwise paths are chosen for each closed contours I and II.

First contour analysis

Figure 3.9(a) shows the first independent contour I with

- rotational joint R between the links 0 and 1 (joint A);
- rotational joint R between the links 1 and 2 (joint B);
- translational joint T between the links 2 and 3 (joint B);
- rotational joint R between the links 3 and 0 (joint C).

The angular velocity ω_{10} of the driver link is known:

$$\omega_{10} = \omega_1 = \omega = \frac{50\pi}{30} \text{ rad/s} = 5.235 \text{ rad/s}.$$

The origin of the reference frame is the point $A(0,0)$.
For the velocity analysis, the following vectorial equations are used

$$\begin{aligned}&\boldsymbol{\omega}_{10}+\boldsymbol{\omega}_{21}+\boldsymbol{\omega}_{03}=\mathbf{0},\\&\mathbf{r}_B\times\boldsymbol{\omega}_{21}+\mathbf{r}_C\times\boldsymbol{\omega}_{03}+\mathbf{v}^{rel}_{B_3B_2}=\mathbf{0},\end{aligned}\tag{3.103}$$

where $\mathbf{r}_B=x_B\mathbf{\imath}+y_B\mathbf{\jmath}$, $\mathbf{r}_C=x_C\mathbf{\imath}+y_C\mathbf{\jmath}$, and

$$\begin{aligned}&\boldsymbol{\omega}_{10}=\omega_{10}\mathbf{k},\ \boldsymbol{\omega}_{21}=\omega_{21}\mathbf{k},\ \boldsymbol{\omega}_{03}=\omega_{03}\mathbf{k},\\&\mathbf{v}^{rel}_{B_3B_2}=\mathbf{v}_{B_{32}}=v_{B_{32}}\cos\phi_2\mathbf{\imath}+v_{B_{32}}\sin\phi_2\mathbf{\jmath}.\end{aligned}$$

The sign of the relative angular velocities is selected as positive as shown in Figs. 3.8(a) and 9(a). The numerical computation will then give the correct orientation of the unknown vectors. The components of the vectors $\mathbf{r}_B$ and $\mathbf{r}_C$, and the angle ϕ_2 are already known from the position analysis of the mechanism. Equation (3.103) becomes

$$\omega_{10}\mathbf{k}+\omega_{21}\mathbf{k}+\omega_{03}\mathbf{k}=\mathbf{0},$$

$$\begin{vmatrix}\mathbf{\imath}&\mathbf{\jmath}&\mathbf{k}\\x_B&y_B&0\\0&0&\omega_{21}\end{vmatrix}+\begin{vmatrix}\mathbf{\imath}&\mathbf{\jmath}&\mathbf{k}\\x_C&y_C&0\\0&0&\omega_{03}\end{vmatrix}+v_{B_{32}}\cos\phi_2\mathbf{\imath}+v_{B_{32}}\sin\phi_2\mathbf{\jmath}=\mathbf{0}.\tag{3.104}$$

The unknown relative velocities are introduced with MATLAB as

```
omega21v = [ 0 0 sym('omega21z','real') ];
omega03v = [ 0 0 sym('omega03z','real') ];
v32v = sym('vB32','real')*[ cos(phi2) sin(phi2) 0];
```

Equation (3.104) represents a system of three equations and with MATLAB commands gives

```
eqIomega = omega10 + omega21v + omega03v;
eqIvz=eqIomega(3);
eqIv = cross(rB,omega21v) + cross(rC,omega03v) + v32v;
eqIvx=eqIv(1);
eqIvy=eqIv(2);
```

To display the equations the following MATLAB statements are uses

```
Ivz=vpa(eqIvz,6);
fprintf('%s = 0 \n', char(Ivz));
Ivx=vpa(eqIvx,6);
fprintf('%s = 0 \n', char(Ivx));
Ivy=vpa(eqIvy,6);
fprintf('%s = 0 \n', char(Ivy));
```

