2 GEARS

2.1 Introduction

Gears are toothed elements that transmit rotary motion from one shaft to another. Gears are generally rugged and durable and their power transmission efficiency is as high as 98 percent. Gears are usually more costly than chains and belts. American Gear Manufacturers Association, AGMA, has established standard tolerances for various degrees of gear manufacturing precision. *Spurs gears* are the simplest and most common type of gears. They are used to transfer motion between parallel shafts, and they have tooth that are parallel to the shaft axes.

2.2 Geometry and nomenclature

The basic requirement of gear-tooth geometry is the condition of angular velocity ratios that are exactly constant, i.e. the angular velocity ratio between a 30-tooth and a 90-tooth gear must be precisely 3 in every position. The action of a pair of gear teeth satisfying this criteria is named conjugate gear-tooth action.

Law of conjugate gear-tooth action

The common normal to the surfaces at the point of contact of two gears in rotation must always intersect the line of centers at the same point P*, called the pitch point.*

The law of conjugate gear-tooth action can be satisfied by various tooth shapes, but the one of current importance is the involute of the circle. An *involute* (of the circle) is the curve generated by any point on a taut thread as it unwinds from a circle, called the base circle (Fig. 2.1(a)). The involute can be defined also as the locus of a point on a taught string that is unwrapped from a cylinder. The circle that represents the cylinder is the *base circle.* Fig. 2.1(b) represents an involute generated from a base circle of radius r_b starting at the point A. The radius of curvature of the involute at any point I is given by

$$\rho = \sqrt{r^2 - r_b^2}, \tag{2.1}$$

where $r = OI$. The involute pressure angle at I is defined as the angle between the normal to the involute IB and the normal to OI, $\phi = \angle IOB$.

In any pair of mating gears, the smaller of the two is called the pinion and the larger one the gear. The term “gear” is used in a general sense to indicate either of the members and also in a specific sense to indicate the larger of the two. The angular velocity ratio between a pinion and a gear is

(Fig. 2.2)

$$i = \omega_p/\omega_g = -d_g/d_p, \tag{2.2}$$

where ω is the angular velocity and d is the *pitch diameter*, and the minus sign indicates that the two gears rotate in opposite directions. The *pitch circles* are the two circles, one for each gear, that remain tangent throughout the engagement cycle. The point of tangency is the pitch point. The diameter of the pitch circle is the pitch diameter. If the angular speed is expressed in rpm then the symbol n is preferred instead of ω. The diameter (without a qualifying adjective) of a gear always refers to its pitch diameter. If other diameters (base, root, outside, etc.) are intended, they are always specified. Similarly, d, without subscripts, refers to pitch diameter. The pitch diameters of a pinion and gear are distinguished by subscripts p and g (d_p and d_g, are their symbols, Fig. 2.2. The *center distance* is

$$c = (d_p + d_g)/2 = r_p + r_g, \tag{2.3}$$

where $r = d/2$ is the *pitch circle radius.*

In Fig. 2.3 line tt is the common tangent to the pitch circles at the pitch point and AB is the common normal to the surfaces at C the point of contact of two gears. point P, and the inclination of AB with the line tt is called the *pressure angle*, ϕ. The most commonly pressure angle used, with both

English and SI units is 20^0. In the United States 25^0 is also standard, and 14.5^0 was formerly an alternative standard value. Pressure angle affects the force that tends to separate mating gears.

The involute profiles are augmented outward beyond the pitch circle by a distance called the *addendum*, a, (Fig. 2.4). The outer circle is usually termed the *addendum circle*, $r_a = r + a$. Similarly, the tooth profiles are extended inward from the pitch circle a distance called the *dedendum*, b. The involute portion can extend inward only to the base circle. A fillet at the base of the tooth merges the profile into the dedendum circle. The fillet decreases the bending stress concentration. The *clearance* is the amount by which the dedendum in a given gear exceeds the addendum of its mating gear.

