Contents

4 Velocity and Acceleration Analysis

4.1 Kinematics of the Rigid Body

The motion of a rigid body (RB) is defined when the position vector, velocity, and acceleration of all points of the rigid body are defined as functions of time with respect to a fixed reference frame with the origin at O_0.

Let $\mathbf{\imath}_0$, $\mathbf{\jmath}_0$, and $\mathbf{k}_0$ be the constant unit vectors of a fixed orthogonal Cartesian reference frame $O_0x_0y_0z_0$ (primary reference frame). The unit vectors $\mathbf{\imath}_0, \mathbf{\jmath}_0$, and $\mathbf{k}_0$ of the primary reference frame are constant with respect to time. Let $\mathbf{\imath}, \mathbf{\jmath}$, and $\mathbf{k}$ be the unit vectors of a mobile orthogonal Cartesian reference frame $Oxyz$ (Fig. 4.1). A reference frame that moves with the rigid body is a *body fixed* (or mobile) reference frame. The unit vectors $\mathbf{\imath}, \mathbf{\jmath}$, and $\mathbf{k}$ of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame. The location of the point O is arbitrary.

The position vector of a point M [$M \in (RB)$], with respect to the fixed reference frame $O_0x_0y_0z_0$, is denoted by $\mathbf{r}_1 = \mathbf{r}_{O_0M}$ and, with respect to the mobile reference frame $Oxyz$, is denoted by $\mathbf{r} = \mathbf{r}_{OM}$. The location of the origin O of the mobile reference frame, with respect to the fixed point O_0, is defined by the position vector $\mathbf{r}_O = \mathbf{r}_{O_0O}$. Thus, the relation between the vectors $\mathbf{r}_1$, $\mathbf{r}$, and $\mathbf{r}_0$ is given by

$$\mathbf{r}_1 = \mathbf{r}_O + \mathbf{r} = \mathbf{r}_O + x\mathbf{\imath} + y\mathbf{\jmath} + z\mathbf{k}, \tag{4.1}$$

where x, y, and z represent the projections of the vector $\mathbf{r}$ on the mobile reference frame. The magnitude of the vector $\mathbf{r} = \mathbf{r}_{OM}$ is a constant, as the distance between the points O and M is constant [$O \in (RB)$ and $M \in (RB)$]. Thus, the x, y, and z components of the vector $\mathbf{r}$ with respect to the mobile reference frame are constant. The unit vectors $\mathbf{\imath}, \mathbf{\jmath}$, and $\mathbf{k}$ are time-dependent vector functions. The vectors $\mathbf{\imath}, \mathbf{\jmath}$, and $\mathbf{k}$ are the unit vector of an orthogonal Cartesian reference frame. Thus, the following relations can be written as

$$\mathbf{\imath} \cdot \mathbf{\imath} = 1, \quad \mathbf{\jmath} \cdot \mathbf{\jmath} = 1, \quad \mathbf{k} \cdot \mathbf{k} = 1, \tag{4.2}$$

$$\mathbf{\imath} \cdot \mathbf{\jmath} = 0, \quad \mathbf{\jmath} \cdot \mathbf{k} = 0, \quad \mathbf{k} \cdot \mathbf{\imath} = 0. \tag{4.3}$$

Velocity of a point on the rigid body

The velocity of an arbitrary point M of the rigid body, with respect to the fixed reference frame $Ox_0y_0z_0$, is the derivative with respect to time of the position vector $\mathbf{r}_1$:

$$\mathbf{v} = \frac{d\mathbf{r}_1}{dt} = \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_O + \dot{\mathbf{r}} = \mathbf{v}_O + x\dot{\boldsymbol{\imath}} + y\dot{\boldsymbol{\jmath}} + z\dot{\mathbf{k}} + \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k}, \tag{4.4}$$

where $\mathbf{v}_O = \dot{\mathbf{r}}_O$ represents the velocity of the origin of the mobile reference frame $O_1x_1y_1z_1$ with respect to the fixed reference frame $Oxyz$. Because all the points in the rigid body maintain their relative position, their velocity relative to the mobile reference frame $Oxyz$ is zero: $\dot{x} = \dot{y} = \dot{z} = 0$.
The velocity of the point M is

$$\mathbf{v} = \mathbf{v}_O + x\dot{\boldsymbol{\imath}} + y\dot{\boldsymbol{\jmath}} + z\dot{\mathbf{k}}.$$

The derivative of the Eqs. (4.2) and (4.3) with respect to time gives

$$\dot{\boldsymbol{\imath}} \cdot \boldsymbol{\imath} = 0, \quad \dot{\boldsymbol{\jmath}} \cdot \boldsymbol{\jmath} = 0, \quad \dot{\mathbf{k}} \cdot \mathbf{k} = 0, \tag{4.5}$$

and

$$\dot{\boldsymbol{\imath}} \cdot \boldsymbol{\jmath} + \dot{\boldsymbol{\jmath}} \cdot \boldsymbol{\imath} = 0, \quad \dot{\boldsymbol{\jmath}} \cdot \mathbf{k} + \dot{\mathbf{k}} \cdot \boldsymbol{\jmath} = 0, \quad \dot{\mathbf{k}} \cdot \boldsymbol{\imath} + \dot{\boldsymbol{\imath}} \cdot \mathbf{k} = 0. \tag{4.6}$$

For Eq. (4.6) the following convention is introduced:

$$\begin{aligned} \dot{\boldsymbol{\imath}} \cdot \boldsymbol{\jmath} &= -\dot{\boldsymbol{\jmath}} \cdot \boldsymbol{\imath} = \omega_z, \\ \dot{\boldsymbol{\jmath}} \cdot \mathbf{k} &= -\dot{\mathbf{k}} \cdot \boldsymbol{\jmath} = \omega_x, \\ \dot{\mathbf{k}} \cdot \boldsymbol{\imath} &= -\dot{\boldsymbol{\imath}} \cdot \mathbf{k} = \omega_y, \end{aligned} \tag{4.7}$$

where ω_x, ω_y and ω_z can be considered as the projections of a vector $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \omega_x\boldsymbol{\imath} + \omega_y\boldsymbol{\jmath} + \omega_z\mathbf{k}.$$

To calculate $\dot{\boldsymbol{\imath}}$, $\dot{\boldsymbol{\jmath}}$, $\dot{\mathbf{k}}$ the following formula is introduced for an arbitrary vector, $\mathbf{d}$,

$$\mathbf{d} = d_x\boldsymbol{\imath} + d_y\boldsymbol{\jmath} + d_z\mathbf{k} = (\mathbf{d} \cdot \boldsymbol{\imath})\,\boldsymbol{\imath} + (\mathbf{d} \cdot \boldsymbol{\jmath})\,\boldsymbol{\jmath} + (\mathbf{d} \cdot \mathbf{k})\,\mathbf{k}. \tag{4.8}$$

Using Eq. (4.8) and the results from Eqs. (4.5) and (4.6) it results

$$\begin{aligned} \dot{\boldsymbol{\imath}} &= (\dot{\boldsymbol{\imath}} \cdot \boldsymbol{\imath})\,\boldsymbol{\imath} + (\dot{\boldsymbol{\imath}} \cdot \boldsymbol{\jmath})\,\boldsymbol{\jmath} + (\dot{\boldsymbol{\imath}} \cdot \mathbf{k})\,\mathbf{k} \\ &= (0)\,\boldsymbol{\imath} + (\omega_z)\,\boldsymbol{\jmath} - (\omega_y)\,\mathbf{k} \\ &= \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix} = \boldsymbol{\omega} \times \boldsymbol{\imath}, \end{aligned}$$

$$\begin{aligned}\dot{\mathbf{j}} &= \left(\dot{\mathbf{j}}\cdot\boldsymbol{\imath}\right)\boldsymbol{\imath}+\left(\dot{\mathbf{j}}\cdot\boldsymbol{\jmath}\right)\boldsymbol{\jmath}+\left(\dot{\mathbf{j}}\cdot\mathbf{k}\right)\mathbf{k}\\ &= \left(-\omega_z\right)\boldsymbol{\imath}+\left(0\right)\boldsymbol{\jmath}+\left(\omega_x\right)\mathbf{k}\\ &= \begin{vmatrix}\boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k}\\ \omega_x & \omega_y & \omega_z\\ 0 & 1 & 0\end{vmatrix} = \boldsymbol{\omega}\times\boldsymbol{\jmath},\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{k}} &= \left(\dot{\mathbf{k}}\cdot\boldsymbol{\imath}\right)\boldsymbol{\imath}+\left(\dot{\mathbf{k}}\cdot\boldsymbol{\jmath}\right)\boldsymbol{\jmath}+\left(\dot{\mathbf{k}}\cdot\mathbf{k}\right)\mathbf{k}\\ &= \left(\omega_y\right)\boldsymbol{\imath}-\left(\omega_x\right)\boldsymbol{\jmath}+\left(0\right)\mathbf{k}\\ &= \begin{vmatrix}\boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k}\\ \omega_x & \omega_y & \omega_z\\ 0 & 0 & 1\end{vmatrix} = \boldsymbol{\omega}\times\mathbf{k}.\end{aligned} \tag{4.9}$$

The relations

$$\dot{\mathbf{i}} = \boldsymbol{\omega}\times\boldsymbol{\imath},\quad \dot{\mathbf{j}} = \boldsymbol{\omega}\times\boldsymbol{\jmath},\quad \dot{\mathbf{k}} = \boldsymbol{\omega}\times\mathbf{k}. \tag{4.10}$$

are known as *Poisson formulas*.
Using Eqs. (4.4) and (4.10), the velocity of M is

$$\mathbf{v} = \mathbf{v}_O + x\boldsymbol{\omega}\times\boldsymbol{\imath} + y\boldsymbol{\omega}\times\boldsymbol{\jmath} + z\boldsymbol{\omega}\times\mathbf{k} = \mathbf{v}_O + \boldsymbol{\omega}\times\left(x\boldsymbol{\imath}+y\boldsymbol{\jmath}+z\mathbf{k}\right),$$

or

$$\mathbf{v} = \mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r}. \tag{4.11}$$

Combining Eqs. (4.4) and (4.11), it results

$$\dot{\mathbf{r}} = \boldsymbol{\omega}\times\mathbf{r}. \tag{4.12}$$

Using Eq. (4.11), the components of the velocity are

$$\begin{aligned}v_x &= v_{Ox} + z\omega_y - y\omega_z,\\ v_y &= v_{Oy} + x\omega_z - z\omega_x,\\ v_z &= v_{Oz} + y\omega_x - x\omega_y.\end{aligned}$$

Acceleration of a point on the rigid body

The acceleration of an arbitrary point $M \in (RB)$, with respect to a fixed reference frame $O_0x_0y_0z_0$, represents the double derivative with respect to time of the position vector $\mathbf{r}_1$:

$$\mathbf{a} = \ddot{\mathbf{r}}_1 = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}) = \frac{d}{dt}\mathbf{v}_O + \frac{d}{dt}\boldsymbol{\omega} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d}{dt}\mathbf{r} = \dot{\mathbf{v}}_O + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}. \tag{4.13}$$

The acceleration of the point O with respect to the fixed reference frame $O_0x_0y_0z_0$, is

$$\mathbf{a}_O = \dot{\mathbf{v}}_O = \ddot{\mathbf{r}}_O. \tag{4.14}$$

The derivative of the vector $\boldsymbol{\omega}$, with respect to the time, is the vector $\boldsymbol{\alpha}$ given by

$$\begin{aligned} \boldsymbol{\alpha} &= \dot{\boldsymbol{\omega}} = \dot{\omega}_x \mathbf{\imath} + \dot{\omega}_y \mathbf{\jmath} + \dot{\omega}_z \mathbf{k} + \omega_x \dot{\mathbf{i}} + \omega_y \dot{\mathbf{j}} + \omega_z \dot{\mathbf{k}} \\ &= \alpha_x \mathbf{\imath} + \alpha_y \mathbf{\jmath} + \alpha_z \mathbf{k} + \omega_x \boldsymbol{\omega} \times \mathbf{\imath} + \omega_y \boldsymbol{\omega} \times \mathbf{\jmath} + \omega_z \boldsymbol{\omega} \times \mathbf{k} \\ &= \alpha_x \mathbf{\imath} + \alpha_y \mathbf{\jmath} + \alpha_z \mathbf{k} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \alpha_x \mathbf{\imath} + \alpha_y \mathbf{\jmath} + \alpha_z \mathbf{k}. \end{aligned} \tag{4.15}$$

where $\alpha_x = \dot{\omega}_x, \alpha_y = \dot{\omega}_y$, and $\alpha_z = \dot{\omega}_z$. In the previous expression the Poisson formulas have been used.

Using Eqs. (4.13), (4.14), and (4.15), the acceleration of the point M is

$$\mathbf{a} = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{4.16}$$

The components of the acceleration are

$$\begin{aligned} a_x &= a_{Ox} + (z\alpha_y - y\alpha_z) + \omega_y (y\omega_x - x\omega_y) + \omega_z (x\omega_x - x\omega_z), \\ a_y &= a_{Oy} + (x\alpha_z - z\alpha_x) + \omega_z (z\omega_y - y\omega_z) + \omega_x (x\omega_y - y\omega_z), \\ a_z &= a_{Oz} + (y\alpha_x - x\alpha_y) + \omega_x (x\omega_z - z\omega_x) + \omega_y (y\omega_z - z\omega_y). \end{aligned}$$

The vector $\boldsymbol{\omega}$ characterizes the rotational motion of the rigid body and is called the *angular velocity*. The vector $\boldsymbol{\alpha}$ is called the *angular acceleration*.