The system of equations can be solved using the MATLAB commands

```
solIv=solve(eqIvz,eqIvx,eqIvy);
omega21 = [ 0 0 eval(solIv.omega21z) ];
omega03 = [ 0 0 eval(solIv.omega03z) ];
vB3B2 = eval(solIv.vB32)*[ cos(phi2) sin(phi2) 0];
```

and the following numerical solutions are obtained

$$\omega_{21} = 0.212 \text{ rad/s}, \ \omega_{03} = -5.448 \text{ rad/s}, \ \text{and } v_{B_{32}} = 0.313 \text{ m/s}.$$

To print the numerical values, the following MATLAB commands are used

```
fprintf('omega21 = [ %g, %g, %g] (rad/s)\n', omega21 );
fprintf('omega03 = [ %g, %g, %g] (rad/s)\n', omega03 );
fprintf('vB32 = %g (m/s)\n', eval(solIv.vB32) );
fprintf('vB3B2 = [ %g, %g, %d] (m/s)\n', vB3B2 );
```

The absolute angular velocities of the links 2 and 3 are

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{30} = -\boldsymbol{\omega}_{03} = 5.448 \ \mathbf{k} \text{ rad/s}.$$

The absolute linear velocity of $D_3 = D_4$ is

$$\mathbf{v}_{D_3} = \mathbf{v}_{D_4} = \mathbf{v}_C + \boldsymbol{\omega}_{30} \times \mathbf{r}_{CD} = 0.067 \ \boldsymbol{\imath} - 0.814 \ \boldsymbol{\jmath} \text{ m/s},$$

where $\mathbf{v}_C = \mathbf{0}$ and $\mathbf{r}_{CD} = \mathbf{r}_D - \mathbf{r}_C$. The MATLAB commands for the absolute velocities are

```
omega30 = - omega03;
omega20 = omega30;
vD3 = cross(omega30,rD-rC);
```

```
fprintf('omega20=omega30=[ %d, %d, %g] (rad/s)\n', omega30 );
fprintf('vD3 = vD4 = [ %g, %g, %g] (m/s)\n', vD3 );
```

For the acceleration analysis, the following vectorial equations are used

$$\begin{gathered}\boldsymbol{\alpha}_{10} + \boldsymbol{\alpha}_{21} + \boldsymbol{\alpha}_{03} = \mathbf{0},\\ \mathbf{r}_B \times \boldsymbol{\alpha}_{21} + \mathbf{r}_C \times \boldsymbol{\alpha}_{03} + \mathbf{a}^{rel}_{B_3B_2} + \mathbf{a}^{cor}_{B_3B_2} - \omega_{10}^2\mathbf{r}_{AB} - \omega_{20}^2\mathbf{r}_{BC} = \mathbf{0}. \quad (3.105)\end{gathered}$$

where

$$\begin{aligned}&\boldsymbol{\alpha}_{10} = \alpha_{10}\mathbf{k},\ \boldsymbol{\alpha}_{21} = \boldsymbol{\alpha}_{21}\mathbf{k},\ \boldsymbol{\alpha}_{03} = \boldsymbol{\alpha}_{03}\mathbf{k},\\ &\mathbf{a}^{rel}_{B_3B_2} = \mathbf{a}_{B_{32}} = a_{B_{32}}\cos\phi_2\boldsymbol{\imath} + a_{B_{32}}\sin\phi_2\boldsymbol{\jmath},\\ &\mathbf{a}^{cor}_{B_3B_2} = \mathbf{a}^{c}_{B_{32}} = 2\,\boldsymbol{\omega}_{20} \times \mathbf{v}_{B_{32}},\end{aligned}$$

The driver link has a constant angular velocity and $\alpha_{10} = \dot{\omega}_{10} = 0$.
The unknown acceleration vectors using the MATLAB commands are

```
alpha21v = [ 0 0 sym('alpha21z','real') ];
alpha03v = [ 0 0 sym('alpha03z','real') ];
a32v = sym('aB32','real')*[ cos(phi2) sin(phi2) 0];
```