The *circular pitch* is designated as p, and measured in inches (English units) or millimeters (SI units). If N is the number of teeth in the gear (or pinion), then

$$p = \pi d/N, \quad p = \pi d_p/N_p, \quad p = \pi d_g/N_g. \tag{2.4}$$

More commonly used indices of gear-tooth size are *diametral pitch*, P_d (used only with English units), and *module*, m (used only with SI). Diametral pitch is defined as the number of teeth per inch of pitch diameter

$$P_d = N/d, \quad P_d = N_p/d_p, \quad P_d = N_g/d_g. \tag{2.5}$$

Module m, which is essentially the complementary of P_d, is defined as the pitch diameter in millimeters divided by the number of teeth (number of millimeters of pitch diameter per tooth)

$$m = d/N, \;\; m = d_p/N_p, \;\; m = d_g/N_g. \tag{2.6}$$

One can easily verify that

$$p\,P_d = \pi \;\; (p \text{ in inches; } P_d \text{ in teeth per inch})$$

$$p/m = \pi \;\; (p \;\; \text{in millimeters;} \, m \text{ in millimeters per tooth})$$

$$m = 25.4/P_d.$$

With English units the word “pitch”, without a qualifying adjective, denotes diametral pitch (a “12-pitch gear” refers to a gear with $P_d = 12$ teeth per inch of pitch diameter). With SI units “pitch” means circular pitch (a “gear of pitch = 3.14 mm” refers to a gear having a circular pitch of $p = 3.14$ mm).

Standard diametral pitches P_d (English units) in common use are

1 to 2 by increments of 0.25,

2 to 4 by increments of 0.5,

4 to 10 by increments of 1,

10 to 20 by increments of 2,

20 to 40 by increments of 4.

With SI units, commonly used standard values of module m are

0.2 to 1.0 by increments of 0.1,

1.0 to 4.0 by increments of 0.25,

4.0 to 5.0 by increments of 0.5.

Addendum, minimum dedendum, and clearance for standard full-depth involute teeth (pressure angle is 20^0) with English units in common use are

addendum $a = 1/P_d$,

minimum dedendum $b = 1.157/P_d$.

For stub involute teeth with the pressure angle equal to 20^0 the standard values are (English units)

addendum $a = 0.8/P_d$,

minimum dedendum $b = 1/P_d$.

For SI units the standard values for full-depth involute teeth with pressure angle of 20^0 are addendum $a = m$,

minimum dedendum $b = 1.25\,m$.

2.3 Interference and contact ratio

The contact of segments of tooth profiles which are not conjugate is called *interference*. The involute tooth form is only defined outside the base circle. In some cases, the dedendum will extend below the base circle, then the portion of tooth below the base circle will not be an involute and will interfere with the tip of the tooth on the mating gear, which is an involute. Interference will occur, preventing rotation of the mating gears, if either of the addendum circles extends beyond tangent points A and B, Fig. 2.5, which are called interference points. In Fig. 2.5 both addendum circles extend beyond the interference points.

The maximum possible addendum circle radius, of pinion or gear, without interference is

$$r_{a(max)} = \sqrt{r_b^2 + c^2 \sin^2 \phi}, \tag{2.7}$$

where $r_b = r \cos \phi$ is the base circle radius of pinion or gear. The base circle diameter is

$$d_b = d \cos \phi. \tag{2.8}$$

The average number of teeth in contact as the gears rotate together is the contact ratio CR, which is calculated from the following equation (for external gears)

$$CR = \frac{\sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c\sin\phi}{p_b}, \tag{2.9}$$

where r_{ap}, r_{ag} are addendum radii of the mating pinion and gear, and r_{bp}, r_{bg} are base circle radii of the mating pinion and gear The base pitch p_b is computed with

$$p_b = \pi d_b/N = p\cos\phi. \tag{2.10}$$

The base pitch is like the circular pitch except that it represents an arc of the base circle rather than an arc of the pitch circle. The acceptable values for contact ratio are $CR > 1.2$.

For internal gears the contact ratio is

$$CR = \frac{\sqrt{r_{ap}^2 - r_{bp}^2} - \sqrt{r_{ag}^2 - r_{bg}^2} + c\sin\phi}{p_b}, \tag{2.11}$$

Gears are commonly specified according to AGMA Class Number, a code which denotes important quality characteristics. Quality numbers denote tooth-elements tolerances. The higher the number, the tighter the tolerance. Gears are heat treated by case hardening, nitriding, precipitation hardening, or through hardening. In general, harder gears are stronger and last longer than soft ones.

Example

Two involute spur gears of module 5, with 19 and 28 teeth operate at a pressure angle of 20°. Determine whether there will be interference when standard full-depth teeth are used. Find the contact ratio.