The angular velocity can also be introduced in another way.
If the orientation of a rigid body RB in a reference frame RF_0 depends on only a single scalar variable $\boldsymbol{\zeta}$, there exists for each value of $\boldsymbol{\zeta}$ a vector $\boldsymbol{\omega}$ such that the derivative with respect to $\boldsymbol{\zeta}$ in RF_0 of every vector $\mathbf{c}$ fixed in rigid body RB is given by

$$\frac{d\,\mathbf{c}}{d\boldsymbol{\zeta}} = \boldsymbol{\omega} \times \mathbf{c}, \tag{4.17}$$

where the vector $\boldsymbol{\omega}$ is the rate of change of orientation of the rigid body RB in the reference frame RF_0 with respect to $\boldsymbol{\zeta}$. The vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}}, \tag{4.18}$$

where $\mathbf{a}$ and $\mathbf{b}$ are any two nonparallel vectors fixed in the rigid body RB.

Proof.

The vectors $\mathbf{a}$ and $\mathbf{b}$ are fixed in the rigid body. The magnitudes $\mathbf{a} \cdot \mathbf{a}$, $\mathbf{b} \cdot \mathbf{b}$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ are independent of $\boldsymbol{\zeta}$

$$\frac{d\,(\mathbf{a} \cdot \mathbf{a})}{d\boldsymbol{\zeta}} = 0, \quad \frac{d(\,\mathbf{b} \cdot \mathbf{b})}{d\boldsymbol{\zeta}} = 0, \quad \frac{d\,(\mathbf{a} \cdot \mathbf{b})}{d\boldsymbol{\zeta}} = 0,$$

or

$$\frac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{a} = 0, \quad \frac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} \cdot \mathbf{b} = 0, \quad \frac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} = 0.$$

Using the vector triple product of three vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{t}$, it results

$$\mathbf{p} \times (\mathbf{q} \times \mathbf{t}) = \mathbf{p} \cdot \mathbf{t}\; \mathbf{q} - \mathbf{p} \cdot \mathbf{q}\; \mathbf{t}, \quad (\mathbf{p} \times \mathbf{q}) \times \mathbf{t} = \mathbf{t} \cdot \mathbf{p}\; \mathbf{q} - \mathbf{t} \cdot \mathbf{q}\; \mathbf{t}.$$

From these expressions it follows that

$$\begin{aligned}
\frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} \times \mathbf{a} \;&=\; \frac{\left(\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}\right) \times \mathbf{a}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} = \frac{\mathbf{a} \cdot \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}\, \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} - \mathbf{a} \cdot \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}\, \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} \\
&=\; \frac{-\mathbf{a} \cdot \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}\, \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} = \frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}\, \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} = \frac{d\,\mathbf{a}}{d\boldsymbol{\zeta}},
\end{aligned} \tag{4.19}$$

and

$$\frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} \times \mathbf{b} = \frac{\left(\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}\right) \times \mathbf{b}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} = \frac{\mathbf{b} \cdot \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} - \mathbf{b} \cdot \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}}$$

$$= \frac{\mathbf{b} \cdot \dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}} = \frac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}. \tag{4.20}$$

The following vector is defined as

$$\boldsymbol{\omega} = \frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \cdot \mathbf{b}},$$

and the Eqs. (4.19) and (4.20) can be written as

$$\frac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} = \boldsymbol{\omega} \times \mathbf{a}, \quad \frac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} = \boldsymbol{\omega} \times \mathbf{b}.$$

In general a given vector $\mathbf{d}$ can be expressed as

$$\mathbf{d} = d_1\mathbf{n}_1 + d_2\mathbf{n}_2 + d_3\mathbf{n}_3,$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are three unit vectors not parallel to the same plane, and d_1, d_2, d_3 are three scalars.

Any vector $\mathbf{c}$ fixed in the rigid body RB can be expressed as

$$\mathbf{c} = c_1\mathbf{a} + c_2\mathbf{b} + c_3\mathbf{a} \times \mathbf{b}, \tag{4.21}$$

where c_1, c_2, and c_3 are constant and independent of $\boldsymbol{\zeta}$. Differentiating Eq. (4.21) with respect to $\boldsymbol{\zeta}$, the following expression is obtained:

$$\begin{aligned}
\frac{d\,\mathbf{c}}{d\boldsymbol{\zeta}} &= c_1\frac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} + c_2\frac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} + c_3\frac{d\,\mathbf{a}}{d\boldsymbol{\zeta}} \times \mathbf{b} + c_3\mathbf{a} \times \frac{d\,\mathbf{b}}{d\boldsymbol{\zeta}} \\
&= c_1\boldsymbol{\omega} \times \mathbf{a} + c_2\boldsymbol{\omega} \times \mathbf{b} + c_3\left[(\boldsymbol{\omega} \times \mathbf{a}) \times \mathbf{b} + \mathbf{a} \times (\boldsymbol{\omega} \times \mathbf{b})\right] \\
&= c_1\boldsymbol{\omega} \times \mathbf{a} + c_2\boldsymbol{\omega} \times \mathbf{b} + c_3\left[\mathbf{b} \cdot \boldsymbol{\omega}\,\mathbf{a} - \mathbf{b} \cdot \mathbf{a}\,\boldsymbol{\omega} + \mathbf{a} \cdot \mathbf{b}\,\boldsymbol{\omega} - \mathbf{a} \cdot \boldsymbol{\omega}\,\mathbf{b}\right]
\end{aligned}$$

$$
\begin{aligned}
&= c_1\boldsymbol{\omega}\times\mathbf{a}+c_2\boldsymbol{\omega}\times\mathbf{b}+c_3\left[\boldsymbol{\omega}\cdot\mathbf{b}\,\mathbf{a}-\mathbf{a}\cdot\mathbf{b}\,\boldsymbol{\omega}+\mathbf{a}\cdot\mathbf{b}\,\boldsymbol{\omega}-\mathbf{a}\cdot\boldsymbol{\omega}\,\mathbf{b}\right]\\
&= c_1\boldsymbol{\omega}\times\mathbf{a}+c_2\boldsymbol{\omega}\times\mathbf{b}+c_3\left[\boldsymbol{\omega}\cdot\mathbf{b}\,\mathbf{a}-\boldsymbol{\omega}\cdot\mathbf{a}\,\mathbf{b}\right]\\
&= c_1\boldsymbol{\omega}\times\mathbf{a}+c_2\boldsymbol{\omega}\times\mathbf{b}+c_3\boldsymbol{\omega}\times(\mathbf{a}\times\mathbf{b})\\
&= \boldsymbol{\omega}\times(c_1\mathbf{a}+c_2\mathbf{b}+c_3\mathbf{a}\times\mathbf{b})\\
&= \boldsymbol{\omega}\times\mathbf{c}. \qquad (4.22)
\end{aligned}
$$

The vector $\boldsymbol{\omega}$ is not associated with any particular point. With the help of $\boldsymbol{\omega}$ the process of differentiation is replaced with that of cross multiplication.

The vector $\boldsymbol{\omega}$ can be expressed in a symmetrical relation in $\mathbf{a}$ and $\mathbf{b}$:

$$
\boldsymbol{\omega}=\frac{1}{2}\left(\frac{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}\times\dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}\cdot\mathbf{b}}+\frac{\dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}\times\dfrac{d\,\mathbf{a}}{d\boldsymbol{\zeta}}}{\dfrac{d\,\mathbf{b}}{d\boldsymbol{\zeta}}\cdot\mathbf{a}}\right). \qquad (4.23)
$$

The first derivatives of a vector $\mathbf{p}$, with respect to a scalar variable $\boldsymbol{\zeta}$ in two reference frames RF_i and RF_j, are related as follows:

$$
\frac{{}^{(j)}d\,\mathbf{p}}{d\boldsymbol{\zeta}}=\frac{{}^{(i)}d\,\mathbf{p}}{d\boldsymbol{\zeta}}+\boldsymbol{\omega}_{ij}\times\mathbf{p}, \qquad (4.24)
$$

where $\boldsymbol{\omega}_{ij}$ is the rate of change of orientation of RF_i in RF_j with respect to $\boldsymbol{\zeta}$ and $\dfrac{{}^{(j)}d\,\mathbf{p}}{d\boldsymbol{\zeta}}$ is the total derivative of $\mathbf{p}$ with respect to $\boldsymbol{\zeta}$ in RF_j.

Proof.

The vector $\mathbf{p}$ can be expressed as

$$
\mathbf{p}=p_1\boldsymbol{\imath}_1+p_2\boldsymbol{\imath}_2+p_3\boldsymbol{\imath}_3,
$$

where $\boldsymbol{\imath}_1$, $\boldsymbol{\imath}_2$, $\boldsymbol{\imath}_3$ are three unit vectors not parallel to the same plane fixed in RF_i, and p_x, p_y, p_z are the scalar measure numbers of $\mathbf{p}$. Differentiating in RF_j, the following expression is obtained:

$$
\begin{aligned}
\frac{{}^{(j)}d\,\mathbf{p}}{d\boldsymbol{\zeta}} &= \frac{{}^{(j)}d}{d\boldsymbol{\zeta}}\left(p_1\boldsymbol{\imath}_1+p_2\boldsymbol{\imath}_2+p_3\boldsymbol{\imath}_3\right)\\
&= \frac{{}^{(j)}d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_1+\frac{{}^{(j)}d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_2+\frac{{}^{(j)}d\,p_3}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_3+p_1\frac{{}^{(j)}d\,\boldsymbol{\imath}_1}{d\boldsymbol{\zeta}}+p_2\frac{{}^{(j)}d\,\boldsymbol{\imath}_2}{d\boldsymbol{\zeta}}+p_3\frac{{}^{(j)}d\,\boldsymbol{\imath}_3}{d\boldsymbol{\zeta}}\\
&= \frac{d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_1+\frac{d\,p_2}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_2+\frac{d\,p_3}{d\boldsymbol{\zeta}}\boldsymbol{\imath}_3+p_1\boldsymbol{\omega}_{ij}\times\boldsymbol{\imath}_1+p_2\boldsymbol{\omega}_{ij}\times\boldsymbol{\imath}_2+p_3\boldsymbol{\omega}_{ij}\times\boldsymbol{\imath}_3
\end{aligned}
$$

$$= \frac{^{(i)}d\,p_2}{d\boldsymbol{\zeta}}\mathbf{1}_1 + \frac{^{(i)}d\,p_2}{d\boldsymbol{\zeta}}\mathbf{1}_2 + \frac{^{(i)}d\,p_3}{d\boldsymbol{\zeta}}\mathbf{1}_3 + \boldsymbol{\omega}_{ij} \times (p_1\mathbf{1}_1 + p_2\mathbf{1}_2 + p_3\mathbf{1}_3)$$
$$= \frac{^{(i)}d\,\mathbf{p}}{d\boldsymbol{\zeta}} + \boldsymbol{\omega}_{ij} \times \mathbf{p}.$$

The *angular velocity* of a rigid body RB in a reference frame RF_0 is the rate of change of orientation with respect to the time t

$$\boldsymbol{\omega} = \frac{1}{2}\left(\frac{\frac{d\,\mathbf{a}}{dt} \times \frac{d\,\mathbf{b}}{dt}}{\frac{d\,\mathbf{a}}{dt}\cdot\mathbf{b}} + \frac{\frac{d\,\mathbf{b}}{dt} \times \frac{d\,\mathbf{a}}{dt}}{\frac{d\,\mathbf{b}}{dt}\cdot\mathbf{a}}\right) = \frac{1}{2}\left(\frac{\dot{\mathbf{a}} \times \dot{\mathbf{b}}}{\dot{\mathbf{a}}\cdot\mathbf{b}} + \frac{\dot{\mathbf{b}} \times \dot{\mathbf{a}}}{\dot{\mathbf{b}}\cdot\mathbf{a}}\right). \qquad (4.25)$$

The direction of $\boldsymbol{\omega}$ is related to the direction of the rotation of the rigid body through a right-hand rule.
Let RF_i, $i = 1, 2, ..., n$ be n reference frames. The angular velocity of a rigid body r in the reference frame RF_n, can be expressed as

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r1} + \boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{23} + ... + \boldsymbol{\omega}_{r,n-1}. \qquad (4.26)$$

Proof.
Let $\mathbf{p}$ be any vector fixed in the rigid body. Then

$$\frac{^{(i)}d\,\mathbf{p}}{dt} = \boldsymbol{\omega}_{ri} \times \mathbf{p}$$
$$\frac{^{(i-1)}d\,\mathbf{p}}{dt} = \boldsymbol{\omega}_{r,i-1} \times \mathbf{p}.$$

On the other hand:

$$\frac{^{(i)}d\,\mathbf{p}}{dt} = \frac{^{(i-1)}d\,\mathbf{p}}{dt} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p}.$$

Hence,

$$\boldsymbol{\omega}_{ri} \times \mathbf{p} = \boldsymbol{\omega}_{r,i-1} \times \mathbf{p} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p},$$

as this equation is satisfied for all $\mathbf{p}$ fixed in the rigid body:

$$\boldsymbol{\omega}_{ri} = \boldsymbol{\omega}_{r,i-1} + \boldsymbol{\omega}_{i,i-1}. \qquad (4.27)$$

With $i = n$, Eq. (4.27) gives

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-1} + \boldsymbol{\omega}_{n,n-1}. \qquad (4.28)$$

With $i = n - 1$, Eq. (4.27) gives

$$\boldsymbol{\omega}_{r,n-1} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2}. \tag{4.29}$$

Substitute Eq. (4.29) into Eq. (4.28) the following expression is obtained:

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2} + \boldsymbol{\omega}_{n,n-1}.$$

Next use Eq. (4.27) with $i = n - 2$, then with $i = n - 3$, and so forth.