Equation (3.105) represents a system of three equations and using MATLAB commands gives

```
eqIalpha = alpha10 + alpha21v + alpha03v;
eqIaz=eqIalpha(3);
eqIa1=cross(rB,alpha21v)+cross(rC,alpha03v)+a32v+2*cross(omega20,vB3B2);
eqIa2=-dot(omega1,omega1)*rB-dot(omega20,omega20)*(rC-rB);
eqIa=eqIa1+eqIa2;
eqIax=eqIa(1);
eqIay=eqIa(2);
```

The equations are displayed with

```
Iaz=vpa(eqIaz,6);
fprintf('%s = 0 \n', char(Iaz));
Iax=vpa(eqIax,6);
fprintf('%s = 0 \n', char(Iax));
```

```
Iay=vpa(eqIay,6);
fprintf('%s = 0 \n', char(Iay));
```

The unknowns are α_{21}, α_{03}, and $a_{B_{32}}$ or `alpha21z`, `alpha03z`, and `aB32`. The system of equations is solved using the MATLAB commands

```
solIa=solve(eqIaz,eqIax,eqIay);
alpha21 = [ 0 0 eval(solIa.alpha21z) ];
alpha03 = [ 0 0 eval(solIa.alpha03z) ];
aB3B2 = eval(solIa.aB32)*[ cos(phi2) sin(phi2) 0];
```

The following numerical solutions are then obtained

$$\alpha_{21} = 14.568 \text{ rad/s}^2, \ \alpha_{03} = -14.568 \text{ rad/s}^2, \text{ and } a_{B_{32}} = -0.140 \text{ m/s}^2.$$

To print the numerical values, the following MATLAB commands are used:

```
fprintf('alpha21 = [ %g, %g, %g] (rad/s^2)\n', alpha21 );
fprintf('alpha03 = [ %g, %g, %g] (rad/s^2)\n', alpha03 );
fprintf('aB32 = %g (m/s^2)\n', eval(solIa.aB32) );
fprintf('aB3B2 = [ %g, %g, %d] (m/s^2)\n', aB3B2 );
```

The absolute angular accelerations of the links 2 and 3 are

$$\boldsymbol{\alpha}_{20} = \boldsymbol{\alpha}_{30} = -\boldsymbol{\alpha}_{03} = 14.568 \ \mathbf{k} \text{ rad/s}^2.$$

The absolute linear acceleration of $D_3 = D_4$ is obtained from the following equation

$$\mathbf{a}_{D_3} = \mathbf{a}_{D_4} = \mathbf{a}_C + \boldsymbol{\alpha}_{30} \times \mathbf{r}_{CD} - \omega_{30}^2 \mathbf{r}_{CD} = 4.617 \ \boldsymbol{\imath} - 1.811 \ \boldsymbol{\jmath} \text{ m/s}^2,$$

where $\mathbf{a}_C = \mathbf{0}$. In MATLAB the absolute accelerations are

```
alpha30 = - alpha03;
alpha20 = alpha30;
aD3 = cross(alpha30,rD-rC)-dot(omega20,omega20)*(rD-rC);
fprintf('alpha20=alpha30=[ %d, %d, %g] (rad/s^2)\n', alpha30 );
fprintf('aD3 = aD4 = [ %g, %g, %g] (m/s^2)\n', aD3 );
```

Second contour analysis
Figure 3.10(a) depicts the second independent contour II

- rotational joint R between the links 0 and 3 (joint C);
- rotational joint R between the links 3 and 4 (joint D);
- translational joint T between the links 4 and 5 (joint D);
- rotational joint R between the links 5 and 0 (joint E).