Solution

A standard full-depth tooth has the addendum of $a = m = 5$ mm. The gears will mesh at their pitch circles, and the pitch circle radii of pinion and gear are

$r_p = m\,N_p/2 = 5\,(19)/2 = 47.5\,\text{mm}$, and

$r_g = m\,N_g/2 = 5\,(28)/2 = 70\,\text{mm}$.

The theoretical center distance is

$c = (d_p + d_g)/2 = r_p + r_g = 47.5 + 70 = 117.5$ mm.

The base circle radii of pinion and gear are

$r_{bp} = r_p \cos\phi = 47.5\,\cos 20° = 44.635$ mm, and

$r_{bg} = r_g \cos\phi = 70\,\cos 20° = 65.778$ mm.

The addendum circle radii of pinion and gear are

$r_{ap} = r_p + a = m(N_p + 2)/2 = 52.5$ mm, and

$r_{ag} = r_g + a = m(N_g + 2)/2 = 75$ mm.

The maximum possible addendum circle radii of pinion and gear, without interference, are

$r_{a(max)p} = \sqrt{r_{bp}^2 + c^2 \sin^2 \phi} = 60.061$ mm $> r_{ap} = 52.5$ mm, and

$r_{a(max)g} = \sqrt{r_{bg}^2 + c^2 \sin^2 \phi} = 77.083$ mm $> r_{ag} = 75$ mm.

Clearly, the use of standard teeth would not cause interference.

The contact ratio is

$$CR = \frac{\sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c \sin \phi}{\pi m \cos \cos \phi} = 1.590,$$

which should be a suitable value ($CR > 1.2$).

2.4 Linkage transformation

The half joint at the contact point of two gear in motion can be substituted for two full joints A and B and an extra link 3, between gears 1 and 2 (Fig. 2.6). The mechanism still has one DOF, and the two gears system (0, 1, and 2) is in fact a fourbar mechanism (0, 1, 2, and 3) in another disguise. The following relations can be written

$$O_1 O_2 = \frac{m}{2}\left(N_1 + N_2\right),$$

$$O_1 A = r_1 \cos \phi,$$

$$O_2B = r_2 \cos\phi,$$

$$AB = AP + PB = \frac{mN_1}{2}\sin\phi + \frac{mN_2}{2}\sin\phi = \frac{m}{2}(N_1 + N_2)\sin\phi.$$

Because m, N_1, N_2 and ϕ are constants, the links of the fourbar mechanism are constant too.

2.5 Ordinary gear trains

A gear train is any collection of two or more meshing gears. Figure 2.7(a) shows a simple gear train with three gears in series. The train ratio is computed with the relation

$$i_{13} = \frac{\omega_1}{\omega_3} = \frac{\omega_1}{\omega_2}\frac{\omega_2}{\omega_3} = \left(-\frac{N_2}{N_1}\right)\left(-\frac{N_3}{N_2}\right) = \frac{N_3}{N_1}. \tag{2.12}$$

Only the sign of the overall ratio is affected by the intermediate gear 2 which is called *idler*.

Figure 2.7(b) shows a compound gear train, without idler gears, with the train ratio

$$i_{14} = \frac{\omega_1}{\omega_2}\frac{\omega_{2'}}{\omega_3}\frac{\omega_{3'}}{\omega_4} = \left(-\frac{N_2}{N_1}\right)\left(-\frac{N_3}{N_{2'}}\right)\left(-\frac{N_4}{N_{3'}}\right) = -\frac{N_2N_3N_4}{N_1N_{2'}N_{3'}}. \tag{2.13}$$

2.6 Epicyclic gear trains

When at least one of the gear axes rotates relative to the frame in addition to the gear's own rotation about its own axes, the train is called a *planetary gear train* or *epicyclic gear train*. The term "epicyclic" comes from the fact that points on gears with moving axes of rotation describe epicyclic paths. When a generating circle (planet gear) rolls on the outside of another circle, called a directing circle (sun gear), each point on the generating circle describes an epicycloid, as shown in Fig. 2.8.

Generally, the more planet gears there are the greater is the torque capacity of the system. For better load balancing, new designs have two sun gears and up to 12 planetary assemblies in one casing.

In the case of simple and compound gears it is not difficult to visualize the motion of the gears and the determination of the speed ratio is relatively easy. In the case of epicyclic gear trains it is often difficult to visualize the motion of the gears. A systematic procedure, using the contour method is presented below. The contour method is applied to determine the distribution of velocities for several epicyclic gear trains.