Motion of a point that moves relative to a rigid body

A reference frame that moves with the rigid body is a body fixed reference frame. Figure 4.2 shows a rigid body (RB) in motion relative to a primary reference frame with its origin at point O_0, $O_0x_0y_0z_0$. The primary reference frame is a fixed reference frame or an earth fixed reference frame. The unit vectors $\mathbf{\imath}_0, \mathbf{\jmath}_0$, and $\mathbf{k}_0$ of the primary reference frame are constant.

The body fixed reference frame (mobile reference frame), $Oxyz$, has its origin at a point O of the rigid body $[O \in (RB)]$, and is a moving reference frame relative to the primary reference. The unit vectors $\mathbf{\imath}, \mathbf{\jmath}$, and $\mathbf{k}$ of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame.

The position vector of a point P of the rigid body $[P \in (RB)]$ relative to the origin O of the body fixed reference frame is the vector $\mathbf{r}_{OP}$. The velocity of P relative to O is

$$\frac{d\mathbf{r}_{OP}}{dt} = \boldsymbol{\omega} \times \mathbf{r}_{OP},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body.

The position vector of a point A (the point A is not assumed to be a point of the rigid body, as shown in Fig. 4.2) relative to the origin O_0 of the primary reference frame is

$$\mathbf{r}_A = \mathbf{r}_O + \mathbf{r},$$

where

$$\mathbf{r} = \mathbf{r}_{OA} = x\mathbf{\imath} + y\mathbf{\jmath} + z\mathbf{k},$$

is the position vector of A relative to the origin O of the body fixed reference frame, and x, y, and z are the coordinates of A in terms of the body fixed reference frame. The velocity of the point A is the time derivative of the position vector $\mathbf{r}_A$:

$$\begin{aligned}\mathbf{v}_A &= \frac{d\mathbf{r}_O}{dt} + \frac{d\mathbf{r}}{dt} = \mathbf{v}_O + \mathbf{v}_{AO} = \\ &\mathbf{v}_O + \frac{dx}{dt}\boldsymbol{\imath} + x\frac{d\boldsymbol{\imath}}{dt} + \frac{dy}{dt}\boldsymbol{\jmath} + y\frac{d\boldsymbol{\jmath}}{dt} + \frac{dz}{dt}\mathbf{k} + z\frac{d\mathbf{k}}{dt}.\end{aligned}$$

Using Poisson formulas, the total derivative of the position vector $\mathbf{r}$ is

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k} + \boldsymbol{\omega} \times \mathbf{r}.$$

The velocity of A relative to the body fixed reference frame is a derivative in the body fixed reference frame:

$$\mathbf{v}_{Arel} = \frac{^{(RB)}d\,\mathbf{r}}{dt} = \frac{dx}{dt}\boldsymbol{\imath} + \frac{dy}{dt}\boldsymbol{\jmath} + \frac{dz}{dt}\mathbf{k} = \dot{x}\boldsymbol{\imath} + \dot{y}\boldsymbol{\jmath} + \dot{z}\mathbf{k}, \tag{4.30}$$

A general formula for the total derivative of a moving vector $\mathbf{r}$ can be written as

$$\frac{d\mathbf{r}}{dt} = \frac{^{(RB)}d\,\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}, \tag{4.31}$$

where $\dfrac{d\mathbf{r}}{dt} = \dfrac{^{(0)}d\,\mathbf{r}}{dt}$ is the derivative in the fixed reference frame (0) $(O_0x_0y_0z_0)$, and $\dfrac{^{(RB)}d\,\mathbf{r}}{dt}$ is the derivative in the mobile reference frame (body fixed reference frame).

The velocity of the point A relative to the primary reference frame is

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}, \tag{4.32}$$

Equation (4.32) expresses the velocity of a point A as the sum of three terms:
- the velocity of a point O of the rigid body,
- the velocity $\mathbf{v}_{Arel}$ of A relative to the rigid body, and
- the velocity $\boldsymbol{\omega} \times \mathbf{r}$ of A relative to O due to the rotation of the rigid body.

The acceleration of the point A relative to the primary reference frame is obtained by taking the time derivative of Eq. (4.32)

$$\begin{aligned}\mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO} \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),\end{aligned} \tag{4.33}$$

where

$$\mathbf{a}_{Arel} = \frac{^{(RB)}d^2\,\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\boldsymbol{\imath} + \frac{d^2y}{dt^2}\boldsymbol{\jmath} + \frac{d^2z}{dt^2}\mathbf{k}, \tag{4.34}$$

is the acceleration of A relative to the body fixed reference frame or relative to the rigid body. The term

$$\mathbf{a}_{Cor} = 2\boldsymbol{\omega} \times \mathbf{v}_{Arel}.$$

is called the *Coriolis acceleration*.

In the case of planar motion, Eq. (4.33) becomes

$$\mathbf{a}_A \;=\; \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} - \boldsymbol{\omega}^2\mathbf{r}. \tag{4.35}$$

The velocity $\mathbf{v}_A$ and the acceleration $\mathbf{a}_A$ of a point A are relative to the primary reference frame. The terms $\mathbf{v}_{Arel}$ and $\mathbf{a}_{Arel}$ are the velocity and acceleration of point A relative to the body fixed reference frame, i.e., they are the velocity and acceleration measured by an observer moving with the rigid body (Fig. 4.3).

If A is a point of the rigid body, $A \in RB$, $\mathbf{v}_{Arel} = \mathbf{0}$, and $\mathbf{a}_{Arel} = \mathbf{0}$.

Inertial reference frames

A reference frame is inertial if Newton's second law is applied in the form $\sum \mathbf{F} = m\mathbf{a}$. Figure 4.4 shows a nonaccelerating, nonrotating reference frame with the origin at O_0, and a secondary nonrotating, earth-centered reference frame with the origin at O. The nonaccelerating, nonrotating reference frame with the origin at O_0 is assumed to be an inertial reference. The acceleration of the earth, due to the gravitational attractions of the sun, moon, etc., is $\mathbf{g}_O$. The earth-centered reference frame has the acceleration $\mathbf{g}_O$, too. Newton's second law for an object A of mass m, using the hypothetical nonaccelerating, nonrotating reference frame with the origin at O_0, can be written as

$$m\mathbf{a}_A = m\mathbf{g}_A + \sum \mathbf{F}, \tag{4.36}$$

where $\mathbf{a}_A$ is the acceleration of A relative to O_0, $\mathbf{g}_A$ is the resulting gravitational acceleration, and $\sum \mathbf{F}$ is the sum of all other external forces acting on A. The acceleration of A relative to O_0 is

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel},$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the earth-centered reference frame. The acceleration of the origin O is equal to the gravitational acceleration of the earth $\mathbf{a}_O = \mathbf{g}_O$. The earth-centered reference frame does not rotate ($\boldsymbol{\omega} = \mathbf{0}$). If the object A is on or near the earth, its gravitational acceleration $\mathbf{g}_A$ due to the attraction of the sun, etc., is nearly equal to the gravitational acceleration of the earth $\mathbf{g}_O$, and Eq. (4.36) becomes

$$\sum \mathbf{F} = m\mathbf{a}_{Arel}. \tag{4.37}$$

Newton's second law can be applied using a nonrotating, earth-centered reference frame if the object is near the earth. In most applications, Newton's second law can be applied using an earth-fixed reference frame. Figure 4.5 shows a nonrotating reference frame with its origin at the center of the earth O and a secondary earth-fixed reference frame with its origin at a point B. The earth-fixed reference frame with the origin at B can be assumed to be an inertial reference and $\sum \mathbf{F} = m\mathbf{a}_{Arel}$, where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the earth-fixed reference frame.

The motion of an object A can be analyzed using a primary inertial reference frame with its origin at the point O (Fig. 4.6). A secondary reference frame with its origin at B undergoes an arbitrary motion with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. Newton's second law for the object A of mass m is $\sum \mathbf{F} = m\mathbf{a}_A$, where $\mathbf{a}_A$ is the acceleration of A relative to O. Newton's second law can be written in the form:

$$\sum \mathbf{F} - m[\mathbf{a}_B + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r}_{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{BA})] = m\mathbf{a}_{Arel}, \tag{4.38}$$

where $\mathbf{a}_{Arel}$ is the acceleration of A relative to the secondary reference frame. The term $\mathbf{a}_B$ is the acceleration of the origin B of the secondary reference frame relative to the primary inertial reference. The term $2\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is the Coriolis acceleration, and the term $-2m\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is called the Coriolis force. This is Newton's second law expressed in terms of a secondary reference frame undergoing an arbitrary motion relative to an inertial primary reference frame.

The classical method for obtaining the velocities and/or accelerations of links and joints is to compute the derivatives of the positions and/or velocities with respect to time.

4.2 Driver Link

For a driver link in rotational motion [see Fig. 3.3(a)], the following position relation can be written:

$$x_B(t) = x_A + L_{AB} \cos \phi(t),$$
$$y_B(t) = y_A + L_{AB} \sin \phi(t). \tag{4.39}$$

Differentiating Eq. (4.39) with respect to time, the following expressions are obtained:

$$v_{Bx} = \dot{x}_B = \frac{d\,x_B(t)}{d\,t} = -L_{AB}\dot{\phi} \sin \phi,$$
$$v_{By} = \dot{y}_B = \frac{d\,y_B(t)}{d\,t} = L_{AB}\dot{\phi} \cos \phi. \tag{4.40}$$

The angular velocity of the driver link is $\omega = \dot{\phi}$.
The time derivative of Eq. (4.40) yields

$$a_{Bx} = \ddot{x}_B = \frac{d\,v_B(t)}{d\,t} = -L_{AB}\dot{\phi}^2 \cos \phi - L_{AB}\ddot{\phi} \sin \phi,$$
$$a_{By} = \ddot{y}_B = \frac{d\,v_B(t)}{d\,t} = -L_{AB}\dot{\phi}^2 \sin \phi + L_{AB}\ddot{\phi} \cos \phi, \tag{4.41}$$

where $\alpha = \ddot{\phi}$ is the angular acceleration of the driver link AB.

4.3 RRR Dyad

For the RRR dyad (see Fig. 3.4), there are two quadratic equations of the form:

$$[x_C(t) - x_A]^2 + [y_C(t) - y_A]^2 = L_{AC}^2 = L_2^2,$$
$$[x_C(t) - x_B]^2 + [y_C(t) - y_B]^2 = L_{BC}^2 = L_3^2. \tag{4.42}$$

Solving the above system of quadratic equations, the coordinates $x_C(t)$ and $y_C(t)$ are obtained.
The derivative of Eq. (4.42) with respect to time yields

$$(x_C - x_A)\,(\dot{x}_C - \dot{x}_A) + (y_C - y_A)\,(\dot{y}_C - \dot{y}_A) = 0,$$
$$(x_C - x_B)\,(\dot{x}_C - \dot{x}_B) + (y_C - y_B)\,(\dot{y}_C - \dot{y}_B) = 0, \tag{4.43}$$

From Eq. (4.43) the velocity vector of the joint C, $\mathbf{v}_C = [\dot{x}_C, \dot{y}_C]^T$, is written in matrix form

$$\mathbf{v}_C = \mathbf{M}_1 \cdot \mathbf{v}, \tag{4.44}$$

where

$$\begin{aligned}
&\mathbf{v} = [\dot{x}_A, \dot{y}_A, \dot{x}_B, \dot{y}_B]^T,\\
&\mathbf{M}_1 = \mathbf{A}_1^{-1} \cdot \mathbf{A}_2,\\
&\mathbf{A}_1 = \begin{bmatrix} x_C - x_A & y_C - y_A \\ x_C - x_B & y_C - y_B \end{bmatrix},\\
&\mathbf{A}_2 = \begin{bmatrix} x_C - x_A & y_C - y_A & 0 & 0 \\ 0 & 0 & x_C - x_B & y_C - y_B \end{bmatrix}.
\end{aligned}$$

Similarly, by differentiating Eq. (4.43), the following acceleration equations are obtained:

$$\begin{aligned}
&(\dot{x}_C - \dot{x}_A)^2 + (x_C - x_A)\,(\ddot{x}_C - \ddot{x}_A) +\\
&(\dot{y}_C - \dot{y}_A)^2 + (y_C - y_A)\,(\ddot{y}_C - \ddot{y}_A) = 0,\\
&(\dot{x}_C - \dot{x}_B)^2 + (x_C - x_B)\,(\ddot{x}_C - \ddot{x}_B) +\\
&(\dot{y}_C - \dot{y}_B)^2 + (y_C - y_B)\,(\ddot{y}_C - \ddot{y}_B) = 0.
\end{aligned} \tag{4.45}$$