For the velocity analysis, the following vectorial equations are used

$$\begin{aligned}&\boldsymbol{\omega}_{30} + \boldsymbol{\omega}_{43} + \boldsymbol{\omega}_{05} = \mathbf{0},\\ &\mathbf{r}_C \times \boldsymbol{\omega}_{30} + \mathbf{r}_D \times \boldsymbol{\omega}_{43} + \mathbf{r}_E \times \boldsymbol{\omega}_{05} + \mathbf{v}^{rel}_{D_5D_4} = \mathbf{0},\end{aligned} \tag{3.106}$$

where $\mathbf{r}_D = x_D\mathbf{\imath} + y_D\mathbf{\jmath}$, $\mathbf{r}_E = x_E\mathbf{\imath} + y_E\mathbf{\jmath}$, and

$$\begin{aligned}&\boldsymbol{\omega}_{30} = \omega_{30}\mathbf{k},\ \boldsymbol{\omega}_{43} = \omega_{43}\mathbf{k},\ \boldsymbol{\omega}_{05} = \omega_{05}\mathbf{k},\\ &\mathbf{v}^{rel}_{D_5D_4} = \mathbf{v}_{D_{54}} = v_{D_{54}}\cos\phi_4\mathbf{\imath} + v_{D_{54}}\sin\phi_4\mathbf{\jmath}.\end{aligned}$$

The sign of the relative angular velocities is selected as positive as shown in Figs. 3.8(a) and 10(a). The numerical computation will then give the correct orientation of the unknown vectors. The components of the vectors $\mathbf{r}_D$ and $\mathbf{r}_E$, and the angle ϕ_4 are already known from the position analysis of the mechanism.

The unknown vectors with MATLAB commands are

```
omega43v = [ 0 0 sym('omega43z','real') ];
omega05v = [ 0 0 sym('omega05z','real') ];
v54v = sym('vD54','real')*[ cos(phi4) sin(phi4) 0];
```

Equation (3.106) becomes

$$\begin{aligned}&\omega_{30}\mathbf{k} + \omega_{43}\mathbf{k} + \omega_{05}\mathbf{k} = \mathbf{0},\\ &\begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ x_C & y_C & 0\\ 0 & 0 & \omega_{30}\end{vmatrix} + \begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ x_D & y_D & 0\\ 0 & 0 & \omega_{43}\end{vmatrix} + \begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ x_E & y_E & 0\\ 0 & 0 & \omega_{05}\end{vmatrix} +\\ &v_{D_{54}}\cos\phi_4\mathbf{\imath} + v_{D_{54}}\sin\phi_4\mathbf{\jmath} = \mathbf{0}.\end{aligned} \tag{3.107}$$

Equation (3.107) projected onto the “fixed” reference frame $Oxyz$ gives

$$\begin{aligned}&\omega_{30} + \omega_{43} + \omega_{05} = 0,\\ &y_C\omega_{30} + y_D\omega_{43} + y_E\omega_{05} + v_{D_{54}}\cos\phi_4 = 0,\\ &-x_C\omega_{30} - x_D\omega_{43} - x_E\omega_{05} + v_{D_{54}}\sin\phi_4 = 0.\end{aligned} \tag{3.108}$$

The above system of equations using the following MATLAB commands becomes

```
eqIIomega = omega30 + omega43v + omega05v;
eqIIvz=eqIIomega(3);
eqIIv=cross(rC,omega30)+cross(rD,omega43v)+cross(rE,omega05v)+v54v;
eqIIvx=eqIIv(1);
eqIIvy=eqIIv(2);
```

Equation (3.108) represents an algebraic system of three equations with three unknowns: ω_{43}, ω_{05}, and $v_{D_{54}}$. The system is solved using the MATLAB commands

```
solIIv=solve(eqIIvz,eqIIvx,eqIIvy);
omega43 = [ 0 0 eval(solIIv.omega43z) ];
omega05 = [ 0 0 eval(solIIv.omega05z) ] ;
vD5D4 = eval(solIIv.vD54)*[ cos(phi4) sin(phi4) 0];
```