The velocity equations for a simple closed kinematic chain are

$$\sum_{(i)} \boldsymbol{\omega}_{i,i-1} = \mathbf{0},$$
$$\sum_{(i)} \mathbf{AA}_i \times \boldsymbol{\omega}_{i,i-1} + \sum_{(i)} \mathbf{v}_{A_{i,i-1}} = \mathbf{0}, \tag{2.14}$$

where $\boldsymbol{\omega}_{i,i-1}$ is the relative angular velocity of the rigid body (i) with respect to the rigid body $(i-1)$, $\mathbf{AA}_i$ is the position vector of the kinematic pair, A_i, between the rigid body (i) and the rigid body $(i-1)$ with respect to a "fixed" reference frame, and $\mathbf{v}_{A_{i,i-1}}$ is the relative velocity of the link (i) with respect to the link $(i-1)$, permitted by the joint at A_i.

The proof of the contour equations is given in Appendix 1.

Example 1

The first epicyclic (planetary) gear train considered is depicted in Fig. 2.9 The system consists of a central gear 2 (sun gear) and another gear 3 in mesh with 2 (planet gear) at B. Gear 3 is carried by the arm 1 hinged at A, as shown. The ring gear 4 meshes with the planet gear 3 and pivots at A. The sun gear, and the ring gear are concentric.

There are four moving bodies 1, 2, 3, and 4, $(n = 4)$ connected by

- four full joints $(c_5 = 4)$: one hinge between the arm 1 and the planet gear

3 at C, one hinge between the frame 0 and the shaft of the sun gear 2 at A, one hinge between the frame 0 and the ring gear 4 at A, and one hinge between the frame 0 and the arm 1 at A;

• two half joints ($c_4 = 2$): one between the sun gear 2 and the planet gear 3, and one between the planet gear 3 and the ring gear 4. The system possesses two DOF

$$M = 3\,n - 2\,c_5 - c_4 = 3 \cdot 4 - 2 \cdot 4 - 2 = 2. \tag{2.15}$$

The sun gear has $N_2 = 40$-tooth external gear, the planet gear has $N_3 = 20$-tooth external gear, and the ring gear has $N_4 = 80$-tooth internal gear. If the arm and the sun gear rotate with input angular speeds $n_1 = 200$ rpm, and $n_2 = 100$ rpm, find the absolute output angular velocity of the ring gear.

Solution

The velocity analysis is carried out using the contour method. The system shown in Fig. 2.9 has a total of five elements ($p = 5$): the frame 0 and four moving links 1, 2, 3 and 4. There are six joints ($l = 6$), four full joints and two half joints. The number of independent loops is given by

$$n_c = l - p + 1 = 6 - 5 + 1 = 2.$$

This gear system has two independent contours. The graph of the kinematic chain is represented in Fig. 2.10.

The angular speeds of the arm and the sun gear expressed in radians per second are

$$\omega_1 = \omega_{10} = \frac{\pi n_1}{30} = \frac{20\pi}{3} \text{ rad/s},$$
$$\omega_2 = \omega_{20} = \frac{\pi n_2}{30} = \frac{10\pi}{3} \text{ rad/s}.$$

First contour

The first contour is formed by the elements 0, 1, 3, 2, and 0 (clockwise path). For the velocity analysis, the following vectorial equations can be written

$$\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{31} + \boldsymbol{\omega}_{23} + \boldsymbol{\omega}_{02} = \mathbf{0},$$
$$\mathbf{AC} \times \boldsymbol{\omega}_{31} + \mathbf{AB} \times \boldsymbol{\omega}_{23} = \mathbf{0}, \tag{2.16}$$

where the input angular velocities are

$$\boldsymbol{\omega}_{10} = [\omega_{10}, 0, 0] = [\omega_1, 0, 0] = \omega_1\,\boldsymbol{\imath} + 0\,\boldsymbol{\jmath} + 0\,\mathbf{k},$$
$$\boldsymbol{\omega}_{02} = [\omega_{02}, 0, 0] = [-\omega_2, 0, 0] = -\omega_2\,\boldsymbol{\imath} + 0\,\boldsymbol{\jmath} + 0\,\mathbf{k},$$

and the unknown angular velocities are

$$\boldsymbol{\omega}_{31} = [\omega_{31}, 0, 0]\,,$$
$$\boldsymbol{\omega}_{23} = [\omega_{23}, 0, 0]\,.$$

The sign of the relative angular velocities is selected positive, and then the numerical computation will give the true orientation of the vectors.