The acceleration vector of the joint C is obtained from the above system of equations:

$$\mathbf{a}_C = [\ddot{x}_C, \ddot{y}_C]^T = \mathbf{M}_1 \cdot \mathbf{a} + \mathbf{M}_2, \tag{4.46}$$

where

$$\begin{aligned}
&\mathbf{a} = [\ddot{x}_A, \ddot{y}_A, \ddot{x}_B, \ddot{y}_B]^T,\\
&\mathbf{M}_2 = -\mathbf{A}_1^{-1} \cdot \mathbf{A}_3,\\
&\mathbf{A}_3 = \begin{bmatrix} (\dot{x}_C - \dot{x}_A)^2 + (\dot{y}_C - \dot{y}_A)^2 \\ (\dot{x}_C - \dot{x}_B)^2 + (\dot{y}_C - \dot{y}_B)^2 \end{bmatrix}.
\end{aligned}$$

To compute the angular velocity and acceleration of the RRR dyad, the following equations are written for the angles $\phi_2(t)$ and $\phi_3(t)$:

$$\begin{aligned}
&y_C(t) - y_A + [x_C(t) - x_A]\ \tan\phi_2(t) = 0,\\
&y_C(t) - y_B + [x_C(t) - x_B]\ \tan\phi_3(t) = 0.
\end{aligned} \tag{4.47}$$

The derivative with respect to time of Eq. (4.47) yields

$$\dot{y}_C - \dot{y}_A - (\dot{x}_C - \dot{x}_A)\ \tan\phi_2 - (x_C - x_A)\frac{1}{\cos^2\phi_2}\,\dot{\phi}_2 = 0,$$
$$\dot{y}_C - \dot{y}_B - (\dot{x}_C - \dot{x}_B)\ \tan\phi_3 - (x_C - x_B)\frac{1}{\cos^2\phi_3}\,\dot{\phi}_3 = 0. \quad (4.48)$$

The angular velocity vector is computed as

$$\boldsymbol{\omega} = [\dot{\phi}_2, \dot{\phi}_3]^T = [\omega_2, \omega_3]^T = \boldsymbol{\Omega}_1 \cdot \mathbf{v} + \boldsymbol{\Omega}_2 \cdot \mathbf{v}_C, \quad (4.49)$$

where

$$\boldsymbol{\Omega}_1 = \begin{bmatrix} \dfrac{x_C - x_A}{L_2^2} & -\dfrac{x_C - x_A}{L_2^2} & 0 & 0 \\ 0 & 0 & \dfrac{x_C - x_B}{L_3^2} & -\dfrac{x_C - x_B}{L_3^2} \end{bmatrix},$$
$$\boldsymbol{\Omega}_2 = \begin{bmatrix} -\dfrac{x_C - x_A}{L_2^2} & \dfrac{x_C - x_A}{L_2^2} \\ -\dfrac{x_C - x_B}{L_3^2} & \dfrac{x_C - x_B}{L_3^2} \end{bmatrix}.$$

Differentiating Eq. (4.49), the angular acceleration vector $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ is

$$\boldsymbol{\alpha} = [\ddot{\phi}_2, \ddot{\phi}_3]^T = [\alpha_2, \alpha_3]^T = \dot{\boldsymbol{\Omega}}_1 \cdot \mathbf{v} + \dot{\boldsymbol{\Omega}}_2 \cdot \mathbf{v}_C + \boldsymbol{\Omega}_1 \cdot \mathbf{a} + \dot{\boldsymbol{\Omega}}_2 \cdot \mathbf{a}_C. \quad (4.50)$$

4.4 RRT Dyad

For the RRT dyad [see Fig. 3.5(a)], the following equations can be written for position analysis

$$[x_C(t) - x_A]^2 + [y_C(t) - y_A]^2 = AC^2 = L_{AC}^2 = L_2^2,$$
$$[x_C(t) - x_B]\sin\alpha - [y_C(t) - y_B]\cos\alpha = \pm\, h. \quad (4.51)$$

From the above system of equations, $x_C(t)$ and $y_C(t)$ can be computed. The time derivative of Eq. (4.51) yields

$$(x_C - x_A)\,(\dot{x}_C - \dot{x}_A) + (y_C - y_A)\,(\dot{y}_C - \dot{y}_A) = 0,$$
$$(\dot{x}_C - \dot{x}_B)\sin\alpha - (\dot{y}_C - \dot{y}_B)\cos\alpha = 0. \quad (4.52)$$

The solution for the velocity vector of the joint C from Eq. (4.52) is

$$\mathbf{v}_C = [\dot{x}_C, \dot{y}_C]^T = \mathbf{M}_3 \cdot \mathbf{v}, \quad (4.53)$$

where

$$\begin{aligned}
\mathbf{M}_3 &= \mathbf{A}_4^{-1} \cdot \mathbf{A_5}, \\
\mathbf{A}_4 &= \begin{bmatrix} x_C - x_A & y_C - y_A \\ \sin\alpha & -\cos\alpha \end{bmatrix}, \\
\mathbf{A}_5 &= \begin{bmatrix} x_C - x_A & y_C - y_A & 0 & 0 \\ 0 & 0 & \sin\alpha & -\cos\alpha \end{bmatrix}.
\end{aligned}$$

Differentiating Eq. (4.52) with respect to time

$$\begin{aligned}
&(\dot{x}_C - \dot{x}_A)^2 + (x_C - x_A)\,(\ddot{x}_C - \ddot{x}_A) + \\
&(\dot{y}_C - \dot{y}_A)^2 + (y_C - y_A)\,(\ddot{y}_C - \ddot{y}_A) = 0, \\
&(\ddot{x}_C - \ddot{x}_B)\sin\alpha - (\ddot{y}_C - \ddot{y}_B)\cos\alpha = 0,
\end{aligned} \tag{4.54}$$

the acceleration vector $\mathbf{a}_C$ is obtained as

$$\mathbf{a}_C = [\ddot{x}_C, \ddot{y}_C]^T = \mathbf{M}_3 \cdot \mathbf{a} + \mathbf{M}_4, \tag{4.55}$$

where

$$\begin{aligned}
\mathbf{M}_4 &= -\mathbf{A}_4^{-1} \cdot \mathbf{A}_6, \\
\mathbf{A}_6 &= \begin{bmatrix} (\dot{x}_C - \dot{x}_A)^2 + (\dot{y}_C - \dot{y}_A)^2 \\ 0 \end{bmatrix}.
\end{aligned} \tag{4.56}$$

The angular position of the element 2 is described by the following equation:

$$y_C(t) - y_A - [x_C(t) - x_A]\tan\phi_2(t) = 0. \tag{4.57}$$

The time derivative of Eq. (4.57) yields

$$\dot{y}_C - \dot{y}_A - (\dot{x}_C - \dot{x}_A)\tan\phi_2 - (x_C - x_A)\frac{1}{\cos^2\phi_2}\,\dot{\phi}_2 = 0, \tag{4.58}$$

and the angular velocity of the element 2 is

$$\omega_2 = \frac{x_C - x_A}{L_2^2}\left[(\dot{y}_C - \dot{y}_A) - (\dot{x}_C - x_A)\tan\phi_2\right]. \tag{4.59}$$

The angular acceleration of the element 2 is $\alpha_2 = \dot{\omega}_2$.

4.5 RTR Dyad

For the RTR dyad [see Fig. 3.6(a)] the position relations are

$$[x_C(t) - x_A]^2 + [y_C(t) - y_A]^2 = L_2^2,$$
$$\tan\alpha = \frac{\dfrac{y_C - y_B}{x_C - x_B} - \dfrac{y_C - y_A}{x_C - x_A}}{1 + \dfrac{y_C - y_B}{x_C - x_B} \cdot \dfrac{y_C - y_A}{x_C - x_A}} =$$
$$\frac{(y_C - y_B)(x_C - x_A) - (y_C - y_A)(x_C - x_B)}{(x_C - x_B)(x_C - x_A) + (y_C - y_B)(y_C - y_A)}. \tag{4.60}$$

The time derivative of Eq. (4.60) yields

$$\begin{aligned}
&(x_C - x_A)\,(\dot{x}_C - \dot{x}_A) + (y_C - y_A)\,(\dot{y}_C - \dot{y}_A) = 0,\\
&\tan\alpha\,[(\dot{x}_C - \dot{x}_B)(x_C - x_A) + (x_C - x_B)(\dot{x}_C - \dot{x}_A)] +\\
&\tan\alpha\,[(\dot{y}_C - \dot{y}_A)(y_C - y_B) + (y_C - y_A)(\dot{y}_C - \dot{y}_B)] +\\
&(\dot{y}_C - \dot{y}_A)(x_C - x_B) + (y_C - y_A)(\dot{x}_C - \dot{x}_B) -\\
&(\dot{y}_C - \dot{y}_B)(x_C - x_A) - (y_C - y_B)(\dot{x}_C - \dot{x}_A) = 0,
\end{aligned} \tag{4.61}$$

or in a matrix form

$$\mathbf{A}_7 \cdot \mathbf{v}_C = \mathbf{A}_8 \cdot \mathbf{v}, \tag{4.62}$$

where

$$\mathbf{A}_7 = \begin{bmatrix} x_C - x_A & y_C - y_A \\ \gamma_1 & \gamma_2 \end{bmatrix},$$
$$\mathbf{A}_8 = \begin{bmatrix} x_C - x_A & y_C - y_A & 0 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \end{bmatrix}.$$

In addition,

$$\begin{aligned}
\gamma_1 &= [(x_C - x_B) + (x_C - x_A)]\tan\alpha - (y_C - y_B) + (y_C - y_A),\\
\gamma_2 &= [(y_C - y_A) + (y_C - y_B)]\tan\alpha - (x_C - x_A) + (x_C - x_B),\\
\gamma_3 &= (x_C - x_B)\tan\alpha + (y_C - y_B),\\
\gamma_4 &= (x_C - x_A)\tan\alpha + (y_C - y_A),\\
\gamma_5 &= (y_C - y_B)\tan\alpha + (x_C - x_B),\\
\gamma_6 &= (y_C - y_A)\tan\alpha - (x_C - x_A).
\end{aligned}$$

The solution for the velocity vector, $\mathbf{v}_C$, of the joint C, from Eq. (4.62) is

$$\mathbf{v}_C = \mathbf{M}_5 \cdot \mathbf{v}, \tag{4.63}$$

where

$$\mathbf{M}_5 = \mathbf{A}_7^{-1} \cdot \mathbf{A}_8.$$

Differentiating Eq. (4.62), the following relation is obtained:

$$\mathbf{A}_7 \cdot \mathbf{a}_C = \mathbf{A}_8 \cdot \mathbf{a} - \mathbf{A}_9, \tag{4.64}$$

where

$$\mathbf{A}_9 = \begin{bmatrix} (\dot{x}_C - \dot{x}_A)^2 + (\dot{y}_C - \dot{y}_A)^2 \\ \gamma_7 \end{bmatrix},$$

$$\begin{aligned} \gamma_7 = \; & 2(\dot{x}_C - \dot{x}_B)(\dot{x}_C - \dot{x}_A)\tan\alpha + 2(\dot{y}_C - \dot{y}_B)(\dot{y}_C - \dot{y}_A)\tan\alpha - \\ & 2(\dot{y}_C - \dot{y}_B)(\dot{x}_C - \dot{x}_A) + 2(\dot{y}_C - \dot{y}_A)(\dot{x}_C - \dot{x}_B). \end{aligned}$$

The acceleration vector of the joint C is

$$\mathbf{a}_C = \mathbf{M}_5 \cdot \mathbf{a} - \mathbf{M}_6, \tag{4.65}$$

where

$$\mathbf{M}_6 = \mathbf{A}_7^{-1} \cdot \mathbf{A}_9.$$

To compute the angular velocities for the RTR dyad, the following equations can be written:

$$\begin{aligned} & y_C(t) - y_A = [x_C(t) - x_A]\tan\phi_2 \\ & \phi_3 = \phi_2 + \alpha. \end{aligned} \tag{4.66}$$

The time derivative of Eq. (4.66) yields

$$\begin{aligned} & (\dot{y}_C - \dot{y}_A) = (\dot{x}_C - \dot{x}_A)\tan\phi_2 + (x_C - x_A)\frac{1}{\cos^2\phi_2}\dot{\phi}_2 \\ & \dot{\phi}_3 = \dot{\phi}_2. \end{aligned} \tag{4.67}$$

The angular velocities of the links 2 and 3 are

$$\omega_2 = \omega_3 = \frac{\cos^2\phi_2}{x_C - x_A}\left[(\dot{y}_C - \dot{y}_A) - (\dot{x}_C - \dot{x}_A)\tan\phi_2\right]. \tag{4.68}$$

The angular accelerations are found to be

$$\alpha_2 = \alpha_3 = \dot{\omega}_2 = \dot{\omega}_3. \tag{4.69}$$

4.6 TRT Dyad

For the TRT dyad (see Fig. 3.7), the two position equations are

$$[x_C(t) - x_A]\sin\alpha - [y_C(t) - y_A]\cos\alpha = \pm\, d,$$
$$[x_C(t) - x_B]\sin\beta - [y_C(t) - y_B]\cos\beta = \pm\, h. \tag{4.70}$$