The following numerical solutions are obtained:

$$\omega_{43} = -4.531 \text{ rad/s}, \ \omega_{05} = -0.917 \text{ rad/s}, \text{ and } v_{D_{54}} = 0.757 \text{ m/s}.$$

To print the numerical values with MATLAB , the following commands are used:

```
fprintf('omega43 = [ %g, %g, %g] (rad/s)\n', omega43 );
fprintf('omega05 = [ %g, %g, %g] (rad/s)\n', omega05 );
fprintf('vD54 = %g (m/s)\n', eval(solIIv.vD54) );
fprintf('vD5D4 = [ %g, %g, %d] (m/s)\n', vD5D4 );
```

The absolute angular velocities of the links 4 and 5 are

$$\boldsymbol{\omega}_{40} = \boldsymbol{\omega}_{50} = -\boldsymbol{\omega}_{05} = 0.917 \ \mathbf{k} \text{ rad/s}, \tag{3.109}$$

and with MATLAB commands, they are

```
omega50 = - omega05;
omega40 = omega50;
```

```
fprintf('omega40=omega50=[ %d, %d, %g] (rad/s)\n', omega50 );
```

For the acceleration analysis, the following vectorial equations are used:

$$\begin{aligned}&\boldsymbol{\alpha}_{30}+\boldsymbol{\alpha}_{43}+\boldsymbol{\alpha}_{05}=\mathbf{0},\\&\mathbf{r}_C\times\boldsymbol{\alpha}_{30}+\mathbf{r}_D\times\boldsymbol{\alpha}_{43}+\mathbf{r}_E\times\boldsymbol{\alpha}_{05}+\mathbf{a}^{rel}_{D_5D_4}+\mathbf{a}^{cor}_{B_5B_4}-\omega^2_{30}\mathbf{r}_{CD}-\omega^2_{40}\mathbf{r}_{DE}=\mathbf{0}.\end{aligned}\tag{3.110}$$

where

$$\begin{aligned}&\boldsymbol{\alpha}_{30}=\alpha_{30}\mathbf{k},\ \boldsymbol{\alpha}_{43}=\boldsymbol{\alpha}_{43}\mathbf{k},\ \boldsymbol{\alpha}_{05}=\boldsymbol{\alpha}_{05}\mathbf{k},\\&\mathbf{a}^{rel}_{D_5D_4}=\mathbf{a}_{D_{54}}=a_{D_{54}}\cos\phi_4\mathbf{\imath}+a_{D_{54}}\sin\phi_4\mathbf{\jmath},\\&\mathbf{a}^{cor}_{D_{54}}=2\,\boldsymbol{\omega}_{40}\times\mathbf{v}_{D_{54}}.\end{aligned}$$

The unknown acceleration vectors using the MATLAB commands are

```
alpha43v = [ 0 0 sym('alpha43z','real') ];
alpha05v = [ 0 0 sym('alpha05z','real') ];
a54v = sym('aD54','real')*[ cos(phi4) sin(phi4) 0];
```

Equation (3.110) becomes

$$\begin{aligned}&\alpha_{30}\mathbf{k}+\alpha_{43}\mathbf{k}+\alpha_{05}\mathbf{k}=\mathbf{0},\\&\begin{vmatrix}\mathbf{\imath}&\mathbf{\jmath}&\mathbf{k}\\x_C&y_C&0\\0&0&\alpha_{30}\end{vmatrix}+\begin{vmatrix}\mathbf{\imath}&\mathbf{\jmath}&\mathbf{k}\\x_D&y_D&0\\0&0&\alpha_{43}\end{vmatrix}+\begin{vmatrix}\mathbf{\imath}&\mathbf{\jmath}&\mathbf{k}\\x_E&y_E&0\\0&0&\alpha_{05}\end{vmatrix}+\\&a_{D_{54}}\cos\phi_4\mathbf{\imath}+a_{D_{54}}\sin\phi_4\mathbf{\jmath}+\begin{vmatrix}\mathbf{\imath}&\mathbf{\jmath}&\mathbf{k}\\0&0&\omega_{40}\\v_{D_{54}}\cos\phi_4&v_{D_{54}}\sin\phi_4&0\end{vmatrix}-\\&\omega^2_{30}[(x_D-x_C)\mathbf{\imath}+(y_D-y_C)\mathbf{\jmath}]-\\&\omega^2_{40}[(x_E-x_D)\mathbf{\imath}+(y_E-y_D)\mathbf{\jmath}]=\mathbf{0}.\end{aligned}\tag{3.111}$$