The vectors **AB**, **AC**, and **AD** are defined as follows

$$\mathbf{AB} = [x_B, y_B, 0]\,, \quad \mathbf{AC} = [x_C, y_C, 0]\,, \quad \mathbf{AD} = [x_D, y_D, 0]\,, \qquad (2.17)$$

where

$$y_B = r_2 = m\,N_2/2,$$

$$y_C = r_2 + r_3 = m\,(N_2 + N_3)/2,$$

$$y_D = r_2 + 2\,r_3 = m\,N_2/2 + m\,N_3.$$

The module of the gears is m. Equation (2.16) becomes

$$\omega_1\,\imath + \omega_{31}\,\imath + \omega_{23}\,\imath - \omega_2\,\imath = \mathbf{0},$$
$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_C & y_C & 0 \\ \omega_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_B & y_B & 0 \\ \omega_{23} & 0 & 0 \end{vmatrix} = \mathbf{0}. \qquad (2.18)$$

Equation (2.18) can be projected on a "fixed" reference frame $xOyz$

$$\omega_1 + \omega_{31} + \omega_{23} - \omega_2 \quad = 0,$$

$$y_C\,\omega_{31} + y_B\,\omega_{23} \quad = 0.$$

Equation (2.19) represents a system of two equations with two unknowns ω_{31}

and ω_{23}. Solving the algebraic equations, the following values are obtained

$$\omega_{31} = N_2\,(\omega_1 - \omega_2)/N_3 = 20\pi/3\ \text{rad/s},$$

$$\omega_{23} = -\omega_1 + \omega_2 - N_2\,(\omega_1 - \omega_2)/N_3 = -10\pi\ \text{rad/s}.$$

Second contour

The second closed contour contains the elements 0, 1, 3, 4, and 0 (Fig. 2.10). The contour velocity equations can be written as (counterclockwise path)

$$\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{31} + \boldsymbol{\omega}_{43} + \boldsymbol{\omega}_{04} = \mathbf{0},$$

$$\mathbf{AC} \times \boldsymbol{\omega}_{31} + \mathbf{AD} \times \boldsymbol{\omega}_{43} = \mathbf{0}, \qquad (2.19)$$

where the known angular velocities are $\boldsymbol{\omega}_{10}$, $\boldsymbol{\omega}_{31}$, and the unknown angular velocities are

$$\boldsymbol{\omega}_{43} = [\omega_{43}, 0, 0]\,,$$

$$\boldsymbol{\omega}_{04} = [\omega_{04}, 0, 0]\,.$$

Equation (2.19) can be written as

$$\omega_1\,\mathbf{\imath} + \omega_{31}\,\mathbf{\imath} + \omega_{43}\,\mathbf{\imath} + \omega_{04}\,\mathbf{\imath} = \mathbf{0},$$

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_C & y_C & 0 \\ \omega_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_D & y_D & 0 \\ \omega_{43} & 0 & 0 \end{vmatrix} = \mathbf{0}. \qquad (2.20)$$

From Eq. (2.20) the absolute angular velocity of the ring gear is

$$\omega_{40} = -\omega_{04} = \frac{2N_2\omega_1 + 2N_3\omega_1 - N_2\omega_2}{N_2 + 2N_3} = 25\pi/3\,\text{rad/s},$$

or n_4=250 rpm.

Example 2

The second planetary gear train considered is shown in Fig. 2.11. The system consists of an input sun gear 1 and a planet gear 2 in mesh with 1 at B. Gear 2 is carried by the arm S fixed on the shaft of the gear 3, as shown. The gear 3 meshes with the output gear 4 at F. The fixed ring gear 4 meshes with the planet gear 2 at D.

There are four moving gears (1, 2, 3, and 4) connected by

- four full joints ($c_5 = 4$): one at A, between the frame 0 and the sun gear 1, one at C, between the arm S and the planet gear 2, one at E, between the frame 0 and the gear 3, and one at G, between the frame 0 and the gear 3;
- three half joints ($c_4 = 3$): one at B, between the sun gear 1 and the planet gear 2, one at D, between the planet gear 2 and the ring gear, and one at F, between the gear 3 and the output gear 4. The module of the gears is $m =$ 5 mm.