The derivative with respect to time of Eq. (4.70) yields

$$(\dot{x}_C - \dot{x}_A)\sin\alpha - (\dot{y}_C - \dot{y}_A)\cos\alpha +$$
$$(x_C - x_A)\dot{\alpha}\cos\alpha + (y_C - y_A)\dot{\alpha}\sin\alpha = 0,$$
$$(\dot{x}_C - \dot{x}_B)\sin\beta - (\dot{y}_C - \dot{y}_B)\cos\beta +$$
$$(x_C - x_B)\dot{\beta}\cos\beta + (y_C - y_B)\dot{\beta}\sin\beta = 0, \tag{4.71}$$

or in a matrix form

$$\mathbf{A}_{10}\cdot\mathbf{v}_C = \mathbf{A}_{11}\cdot\mathbf{v}_1, \tag{4.72}$$

where

$$\mathbf{v}_1 = [\dot{x}_A, \dot{y}_A, \dot{\alpha}, \dot{x}_B, \dot{y}_B, \dot{\beta}]^T,$$
$$\mathbf{A}_{10} = \begin{bmatrix} -\sin\alpha & -\cos\alpha \\ \sin\beta & -\cos\beta \end{bmatrix},$$
$$\mathbf{A}_{11} = \begin{bmatrix} \sin\alpha & -\cos\alpha & \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\beta & -\cos\beta & \xi_2 \end{bmatrix},$$
$$\xi_1 = (x_A - x_C)\cos\alpha + (y_A - y_C)\sin\alpha,$$
$$\xi_2 = (x_B - x_C)\cos\beta + (y_B - y_C)\sin\beta.$$

The solution of Eq. (4.72) gives the velocity of the joint C as

$$\mathbf{v}_C = \mathbf{M}_7\cdot\mathbf{v}_1, \tag{4.73}$$

where

$$\mathbf{M}_7 = \mathbf{A}_{10}^{-1}\cdot\mathbf{A}_{11}.$$

Differentiating Eq. (4.72), with respect to time, gives

$$\mathbf{A}_{10}\cdot\mathbf{a}_C = \mathbf{A}_{11}\cdot\mathbf{a}_1 - \mathbf{A}_{12}, \tag{4.74}$$

where

$$\mathbf{a}_1 = [\ddot{x}_A, \ddot{y}_A, \ddot{\alpha}, \ddot{x}_B, \ddot{y}_B, \ddot{\beta}]^T,$$

$$\mathbf{A}_{12} = \begin{bmatrix} \xi_3 \\ \xi_4 \end{bmatrix},$$

$$\begin{aligned} \xi_3 = \;& 2(\dot{x}_C - \dot{x}_A)\dot{\alpha}\cos\alpha + 2(\dot{y}_C - \dot{y}_A)\dot{\beta}\sin\alpha - \\ & (x_C - x_A)\dot{\alpha}^2\sin\alpha + (y_C - y_A)\dot{\alpha}^2\cos\alpha, \\ \xi_4 = \;& 2(\dot{x}_C - \dot{x}_B)\dot{\beta}\cos\beta + 2(\dot{y}_C - \dot{y}_B)\dot{\beta}\sin\beta - \\ & (x_C - x_B)\dot{\beta}^2\sin\beta + (y_C - y_B)\dot{\beta}^2\cos\beta. \end{aligned}$$

The solution of Eq. (4.74) gives the acceleration vector of joint C as

$$\mathbf{a}_C = \mathbf{M}_7 \cdot \mathbf{a} + \mathbf{M}_8. \tag{4.75}$$

where

$$\mathbf{M}_8 = \mathbf{A}_{10}^{-1} \cdot \mathbf{A}_{12}.$$

4.7 Examples

Example 4.1: R-TRR mechanism.
The following dimensions are given for the mechanism shown in Fig. 4.7: $AC = a = 0.100$ m and $BC = 0.300$ m. The angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 45°$. The coordinates of joint B are $x_B = y_B = 0.256$ m. The driver link 1 rotates with a constant speed of $n_1 = 30$ rpm. Find the velocities and the accelerations of the mechanism.

Solution

A cartesian reference frame with the origin at A is selected. The coordinates of joint A are

$$x_A = y_A = 0.$$

the coordinates of the joint C are

$$x_C = AC = 0.100 \text{ m} \quad \text{and} \quad y_C = 0,$$

and the coordinates of joint B are

$$x_B = 0.256 \,\text{m} \quad \text{and} \quad y_B = 0.256 \,\text{m}.$$

The position of joint B is calculated from the equations

$$\tan\phi(t) = \frac{y_B(t)}{x_B(t)} \quad \text{and} \quad [x_B(t) - x_C]^2 + [y_B(t) - y_C]^2 = BC^2,$$

or

$$\begin{aligned} x_B(t) \sin\phi(t) &= y_B(t) \cos\phi(t), \\ [x_B(t) - x_C]^2 + [y_B(t) - y_C]^2 &= BC^2. \end{aligned} \tag{4.76}$$

The linear velocity of point B on link 3 or 2 is

$$\mathbf{v}_B = \mathbf{v}_{B_3} = \mathbf{v}_{B_2} = \dot{x}_B\mathbf{\imath} + \dot{y}_B\mathbf{\jmath},$$

where

$$\dot{x}_B = \frac{dx_B}{dt} \quad \text{and} \quad \dot{y}_B = \frac{dy_B}{dt}.$$

The velocity analysis is carried out differentiating Eq. (4.76):

$$\begin{aligned} \dot{x}_B \sin\phi + x_B\,\dot{\phi}\,\cos\phi &= \dot{y}_B \cos\phi - y_B\,\dot{\phi}\,\sin\phi, \\ \dot{x}_B(x_B - x_C) + \dot{y}_B(y_B - y_C) &= 0, \end{aligned}$$

or

$$\dot{x}_B \sin\phi + x_B\,\omega \cos\phi = \dot{y}_B \cos\phi - y_B\,\omega \sin\phi,$$
$$\dot{x}_B(x_B - x_C) + \dot{y}_B(y_B - y_C) = 0. \tag{4.77}$$

The magnitude of the angular velocity of the driver link 1 is

$$\omega = \omega_1 = \dot{\phi} = \frac{\pi\, n_1}{30} = \frac{\pi\,(30 \text{ rpm})}{30} = 3.141 \text{ rad/s}. \tag{4.78}$$

The angular velocity of link 1 is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 = \omega\,\mathbf{k} = 3.141\,\mathbf{k} \quad \text{rad/s}.$$

The link 2 and the driver link 1 have the same angular velocity $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$. For the given numerical data Eq. (4.77) becomes

$$\dot{x}_B \sin 45^\circ + 0.256\,(3.141) \cos 45^\circ = \dot{y}_B \cos 45^\circ - 0.256\,(3.141) \sin 45^\circ,$$
$$\dot{x}_B(0.256 - 0.1) + \dot{y}_B(0.256 - 0) = 0. \tag{4.79}$$

The solution of Eq. (4.79) gives

$$\dot{x}_B = -0.999 \text{ m/s} \quad \text{and} \quad \dot{y}_B = 0.609 \text{ m/s}.$$

The velocity of B is

$$\mathbf{v}_B = \mathbf{v}_{B_3} = \mathbf{v}_{B_2} = -0.999\,\mathbf{\imath} + 0.609\,\mathbf{\jmath} \quad \text{m/s},$$

$$|\mathbf{v}_B| = |\mathbf{v}_{B_3}| = |\mathbf{v}_{B_2}| = \sqrt{(-0.999)^2 + (0.609)^2} = 1.171 \quad \text{m/s}.$$

The acceleration analysis is obtained using the derivative of the velocities given by Eq. (4.77):

$$\ddot{x}_B \sin\phi + \dot{x}_B\,\omega \cos\phi + \dot{x}_B\,\omega \cos\phi - x_B\,\omega^2 \sin\phi =$$
$$\ddot{y}_B \cos\phi - \dot{y}_B\,\omega \sin\phi - \dot{y}_B\,\omega \sin\phi + y_B\,\omega^2 \cos\phi,$$
$$\ddot{x}_B(x_B - x_C) + \dot{x}_B^2 + \ddot{y}_B(y_B - y_C) + \dot{y}_B^2 = 0. \tag{4.80}$$

The magnitude of the angular acceleration of the driver link 1 is

$$\alpha = \dot{\omega} = \ddot{\phi} = 0.$$

Numerically, Eq. (4.80) gives

$$\ddot{x}_B \sin 45^\circ + 2\,(-0.999)\,(3.141)\,\cos 45^\circ - 0.256\,(3.141)^2 \sin 45^\circ =$$
$$\ddot{y}_B \cos 45^\circ - 2\,(0.609)\,(3.141)\,\sin 45^\circ + 0.256\,(3.141)^2 \cos 45^\circ,$$
$$\ddot{x}_B(0.256 - 0.1) + (-0.999)^2 + \ddot{y}_B(0.256) + 0.609^2 = 0. \tag{4.81}$$

The solution of Eq. (4.81) is

$$\ddot{x}_B = -1.802 \text{ m/s}^2 \quad \text{and} \quad \ddot{y}_B = -4.255 \text{ m/s}^2.$$

The acceleration of B on link 3 or 2 is

$$\mathbf{a}_B = \mathbf{a}_{B_3} = \mathbf{a}_{B_2} = \ddot{x}_B\,\mathbf{\imath} + \ddot{y}_B\,\mathbf{\jmath} = -1.802\,\mathbf{\imath} - 4.255\,\mathbf{\jmath} \quad \text{m/s}^2,$$

$$|\mathbf{a}_B| = |\mathbf{a}_{B_3}| = |\mathbf{a}_{B_2}| = \sqrt{(-1.802)^2 + (-4.255)^2} = 4.620 \quad \text{m/s}^2.$$

The slope of the link 3 (the points B and C are on the straight line BC) is

$$\tan\phi_3(t) = \frac{y_B(t) - y_C}{x_B(t) - x_C},$$

or

$$[x_B(t) - x_C]\,\sin\phi_3(t) = [y_B(t) - y_C]\,\cos\phi_3(t). \tag{4.82}$$

The angle ϕ_3 is computed as follows:

$$\phi_3 = \arctan\frac{y_B - y_C}{x_B - x_C} = \arctan\frac{0.256}{0.256 - 0.1} = 1.023 \text{ rad} = 58.633^\circ.$$

The derivative of Eq. (4.82) yields

$$\dot{x}_B \sin\phi_3 + (x_B - x_C)\,\dot{\phi}_3 \cos\phi_3 = \dot{y}_B \cos\phi_3 - (y_B - y_C)\,\dot{\phi}_3 \sin\phi_3,$$

or

$$\dot{x}_B \sin\phi_3 + (x_B - x_C)\,\omega_3 \cos\phi_3 = \dot{y}_B \cos\phi_3 - (y_B - y_C)\,\omega_3 \sin\phi_3, \tag{4.83}$$

where $\omega_3 = \dot{\phi}_3$.
Numerically Eq. (4.83) gives

$$-0.999 \sin 58.633^\circ + (0.256 - 0.1)\,\omega_3 \cos 58.633^\circ =$$
$$0.609 \cos 58.633^\circ - 0.256\,\omega_3 \sin 58.633^\circ,$$

with the solution $\omega_3 = 3.903$ rad/s.
The angular velocity of link 3 is

$$\boldsymbol{\omega}_3 = \omega_3\,\mathbf{k} = 3.903\,\mathbf{k} \quad \text{rad/s}.$$

The angular acceleration of link 3, $\alpha_3 = \dot{\omega}_3 = \ddot{\phi}_3$, is obtained using the derivative of the Eq. (4.83):

$$\begin{gathered}
\ddot{x}_B \sin\phi_3 + \dot{x}_B\,\omega_3 \cos\phi_3 + \\
\dot{x}_B\,\omega_3 \cos\phi_3 + (x_B - x_C)\,\dot{\omega}_3 \cos\phi_3 - (x_B - x_C)\,\omega_3^2 \sin\phi_3 = \\
\ddot{y}_B \cos\phi_3 - \dot{y}_B\,\omega_3 \sin\phi_3 - \\
\dot{y}_B\,\omega_3 \sin\phi_3 - (y_B - y_C)\,\dot{\omega}_3 \sin\phi_3 - (y_B - y_C)\,\omega_3^2 \cos\phi_3,
\end{gathered}$$

or

$$\begin{gathered}
\ddot{x}_B \sin\phi_3 + 2\,\dot{x}_B\,\omega_3 \cos\phi_3 + (x_B - x_C)\,\alpha_3 \cos\phi_3 - (x_B - x_C)\,\omega_3^2 \sin\phi_3 = \\
\ddot{y}_B \cos\phi_3 - 2\,\dot{y}_B\,\omega_3 \sin\phi_3 - (y_B - y_C)\,\alpha_3 \sin\phi_3 - (y_B - y_C)\,\omega_3^2 \cos\phi_3.
\end{gathered}$$

Numerically, the previous equation becomes

$$\begin{gathered}
-1.802 \sin 58.633^\circ + 2\,(-0.999)\,(3.903) \cos 58.633^\circ + \\
(0.256 - 0.1)\,\alpha_3 \cos 58.633^\circ - (0.256 - 0.1)\,(3.903)^2 \sin 58.633^\circ = \\
-4.255 \cos 58.633^\circ - 2\,(0.609)\,(3.903) \sin 58.633^\circ - \\
0.256\,\alpha_3 \sin 58.633^\circ - 0.256\,(3.903)^2 \cos 58.633^\circ,
\end{gathered}$$

with the solution $\alpha_3 = -2.252$ rad/s^2. The angular acceleration of link 3 is

$$\boldsymbol{\alpha}_3 = \alpha_3\,\mathbf{k} = -2.252\,\mathbf{k} \quad \text{rad/s}^2.$$