Equation (3.111) can be rewritten as

$$\begin{aligned}&\alpha_{30}+\alpha_{43}+\alpha_{05}=0,\\&y_C\alpha_{30}+y_D\alpha_{43}+y_E\alpha_{05}+a_{54}\cos\phi_4-2\omega_{40}v_{54}\sin\phi_4-\\&\omega^2_{30}(x_D-x_C)-\omega^2_{40}(x_E-x_D)=0,\\&-x_C\alpha_{30}-x_D\alpha_{43}-x_E\alpha_{05}+a_{54}\sin\phi_4+2\omega_{40}v_{54}\cos\phi_4-\\&\omega^2_{30}(y_D-y_C)-\omega^2_{40}(y_E-y_D)=0.\end{aligned}\tag{3.112}$$

The contour acceleration equations using MATLAB commands are

```
eqIIalpha = alpha30 + alpha43v + alpha05v;
eqIIaz=eqIIalpha(3);
eqIIa1=cross(rC,alpha30)+cross(rD,alpha43v)+cross(rE,alpha05v)+a54v;
eqIIa2=2*cross(omega40,vD5D4);
eqIIa3=-dot(omega30,omega30)*(rD-rC)-dot(omega40,omega40)*(rE-rD);
eqIIa=eqIIa1+eqIIa2+eqIIa3;
eqIIax=eqIIa(1);
eqIIay=eqIIa(2);
```

The unknowns in Eq. (3.112) are α_{43}, α_{05}, and $a_{D_{54}}$. To solve the system, the following MATLAB command is used:

```
solIIa=solve(eqIIaz,eqIIax,eqIIay);
alpha43 = [ 0 0 eval(solIIa.alpha43z) ];
alpha05 = [ 0 0 eval(solIIa.alpha05z) ] ;
aD5D4 = eval(solIIa.aD54)*[ cos(phi4) sin(phi4) 0];
```

The following numerical solutions are obtained:

$$\alpha_{43} = -20.339 \text{ rad/s}^2,\ \alpha_{05} = 5.771 \text{ rad/s}^2,\ \text{and } a_{D_{54}} = 3.411 \text{ m/s}^2.$$

The MATLAB commands for displaying the solutions are

```
fprintf('alpha43 = [ %g, %g, %g] (rad/s^2)\n', alpha43 );
fprintf('alpha05 = [ %g, %g, %g] (rad/s^2)\n', alpha05 );
fprintf('aD54 = %g (m/s^2)\n', eval(solIIa.aD54) );
fprintf('aD5D4 = [ %g, %g, %d] (m/s^2)\n', aD5D4 );
```

The absolute angular accelerations of the links 4 and 5 are

$$\boldsymbol{\alpha}_{40} = \boldsymbol{\alpha}_{50} = -\boldsymbol{\alpha}_{05} = -5.771\ \mathbf{k}\ \text{rad/s}^2,$$

and with MATLAB they are

```
alpha50 = - alpha05;
alpha40 = alpha50;
```

```
fprintf('alpha40=alpha50=[ %d, %d, %g] (rad/s^2)\n', alpha50 );
```

The MATLAB program and results for the velocity and acceleration analysis using the contour method are given in Program 3.8.