The system possesses one DOF

$$M = 3\,n - 2\,c_5 - c_4 = 3 \cdot 4 - 2 \cdot 4 - 3 = 1. \tag{2.21}$$

The sun gear has $N_1 = 19$-tooth external gear, the planet gear has $N_2 = 28$-tooth external gear, and the ring gear has $N_5 = 75$-tooth internal gear. The gear 3 has $N_3 = 18$-tooth external gear, and the output gear has $N_4 = 36$-tooth external gear. The sun gear rotates with input angular speed $n_1 = 2970$ rpm ($\omega_1 = \omega_{10} = \pi n_1/30 = 311.018$ rad/s). Find the absolute output angular velocity of the gear 4, the velocities of the pitch points B and F, and the velocity of joint C.

Solution

The velocity analysis is carried out using the contour equation method. The system shown in Fig. 2.11 has five elements (0, 1, 2, 3, 4) and seven joints. The number of independent loops is given by

$$n_c = l - p + 1 = 7 - 5 + 1 = 3.$$

This gear system has three independent contours. The graph of the kinematic chain and the independent contours are represented in Fig. 2.12.

The position vectors **AB**, **AC**, **AD**, **AF**, and **AG** are defined as follow

$$\mathbf{AB} = [x_B, y_B, 0] = [x_B, r_1, 0] = [x_B, m\,N_1/2, 0]\,,$$

$$\mathbf{AC} = [x_C, y_C, 0] = [x_C, r_1 + r_2, 0] = [x_C, m\,(N_1 + N_2)/2, 0]\,,$$

$$\mathbf{AD} = [x_D, y_D, 0] = [x_D, r_1 + 2r_2, 0] = [x_D, m\,(N_1 + 2N_2)/2, 0]\,,$$

$$\mathbf{AF} = [x_F, y_F, 0] = [x_F, r_3, 0] = [x_F, m\,N_3/2, 0]\,,$$

$$\mathbf{AG} = [x_G, y_G, 0] = [x_C, r_3 + r_4, 0] = [x_G, m\,(N_3 + N_4)/2, 0]\,.$$

First contour

The first closed contour contains the elements 0, 1, 2, and 0 (clockwise path). For the velocity analysis, the following vectorial equations can be written

$$\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{02} = \mathbf{0},$$

$$\mathbf{AB} \times \boldsymbol{\omega}_{21} + \mathbf{AD} \times \boldsymbol{\omega}_{02} = \mathbf{0}, \tag{2.22}$$

where the input angular velocity is

$$\boldsymbol{\omega}_{10} = [\omega_{10}, 0, 0] = [\omega_1, 0, 0]\,,$$

and the unknown angular velocities are

$$\boldsymbol{\omega}_{21} = [\omega_{21}, 0, 0]\,,$$

$$\boldsymbol{\omega}_{02} = [\omega_{02}, 0, 0]\,.$$

The sign of the relative angular velocities is selected positive, and then the numerical results will give the real orientation of the vectors.

Equation (2.33) becomes

$$\omega_1 \, \imath + \omega_{21} \, \imath + \omega_{02} \, \imath = \mathbf{0},$$
$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_B & y_B & 0 \\ \omega_{21} & 0 & 0 \end{vmatrix} + \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_D & y_D & 0 \\ \omega_{02} & 0 & 0 \end{vmatrix} = \mathbf{0}. \tag{2.23}$$

Equation (2.34) projected on a "fixed" reference frame $xOyz$ is

$$\begin{aligned} \omega_1 + \omega_{21} + \omega_{02} &= 0, \\ y_B \, \omega_{21} + y_D \, \omega_{02} &= 0. \end{aligned} \tag{2.24}$$

Equation (2.35) represents a system of two equations with two unknowns ω_{21} and ω_{02}. Solving the algebraic equations, the following value is obtained the absolute angular velocity of planet gear 2

$$\omega_{20} = -\,\omega_{02} = -\,\frac{N_1 \, \omega_1}{2 \, N_2} = -105.524 \, \mathrm{rad/s}. \tag{2.25}$$