The velocity of the point B_1 on link 1 is calculated with the expression of velocity field of two points (B_1 and A) on the same rigid body (link 1):

$$\mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} = \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ x_B & y_B & 0 \end{vmatrix} =$$

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & 3.141 \\ 0.256 & 0.256 & 0 \end{vmatrix} = -0.804\ \mathbf{\imath} + 0.804\ \mathbf{\jmath} \quad \text{m/s}.$$

The velocity field of two points (B_1 and B_2) not situated on the same rigid body (B_1 is on link 1 and B_2 is on link 2) is calculated with

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_1} + \mathbf{v}^r_{B_2B_1} = \mathbf{v}_{B_1} + \mathbf{v}^r_{B_{21}},$$

where $\mathbf{v}^r_{B_{21}}$ is the relative velocity of the point B_2 on link 2 with respect to the point B_1 on link 1:

$$\begin{aligned}\mathbf{v}^r_{B_{21}} &= \mathbf{v}_{B_2} - \mathbf{v}_{B_1} = -0.999\,\imath + 0.609\,\jmath - (-0.804\ \imath + 0.804\ \jmath)\\ &= -0.195\,\imath - 0.195\,\jmath \quad \text{m/s}.\end{aligned}$$

The relation between the angular velocities of link 2 and link 3 is

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_3 + \boldsymbol{\omega}_{23},$$

and the relative angular velocity of link 2 with respect to link 3 is

$$\boldsymbol{\omega}_{23} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_3 = 3.141\,\mathbf{k} - 3.903\,\mathbf{k} = -0.762\,\mathbf{k} \quad \text{rad/s}.$$

The acceleration of the point B_1 on link 1 is

$$\begin{aligned}\mathbf{a}_{B_1} &= \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_{AB} - \omega_1^2\,\mathbf{r}_{AB} = -\omega_1^2\,\mathbf{r}_{AB} = -\omega_1^2\,(x_B\,\imath + y_B\,\jmath)\\ &= -3.141^2\,(0.256\ \imath + 0.256\ \jmath) = -2.528\,\imath - 2.528\,\jmath \quad \text{m/s}^2.\end{aligned}$$

The acceleration of B_2 in terms of B_1 is

$$\mathbf{a}_{B_2} = \mathbf{a}_{B_1} + \mathbf{a}^r_{B_{21}} + 2\,\boldsymbol{\omega}_1 \times \mathbf{v}^r_{B_{21}},$$

where $\mathbf{a}^r_{B_{21}}$ is the relative acceleration of the point B_2 on link 2 with respect to the point B_1 on link 1 and $2\,\boldsymbol{\omega}_1 \times \mathbf{v}^r_{B_{21}}$ is the Coriolis acceleration:

$$\mathbf{a}^c_{B_{21}} = 2\,\boldsymbol{\omega}_1 \times \mathbf{v}^r_{B_{21}} = 2\,\boldsymbol{\omega}_2 \times \mathbf{v}^r_{B_{21}} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \omega_1 \\ v^r_{B_{21}x} & v^r_{B_{21}y} & 0 \end{vmatrix} =$$

$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & 3.141 \\ -0.195 & -0.195 & 0 \end{vmatrix} = 1.226\ \imath - 1.226\ \jmath \quad \text{m/s}^2.$$

The relative acceleration of B_2 with respect to B_1 is

$$\begin{aligned}&\mathbf{a}^r_{B_{21}} = \mathbf{a}_{B_2} - \mathbf{a}_{B_1} - \mathbf{a}^c_{B_{21}} =\\ &-1.802\ \imath - 4.255\ \jmath - (-2.528\,\imath - 2.528\,\jmath) - (1.226\ \imath - 1.226\ \jmath) =\\ &-0.5\ \imath - 0.5\ \jmath \quad \text{m/s}^2.\end{aligned}$$

The relative angular acceleration of link 2 with respect to link 3 is

$$\boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3 = -\boldsymbol{\alpha}_3 = 2.252\,\mathbf{k} \quad \text{rad/s}^2,$$

where $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_1 = \mathbf{0}$.

Example 4.2: R-RTR-RRT mechanism
The mechanism shown in Fig. 4.8 and has the dimensions: $AB = 0.100$ m, $AC = 0.150$ m, $CD = 0.075$ m, and $DE = 0.200$ m. The angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 45°$, and the angular speed of the driver link 1 is $\omega = \omega_1 = 4.712$ rad/s. Find the velocities and the accelerations of the mechanism.

Solution.

The origin of the fixed reference frame is at $C \equiv 0$. The position of the fixed joint A is

$$x_A = 0, \quad y_A = AC = 0.150 \ \text{ m}.$$

The position of joint B is

$$x_B(t) = x_A + AB\cos\phi(t), \quad y_B(t) = y_A + AB\sin\phi(t),$$

and for $\phi = 45°$, the position is

$$x_B = 0 + 0.100\ \cos 45° = 0.070 \text{ m}, \ \ y_B = 0.150 + 0.100\ \sin 45° = 0.220 \text{ m}.$$

The linear velocity vector of B is

$$\mathbf{v}_B = \dot{x}_B\boldsymbol{\imath} + \dot{y}_B\boldsymbol{\jmath},$$

where

$$\dot{x}_B = \frac{dx_B}{dt} = -AB\,\dot{\phi}\ \sin\phi, \ \ \dot{y}_B = \frac{dy_B}{dt} = AB\,\dot{\phi}\ \cos\phi.$$

With $\phi = 45°$ and $\dot{\phi} = \omega = 4.712$ rad/s:

$$\dot{x}_B = -0.100\,(4.712)\ \sin 45° = -0.333 \ \text{ m/s},$$
$$\dot{y}_B = 0.100\,(4.712)\ \cos 45° = 0.333 \ \text{ m/s},$$
$$v_B = |\mathbf{v}_B| = \sqrt{\dot{x}_B^2 + \dot{y}_B^2} = \sqrt{(-0.333)^2 + 0.333^2} = 0.471 \ \text{ m/s}.$$

The linear acceleration vector of B is

$$\mathbf{a}_B = \ddot{x}_B \imath + \ddot{y}_B \jmath,$$

where

$$\ddot{x}_B = \frac{d\dot{x}_B}{dt} = -AB\dot{\phi}^2 \cos\phi - AB\ddot{\phi}\sin\phi,$$
$$\ddot{y}_B = \frac{d\dot{y}_B}{dt} = -AB\dot{\phi}^2 \sin\phi + AB\ddot{\phi}\cos\phi.$$

The angular acceleration of link 1 is $\ddot{\phi} = \dot{\omega} = 0$. The numerical values for the acceleration of B are

$$\ddot{x}_B = -0.100\,(4.712)^2 \cos 45^\circ = -1.569 \ \ \text{m/s}^2,$$
$$\ddot{y}_B = -0.100\,(4.712)^2 \sin 45^\circ = -1.569 \ \ \text{m/s}^2,$$
$$a_B = |\mathbf{a}_B| = \sqrt{\ddot{x}_B^2 + \ddot{y}_B^2} = \sqrt{(-1.569)^2 + (-1.569)^2} = 2.220 \ \ \text{m/s}^2.$$

The velocity and acceleration of point B on link 1 (or on link 2) can also be calculated with the relations

$$\mathbf{v}_B = \mathbf{v}_{B_1} = \mathbf{v}_{B_2} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \omega_1 \\ x_B - x_A & y_B - y_A & 0 \end{vmatrix} =$$
$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & 4.712 \\ 0.070 - 0.15 & 0.220 & 0 \end{vmatrix} = -0.333 \ \imath + 0.333 \ \jmath \quad \text{m/s},$$

$$\mathbf{a}_B = \mathbf{a}_{B_1} = \mathbf{a}_{B_2} = \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_{AB} - \omega_1^2 \mathbf{r}_{AB} = -\omega_1^2 \mathbf{r}_{AB} =$$
$$4.712^2[(0.070 - 0.15) \ \imath + 0.220 \ \jmath] = -1.569 \ \imath - 1.569 \ \jmath \quad \text{m/s}^2,$$

where $\boldsymbol{\omega}_1 = \omega_1\,\mathbf{k} = \omega\,\mathbf{k}$ and $\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\omega}}_1 = \mathbf{0}$.

The points B and C are located on the same straight line BD:

$$y_B(t) - y_C - [x_B(t) - x_C] \ \tan\phi_3(t) = 0. \tag{4.84}$$

The angle $\phi_3 = \phi_2$ is computed as follows:

$$\phi_3 = \phi_2 = \arctan\frac{y_B - y_C}{x_B - x_C},$$

and for $\phi = 45°$ is obtained by

$$\phi_3 = \arctan\frac{0.22}{0.07} = 72.235°.$$

The derivative of Eq. (4.84) yields

$$\dot{y}_B - \dot{y}_C - (\dot{x}_B - \dot{x}_C)\ \tan\phi_3 - (x_B - x_C)\frac{1}{\cos^2\phi_3}\ \dot{\phi}_3 = 0. \tag{4.85}$$

The angular velocity of link 3, $\omega_3 = \omega_2 = \dot{\phi}_3$, is computed as follows

$$\omega_3 = \omega_2 = \frac{\cos^2\phi_3[\dot{y}_B - \dot{y}_C - (\dot{x}_B - \dot{x}_C)\ \tan\phi_3]}{x_B - x_C},$$

and

$$\omega_3 = \frac{\cos^2 72.235°(0.333 + 0.333\tan 72.235°)}{0.07} = 1.807\,\text{rad/s}.$$

The angular acceleration of link 3, $\alpha_3 = \alpha_2 = \ddot{\phi}_3$, is computed from the time derivative of Eq. (4.85)

$$\ddot{y}_B - \ddot{y}_C - (\ddot{x}_B - \ddot{x}_C)\tan\phi_3 - 2(\dot{x}_B - \dot{x}_C)\frac{1}{\cos^2\phi_3}\dot{\phi}_3 -$$
$$2(x_B - x_C)\frac{\sin\phi_3}{\cos^3\phi_3}\ \dot{\phi}_3^2 - (x_B - x_C)\frac{1}{\cos^2\phi_3}\ \ddot{\phi}_3 = 0.$$

The solution of the previous equation is

$$\alpha_3 = \alpha_2 = [\ddot{y}_B - \ddot{y}_C - (\ddot{x}_B - \ddot{x}_C)\tan\phi_3 - 2(\dot{x}_B - \dot{x}_C)\frac{1}{\cos^2\phi_3}\dot{\phi}_3 -$$
$$2(x_B - x_C)\frac{\sin\phi_3}{\cos^3\phi_3}\ \dot{\phi}_3^2]\frac{\cos^2\phi_3}{x_B - x_C},$$

and for the given numerical data:

$$\alpha_3 = \alpha_2 = [-1.569 + 1.569\tan 72.235° + 2(0.333)\frac{1}{\cos^2 72.235°}1.807 -$$
$$2(0.07)\frac{\sin 72.235°}{\cos^3 72.235°}(1.807)^2]\frac{\cos^2 72.235°}{0.07} = 1.020\,\text{rad/s}^2.$$

The links 2 and 3 have the same angular velocity $\boldsymbol{\omega}_3 = \boldsymbol{\omega}_2 = \omega_3\,\mathbf{k}$ and the same angular acceleration $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_2 = \alpha_3\,\mathbf{k}$. The relative angular velocity of link 2 relative to link 1 is

$$\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 = (1.807 - 4.712)\,\mathbf{k} = -2.905\,\mathbf{k}\ \text{rad/s},$$

and the relative angular acceleration of link 2 relative to link 1 is

$$\boldsymbol{\alpha}_{21} = \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = 1.020\,\mathbf{k} \text{ rad/s}^2.$$

The velocity and acceleration of point B on link 3 are calculated with

$$\mathbf{v}_{B_3} = \mathbf{v}_C + \boldsymbol{\omega}_3 \times \mathbf{r}_{CB} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_B & y_B & 0 \end{vmatrix} =$$

$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & 1.807 \\ 0.070 & 0.220 & 0 \end{vmatrix} = -0.398\ \imath + 0.127\ \jmath \quad \text{m/s},$$

$$\mathbf{a}_{B_3} = \mathbf{a}_C + \boldsymbol{\alpha}_3 \times \mathbf{r}_{CB} - \omega_3^2 \mathbf{r}_{CB} =$$

$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \alpha_3 \\ x_B & y_B & 0 \end{vmatrix} - \omega_3^2 (x_B\,\imath + y_B\,\jmath) =$$

$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & 1.020 \\ 0.070 & 0.220 & 0 \end{vmatrix} - 1.807^2 (0.070\,\imath + 0.220\,\jmath) =$$

$$-0.456\ \imath - 0.649\ \jmath \quad \text{m/s}^2.$$

The velocity field of two points (B_2 and B_3) not situated on the same rigid body (B_2 is on link 2 and B_3 is on link 3) is expressed by

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_3} + \mathbf{v}^r_{B_{23}},$$

and

$$\begin{aligned} \mathbf{v}^r_{B_{23}} &= \mathbf{v}_{B_2} - \mathbf{v}_{B_3} = -0.333\ \imath + 0.333\ \jmath - (-0.398\ \imath + 0.127\ \jmath) \\ &= 0.065\ \imath + 0.205\ \jmath \quad \text{m/s}. \end{aligned}$$

The expression for the Coriolis acceleration is

$$\mathbf{a}^c_{B_{23}} = 2\,\boldsymbol{\omega}_2 \times \mathbf{v}^r_{B_{23}} = 2\,\boldsymbol{\omega}_3 \times \mathbf{v}^r_{B_{23}} =$$

$$2\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \omega_3 \\ v^r_{B_{23x}} & v^r_{B_{21y}} & 0 \end{vmatrix} = 2\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & 1.807 \\ 0.065 & 0.205 & 0 \end{vmatrix} =$$

$$-0.742\ \imath + 0.237\ \jmath \ \text{m/s}^2.$$

The relative acceleration of B_2 with respect to B_3 is

$$\mathbf{a}^r_{B_{23}} = \mathbf{a}_{B_2} - \mathbf{a}_{B_3} - \mathbf{a}^c_{B_{23}} =$$
$$-1.569\ \imath - 1.569\ \jmath - (-0.456\,\imath - 0.649\,\jmath) - (-0.742\ \imath + 0.237\ \jmath) =$$
$$-0.5\ \imath - 0.5\ \jmath \quad \text{m/s}^2.$$

The position of the joint D is given by the following quadratic equations:

$$[x_D(t) - x_C]^2 + [y_D(t) - y_C]^2 = CD^2,$$
$$[x_D(t) - x_C]\ \sin\phi_3(t) - [y_D(t) - y_C]\ \cos\phi_3(t) = 0,$$

The previous equations are rewritten as follows:

$$x_D^2(t) + y_D^2(t) = CD^2,$$
$$x_D(t)\ \sin\phi_3(t) - y_D(t)\ \cos\phi_3(t) = 0. \qquad (4.86)$$

For $\phi = 45°$, the coordinates of joint D are

$$x_D = \pm\frac{CD}{\sqrt{1+\tan^2\phi_3}} = \pm\frac{0.075}{\sqrt{1+\tan^2 72.235°}} = -0.023\,\text{m},$$
$$y_D = x_D \tan\phi_3 = -0.023\tan 72.235° = -0.071\,\text{m}.$$

The negative value for x_D was selected for this position of the mechanism.

The velocity analysis is carried out differentiating Eq. (4.86)

$$x_D\dot{x}_D + y_D\dot{y}_D = 0,$$
$$\dot{x}_D \sin\phi_3 + x_D\cos\phi_3\ \dot{\phi}_3 - \dot{y}_D\cos\phi_3 + y_D\sin\phi_3\ \dot{\phi}_3 = 0. \qquad (4.87)$$

For the given data, Eq. (4.87) becomes

$$-0.023\dot{x}_D - 0.071\dot{y}_D = 0,$$
$$0.952\,\dot{x}_D - 0.023(0.305)(1.807) - 0.305\,\dot{y}_D - 0.071(0.952)(1.807) = 0.$$

The solution is

$$\dot{x}_D = 0.129\,\text{m/s}, \quad \dot{y}_D = -0.041\,\text{m/s}.$$

The magnitude of the velocity of joint D is

$$v_D = |\mathbf{v}_D| = \sqrt{\dot{x}_D^2 + \dot{y}_D^2} = \sqrt{0.129^2 + (-0.041)^2} = 0.135\,\text{m/s}.$$

The acceleration analysis is obtained using the derivative of the velocity given by Eq. (4.87):

$$\dot{x}_D^2 + x_D\ddot{x}_D + \dot{y}_D^2 + y_D\ddot{y}_D = 0,$$
$$\ddot{x}_D \sin\phi_3 + 2\dot{x}_D\dot{\phi}_3 \cos\phi_3 - x_D\dot{\phi}_3^2 \sin\phi_3 + x_D\ddot{\phi}_3 \cos\phi_3 -$$
$$\ddot{y}_D \cos\phi_3 + 2\dot{y}_D\dot{\phi}_3 \sin\phi_3 + y_D\dot{\phi}_3^2 \cos\phi_3 + y_D\ddot{\phi}_3^2 \sin\phi_3 = 0,$$

or

$$0.129^2 + (-0.022)\ddot{x}_D + (-0.041)^2 + (-0.071)\ddot{y}_D = 0,$$
$$\ddot{x}_D \sin 72.235^\circ + 2(0.129)(1.807) \cos 72.235^\circ - (-0.022)(1.807)^2 \sin 72.235^\circ +$$
$$(-0.022)(1.020) \cos 72.235^\circ -$$
$$\ddot{y}_D \cos 72.235^\circ + 2(-0.041)(1.807) \sin 72.235^\circ + (-0.071)(1.807)^2 \cos 72.235^\circ +$$
$$(-0.071)(1.020)^2 \sin 72.235^\circ = 0.$$

The solution of the previous system is

$$\ddot{x}_D = 0.147 \quad \text{m/s}^2, \quad \ddot{y}_D = 0.210 \quad \text{m/s}^2.$$

The absolute acceleration of joint D is

$$a_D = |\mathbf{a}_D| = \sqrt{\ddot{x}_D^2 + \ddot{y}_D^2} = \sqrt{(0.150)^2 + (0.212)^2} = 0.256 \quad \text{m/s}^2.$$

The position of joint E is determined from the following equation:

$$[x_E(t) - x_D(t)]^2 + [y_E(t) - y_D(t)]^2 = DE^2,$$

and with the coordinate $y_E = 0$:

$$[x_E(t) - x_D(t)]^2 + y_D^2(t) = DE^2. \tag{4.88}$$

With the given numerical values Eq. (4.88) becomes

$$(x_E + 0.023)^2 + (0.071)^2 = 0.2^2,$$

with the correct solution $x_E = 0.164$ m.
The velocity of joint E is determined by differentiating Eq. (4.88) as follows

$$2(\dot{x}_E - \dot{x}_D)(x_E - x_D) + 2y_D\dot{y}_D = 0, \tag{4.89}$$

or

$$\dot{x}_E - \dot{x}_D = -\frac{y_D \dot{y}_D}{x_E - x_D}.$$

The solution of the above equation is

$$\dot{x}_E = 0.129 - \frac{(-0.071)(-0.041)}{0.164 + 0.023} = 0.113\,\text{m/s}.$$

The derivative of Eq. (4.89) yields

$$(\ddot{x}_E - \ddot{x}_D)(x_E - x_D) + (\dot{x}_E - \dot{x}_D)^2 + \dot{y}_D^2 + y_D \ddot{y}_D = 0,$$

with the solution

$$\ddot{x}_E = \ddot{x}_D - \frac{\dot{y}_D^2 + y_D \ddot{y}_D + (\dot{x}_E - \dot{x}_D)^2}{x_E - x_D},$$

or with numerical values

$$\ddot{x}_E = 0.150 - \frac{(-0.041)^2 + (-0.071)(0.21) + (0.112 - 0.129)^2}{0.164 + 0.023} = 0.217 \quad \text{m/s}^2.$$

The angle ϕ_4 is determined from the following equation:

$$y_E - y_D(t) - [x_E(t) - x_D(t)]\ \tan\phi_4(t) = 0, \tag{4.90}$$

where $y_E = 0$. The above equation can be rewritten as

$$-y_D(t) - [x_E(t) - x_D(t)]\ \tan\phi_4(t) = 0, \tag{4.91}$$

and the solution is

$$\phi_4 = \arctan\left(\frac{-y_D}{x_E - x_D}\right) = \arctan\left(\frac{0.071}{0.164 + 0.023}\right) = 20.923°.$$

The derivative of Eq. (4.91) yields

$$-\dot{y}_D - (\dot{x}_E - \dot{x}_D)\ \tan\phi_4 - (x_E - x_D)\frac{1}{\cos^2\phi_4}\dot{\phi}_4 = 0. \tag{4.92}$$

Hence,

$$\begin{aligned}
\omega_4 &= \dot{\phi}_4 = -\frac{\cos^2\phi_4[\dot{y}_D + (\dot{x}_E - \dot{x}_D)\tan\phi_4]}{x_E - x_D} \\
&= -\frac{\cos^2 20.923°[-0.041 + (0.113 - 0.129)\tan 20.923°]}{0.164 - (-0.022)} \\
&= 0.221 \quad \text{rad/s.}
\end{aligned}$$

The angular acceleration of link 4 is determined by differentiating Eq. (4.92) as follows:

$$-\ddot{y}_D - (\ddot{x}_E - \ddot{x}_D)\tan\phi_4 - 2(\dot{x}_E - \dot{x}_D)\frac{1}{\cos^2\phi_4}\dot{\phi}_4 -$$
$$2(x_E - x_D)\frac{\sin\phi_4}{\cos^3\phi_4}\dot{\phi}_4^2 - (x_E - x_D)\frac{1}{\cos^2\phi_4}\ddot{\phi}_4 = 0,$$

or

$$-0.210 - (0.217 - 0.147)\tan 20.923^\circ - 2(0.113 - 0.129)\frac{1}{\cos^2 20.923^\circ}0.221 -$$
$$2(0.164 + 0.022)\frac{\sin 20.923^\circ}{\cos^3 20.923^\circ}0.221^2 - (0.164 + 0.022)\frac{1}{\cos^2 20.923^\circ}\ddot{\phi}_4 = 0,$$

The solution of the previous equation is

$$\alpha_4 = \ddot{\phi}_4 = -1.105 \quad \text{rad/s}^2.$$

4.8 Problems

4.1 The four-bar mechanism shown in Fig. 3.16 has the dimensions: $AB = CD = 0.04$ m and $AD = BC = 0.09$ m. The driver link AB rotates with a constant angular speed of 120 rpm. Find the velocities and the accelerations of the four-bar mechanism for the case when the angle of the driver link AB with the horizontal axis is $\phi = 30°$.

4.2 The constant angular speed of the driver link 1 of the mechanism shown in Fig. 4.9, is $\omega = \omega_1 = 10$ rad/s. The distance from link 3 to the horizontal axis Ax is $a = 55$ mm. Find the velocity and the acceleration of point C on link 3 for $\phi = 45°$.

4.3 The slider crank mechanism shown in Fig. 4.10 has the dimensions: $AB = 0.1$ m and $BC = 0.2$ m. The driver link 1 rotates with a constant angular speed of $n = 60$ rpm. Find the velocity and acceleration of the slider 3 when the angle of the driver link with the horizontal axis is $\phi = 45°$.

4.4 The planar mechanism considered is shown in Fig. 3.19. The following data are given: AB=0.150 m, BC=0.400 m, CD=0.370 m, CE=0.230 m, EF=CE, L_a=0.300 m, L_b=0.450 m, and L_c=CD. The constant angular speed of the driver link 1 is 60 rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30°$.

4.5 The R-RRR-RTT mechanism is shown in Fig. 3.20. The following data are given: AB=0.080 m, BC=0.350 m, CE=0.200 m, CD=0.150 m, L_a=0.200 m, L_b=0.350 m, and L_c=0.040 m. The driver link 1 rotates with a constant angular speed of $n = 300$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link with the horizontal axis is $\phi = 155°$.

4.6 The mechanism shown in Fig. 3.21 has the following dimensions: $AB = 60$ mm, $AD = 200$ mm, $BC = 140$ mm, $CE = 50$ mm, $EF = 170$ mm, and $a = 130$ mm. The constant angular speed of the driver link 1 is $n = 300$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 30°$.

4.7 The dimensions for the mechanism shown in Fig. 3.22 are: $AB = 120$ mm, $BD = 320$ mm, $BC = 110$ mm, $CD = 300$ mm, $DE =$

200 mm, $CF = 400$ mm, $AE = 320$ mm, and $b = 80$ mm. The constant angular speed of the driver link 1 is $n = 30$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30°$.

4.8 The mechanism in Fig. 3.23 has the dimensions: $AB = 50$ mm, $AC = 25$ mm, $BD = 100$ mm, $DE = 140$ mm, $EF = 80$ mm, $L_a = 130$ mm, and $L_b = 30$ mm. Find the velocities and the accelerations of the mechanism if the constant angular speed of the driver link 1 is $n = 100$ rpm and for $\phi = \phi_1 = 150°$.

4.9 The dimensions for the mechanism shown in Fig. 3.24 are: $AB = 180$ mm, $BC = 470$ mm, $AD = 430$ mm, $CD = 270$ mm, $DE = 180$ mm, $EF = 400$ mm, and $L_a = 70$ mm. The constant angular speed of the driver link 1 is $n = 220$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 45°$.

4.10 The mechanism in Fig. 3.25 has the dimensions: $AB = 200$ mm, $AC = 600$ mm, $BD = 1000$ mm, $L_a = 150$ mm, and $L_b = 250$ mm. The driver link 1 rotates with a constant angular speed of $n = 60$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120°$.

4.11 Figure 3.26 shows a mechanism with the following dimensions: $AB = 250$ mm, $BD = 900$ mm, and $L_a = 300$ mm. The constant angular speed of the driver link 1 is $n = 500$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = 240°$.

4.12 The mechanism in Fig. 3.27 has the dimensions: $AB = 150$ mm, $AC = 350$ mm, $BD = 530$ mm, $DE = 300$ mm, $EF = 200$ mm, $L_a = 55$ mm, and $L_b = 250$ mm. The constant angular speed of the driver link 1 is $n = 30$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120°$.

4.13 Figure 3.28 shows a mechanism with the following dimensions: $AB = 150$ mm, $BC = 550$ mm, $CD = DE = 220$ mm, $EF = 400$ mm, $L_a = 530$ mm, and $L_b = L_c = 180$ mm. Find the velocities and the accelerations of the mechanism if the constant angular speed of the driver link 1 is $n = 1000$ rpm and for $\phi = \phi_1 = 150°$.

4.14 Figure 3.29 shows a mechanism with the following dimensions: AB = 250 mm, BC = 1200 mm, CE = 400 mm, CD = 800 mm, EF = 700 mm, L_a = 650 mm, L_b = 1000 mm, and L_c = 1200 mm. The constant angular speed of the driver link 1 is n = 70 rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120°$.

4.15 Figure 3.30 shows a mechanism with the following dimensions: AB = 100 mm, BC = 270 mm, CF = 260 mm, CD = 90 mm, DE = 300 mm, L_a = 350 mm, L_b = 200 mm, and L_c = 120 mm. The constant angular speed of the driver link 1 is n = 100 rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = 60°$.

4.16 Figure 3.31 shows a mechanism with the following dimensions: AB = 40 mm, BC = 100 mm, AD = 50 mm, and BE = 110 mm. The constant angular speed of the driver link 1 is n = 250 rpm. Find the velocities and the accelerations of the mechanism if the angle of the driver link 1 with the horizontal axis is $\phi = 30°$.

4.17 The dimensions of the mechanism shown in Fig. 3.32 are: AB = 100 mm, BC = 200 mm, BE = 400 mm, CE = 600 mm, CD = 220 mm, EF = 800 mm, L_a = 250 mm, L_b = 150 mm, and L_c = 100 mm. The constant angular speed of the driver link 1 is n = 100 rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 150°$.

4.18 The dimensions of the mechanism shown in Fig. 3.33 are: AB = 200 mm, AC = 300 mm, CD = 500 mm, DE = 250 mm, and L_a = 400 mm. Find the positions of the joints and the angles of the links. The constant angular speed of the driver link 1 is n = 40 rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = 60°$.

4.19 The dimensions of the mechanism shown in Fig. 3.34 are: AB = 160 mm, AC = 90 mm, CD = 150 mm, and DE = 400 mm. The constant angular speed of the driver link 1 is n = 70 rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 45°$.

4.20 The dimensions of the mechanism shown in Fig. 3.35 are: AB = 150 mm, AC = 250 mm, and CD = 450 mm. For the distance b

select a suitable value. The constant angular speed of the driver link 1 is $n = 80$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30°$.

4.21 The dimensions of the mechanism shown in Fig. 3.36 are: $AB =$ 180 mm, $AC = 90$ mm, and $CD = 200$ mm. The constant angular speed of the driver link 1 is $n = 180$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 60°$.

4.22 The dimensions of the mechanism shown in Fig. 3.37 are: $AB =$ 180 mm, $AC = 500$ mm, $BD = L_a = 770$ mm, and $DE = 600$ mm. The constant angular speed of the driver link 1 is $n = n_1 = 700$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 =$ 45°.

4.23 The dimensions of the mechanism shown in Fig. 3.38 are: $AB =$ 220 mm, $AD = 600$ mm, and $BC = 250$ mm. The constant angular speed of the driver link 1 is $n = 700$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120°$. Select a suitable value for the distance a.

4.24 Referr to Example 3.1. The mechanism in Fig. 3.11(a) has the dimensions: $AB = 0.20$ m, $AD = 0.40$ m, $CD = 0.70$ m, $CE = 0.30$ m, and $y_E = 0.35$ m. The constant angular speed of the driver link 1 is $n = 1000$ rpm. Find the velocities and the accelerations of the mechanism for the given input angle $\phi = \phi_1$=60°.

4.25 Referr to Example 3.2. The mechanism in Fig. 3.12 has the dimensions: $AB = 0.02$ m, $BC = 0.03$ m, $CD = 0.06$ m, $AE = 0.05$ m, and $L_a =$ 0.02 m. The constant angular speed of the driver link 1 is $n = 600$ rpm. Find the velocities and the accelerations of the mechanism for the given input angle $\phi = \phi_1 = \pi/3$.

4.26 Referr to Example 3.3. The mechanism in Fig. 3.15 has the dimensions: $AC = 0.100$ m, $BC = 0.300$ m, $BD = 0.900$ m, and $L_a = 0.100$ m. The constant angular speed of the driver link 1 is $n = 100$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = 30°$.

References

[1] P. Antonescu, *Mechanisms*, Printech, Bucharest, 2003.

[2] P. Appell, *Traité de Mécanique Rationnelle*, Gautier-Villars, Paris, 1941.

[3] I.I. Artobolevski, *Mechanisms in Modern Engineering Design*, MIR, Moscow, 1977.

[4] M. Atanasiu, *Mecanica*, EDP, Bucharest, 1973.

[5] H. Baruh, *Analytical Dynamics*, WCB/McGraw-Hill, Boston, 1999.

[6] A. Bedford and W. Fowler, *Dynamics*, Addison Wesley, Menlo Park, 1999.

[7] A. Bedford and W. Fowler, *Statics*, Addison Wesley, Menlo Park, 1999.

[8] M.I. Buculei, *Mechanisms*, University of Craiova Press, Craiova, 1976.

[9] M.I. Buculei, D. Bagnaru, G. Nanu, D.B. Marghitu, *Analysis of Mechanisms with Bars*, Scrisul romanesc, Craiova, 1986.

[10] T. Demian et al., *Mechanisms - problems*, EDP, Bucharest, 1969.

[11] A.G. Erdman, and G.N. Sandor, *Mechanisms Design*, Prentice-Hall, Upper Saddle River, 1984.

[12] A. Ertas and J.C. Jones, *The Engineering Design Process*, John Wiley & Sons, New York, 1996.

[13] F. Freudenstein, "An Application of Boolean Algebra to the Motion of Epicyclic Drives," *Transaction of the ASME, Journal of Engineering for Industry*, pp.176-182, 1971.

[14] J.H. Ginsberg, *Advanced Engineering Dynamics*, Cambridge University Press, Cambridge, 1995.

[15] D.T. Greenwood, *Principles of Dynamics*, Prentice-Hall, Englewood Cliffs, 1998.

[16] A.S. Hall, Jr., A.R. Holowenko, and H.G. Laughlin, *Theory and problems of machine design*, McGraw-Hill, New York, 1961.

[17] R.C. Hibbeler, *Engineering Mechanics - Statics and Dynamics*, Prentice-Hall, Upper Saddle River, New Jersey, 1995.

[18] R.C. Juvinall and K.M. Marshek, *Fundamentals of Machine Component Design*, John Wiley & Sons, New York, 1983.

[19] T.R. Kane, *Analytical Elements of Mechanics*, Vol. 1, Academic Press, New York, 1959.

[20] T.R. Kane, *Analytical Elements of Mechanics*, Vol. 2, Academic Press, New York, 1961.

[21] T.R. Kane and D.A. Levinson, "The Use of Kane's Dynamical Equtions in Robotics", *MIT International Journal of Robotics Research*, No. 3, pp. 3-21, 1983.

[22] T.R. Kane, P.W. Likins, and D.A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, New York, 1983.

[23] T.R. Kane and D.A. Levinson, *Dynamics*, McGraw-Hill, New York, 1985.

[24] J.T. Kimbrell, *Kinematics Analysis and Synthesis*, McGraw-Hill, New York, 1991.

[25] R. Maeder, *Programming in Mathematica*, Addison–Wesley Publishing Company, Redwood City, California, 1990.

[26] N.H. Madsen, *Statics and Dynamics*, class notes, www.eng.auburn.edu/users/nmadsen/, 2004.

[27] N.I. Manolescu, F. Kovacs, and A. Oranescu, *The Theory of Mechanisms and Machines*, EDP, Bucharest, 1972.

[28] D.B. Marghitu, *Mechanical Engineer's Handbook*, Academic Press, San Diego, California, 2001.

[29] D.B. Marghitu and M.J. Crocker, *Analytical Elements of Mechanisms*, Cambridge University Press, Cambridge, 2001.

[30] D.B. Marghitu and E.D. Stoenescu, *Kinematics and Dynamics of Machines and Machine Design*, class notes, www.eng.auburn.edu/users/marghitu/, 2004.

[31] J.L. Meriam and L.G. Kraige, *Engineering Mechanics: Dynamics*, John Wiley & Sons, New York, 1997.

[32] D.J. McGill and W.W. King, *Engineering Mechanics: Statics and an Introduction to Dynamics*, PWS Publishing Company, Boston, 1995.

[33] R.L. Mott, *Machine elements in mechanical design*, Prentice Hall, Upper Saddle River, New Jersey, 1999.

[34] D.H. Myszka, *Machines and Mechanisms*, Prentice-Hall, Upper Saddle River, New Jersey, 1999.

[35] R.L. Norton, *Machine Design*, Prentice-Hall, Upper Saddle River, New Jersey, 1996.

[36] R.L. Norton, *Design of Machinery*, McGraw-Hill, New York, 1999.

[37] W.C. Orthwein, *Machine Component Design*, West Publishing Company, St. Paul, 1990.

[38] L.A. Pars, *A treatise on analytical dynamics*, Wiley, New York, 1965.

[39] R.M. Pehan, *Dynamics of Machinery*, McGraw-Hill, New York, 1967.

[40] I. Popescu, *Mechanisms*, University of Craiova Press, Craiova, 1990.

[41] I. Popescu and C. Ungureanu, *Structural Synthesis and Kinematics of Mechanisms with Bars*, Universitaria Press, Craiova, 2000.

[42] I. Popescu and D.B. Marghitu, "Dyad Classification for Mechanisms," *World Conference on Integrated Design and Process Technology*, Austin, Texas, December 3-5, 2003.

[43] I. Popescu, E.D. Stoenescu, and D.B. Marghitu, "Analysis of Spatial Kinematic Chains Using the System Groups," *8th International Congress on Sound and Vibration*, St. Petersburg, Russia, July 5-8, 2004.

[44] M. Radoi and E. Deciu, *Mecanica*, EDP, Bucharest, 1981.

[45] F. Reuleaux, *The Kinematics of Machinery*, Dover, New York, 1963.

[46] C.A. Rubin, *The Student Edition of Working Model*, Addison–Wesley Publishing Company, Reading, Massachusetts, 1995.

[47] I.H. Shames, *Engineering Mechanics - Statics and Dynamics*, Prentice-Hall, Upper Saddle River, New Jersey, 1997.

[48] J.E. Shigley and C.R. Mischke, *Mechanical Engineering Design*, McGraw-Hill, New York, 1989.

[49] J.E. Shigley and J.J. Uicker, *Theory of Machines and Mechanisms*, McGraw-Hill, New York, 1995.

[50] R.W. Soutas-Little and D.J. Inman, *Engineering Mechanics: Statics and Dynamics*, Prentice-Hall, Upper Saddle River, New Jersey, 1999.

[51] A. Stan and M. Grumarescu, *Mechanics Problems*, EDP, Bucharest, 1973.

[52] A. Stoenescu, A. Ripianu, and M. Atanasiu, *Theoretical Mechanics Problems*, EDP, Bucharest, 1965.

[53] A. Stoenescu and G. Silas, *Theoretical Mechanics*, ET, Bucharest, 1957.

[54] E.D. Stoenescu, *Dynamics and Synthesis of Kinematic Chains with Impact and Clearance*, Ph.D. Dissertation, Mechanical Engineering, Auburn University, 2005.

[55] L. W. Tsai, *Mechanism Design: Enumeration of Kinematic Structures According to Function*, CRC Press, Boca Raton, Florida, 2001.

[56] R. Voinea, D. Voiculescu, and V. Ceausu, *Mecanica*, EDP, Bucharest, 1983.

[57] K.J. Waldron and G.L. Kinzel, *Kinematics, Dynamics, and Design of Machinery*, John Wiley&Sons, New York, 1999.

[58] C.E. Wilson and J.P. Sadler, *Kinematics and Dynamics of Machinery*, Harper Collins College Publishers, 1991.

[59] C.W. Wilson, *Computer integrated macine design*, Prentice Hall, Inc., Upper Saddle River, New Jersey, 1997.

[60] S. Wolfram, *Mathematica*, Wolfram Media/Cambridge University Press, Cambridge, 1999.

[61] * * * ,*The theory of mechanisms and machines (Teoria mehanizmov i masin)*, Vassaia scola, Minsc, 1970.

[62] * * * ,*Working Model 2D, Users Manual*, Knowledge Revolution, San Mateo, California, 1996.

Figure captions

Figure 4.1. Mobile reference frame ($\mathbf{\imath}, \mathbf{\jmath}, \mathbf{k}$) that moves with the rigid body (RB).

Figure 4.2. Motion of a point A that moves relative to a rigid body (RB).

Figure 4.3. Velocity ($\mathbf{v}_{Arel}$) and acceleration ($\mathbf{a}_{Arel}$) of A relative to the rigid body.

Figure 4.4. Nonaccelerating, nonrotating reference frame with the origin at O_0, and a secondary nonrotating, earth-centered reference frame with the origin at O.

Figure 4.5. Nonrotating reference frame with the origin at the center of the earth O and a secondary earth-fixed reference frame with the origin at B.

Figure 4.6. Primary inertial reference frame with the origin at O and a secondary reference frame with the origin at B.

Figure 4.7. R-TRR mechanism for Example 4.1.

Figure 4.8. R-RTR-RRT mechanism for Example 4.2.

Figure 4.9. Mechanism for Problem 4.2.

Figure 4.10. Mechanism for Problem 4.3