Second contour

The second closed contour contains the elements 0, 3, 2, and 0 (counterclockwise path). For the velocity analysis, the following vectorial equations can be written

$$\begin{aligned} \boldsymbol{\omega}_{30} + \boldsymbol{\omega}_{23} + \boldsymbol{\omega}_{02} &= \mathbf{0}, \\ \mathbf{AE} \times \boldsymbol{\omega}_{30} + \mathbf{AC} \times \boldsymbol{\omega}_{23} + \mathbf{AD} \times \boldsymbol{\omega}_{02} &= \mathbf{0}, \end{aligned} \tag{2.26}$$

The unknown angular velocities are

$$\boldsymbol{\omega}_{30} = [\omega_{21}, 0, 0]\,,$$

$$\boldsymbol{\omega}_{23} = [\omega_{23}, 0, 0]\,.$$

Solving Eq. (2.37) the following value is obtained for the absolute angular velocity of the gear 3 and the arm S

$$\omega_{30} = \frac{N_1\,\omega_1}{2\,(N_1 + N_2)} = 62.865\ \text{rad/s}. \tag{2.27}$$

Third contour

The third closed contour contains the links 0, 4, 3, and 0 (counterclockwise path). The velocity vectorial equations are

$$\boldsymbol{\omega}_{40} + \boldsymbol{\omega}_{34} + \boldsymbol{\omega}_{03} = \mathbf{0},$$

$$\mathbf{AG} \times \boldsymbol{\omega}_{40} + \mathbf{AF} \times \boldsymbol{\omega}_{34} + \mathbf{AE} \times \boldsymbol{\omega}_{03} = \mathbf{0}, \tag{2.28}$$

or

$$\omega_{40}\,\mathbf{\imath} + \omega_{34}\,\mathbf{\imath} - \omega_{30}\,\mathbf{\imath} = \mathbf{0},$$

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_G & y_G & 0 \\ \omega_{40} & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_F & y_F & 0 \\ \omega_{34} & 0 & 0 \end{vmatrix} = \mathbf{0}. \tag{2.29}$$

The unknown angular velocities are

$$\boldsymbol{\omega}_{40} = [\omega_{40}, 0, 0]\,,$$

$$\boldsymbol{\omega}_{34} = [\omega_{34}, 0, 0] .$$

The absolute angular velocity of the output gear 4 is

$$\omega_{40} = -\frac{N_1 N_3 \omega_1}{2 (N_1 + N_2) N_4} = -31.432 \,\text{rad/s}. \tag{2.30}$$

Linear velocities of pitch points

The velocity of the pitch point B is

$$v_B = \omega_{10} r_1 = 14.773 \,\text{m/s},$$

and the velocity of the pitch point F is

$$v_F = \omega_{40} r_4 = 2.828 \,\text{m/s}.$$

The velocity of the joint C is

$$v_C = \omega_{30}(r_1 + r_2) = 7.386 \,\text{m/s}.$$

Gear geometrical dimensions

For standard external gear teeth the addendum is $a = m$.

Gear 1

pitch circle diameter $d_1 = mN_1 = 95.0$ mm;

addendum circle diameter $d_{a1} = m(N_1 + 2) = 105.0$ mm;

dedendum circle diameter $d_{d1} = m(N_1 - 2.5) = 82.5$ mm.

Gear 2

pitch circle diameter $d_2 = mN_2 = 140.0$ mm;

addendum circle diameter $d_{a2} = m(N_2 + 2) = 150.0$ mm;

dedendum circle diameter $d_{d2} = m(N_2 - 2.5) = 127.5$ mm.

Gear 3

pitch circle diameter $d_3 = mN_3 = 90.0$ mm;

addendum circle diameter $d_{a3} = m(N_3 + 2) = 100.0$ mm;

dedendum circle diameter $d_{d3} = m(N_3 - 2.5) = 77.5$ mm.

Gear 4

pitch circle diameter $d_4 = mN_4 = 180.0$ mm;

addendum circle diameter $d_{a4} = m(N_4 + 2) = 190.0$ mm;

dedendum circle diameter $d_{d4} = m(N_4 - 2.5) = 167.5$ mm.

Gear 5 (internal gear)

pitch circle diameter $d_5 = mN_5 = 375.0$ mm;

addendum circle diameter $d_{a5} = m(N_5 - 2) = 365.0$ mm;

dedendum circle diameter $d_{d5} = m(N_5 + 2.5) = 387.5$ mm.

Number of planet gears

The number of necessary planet gears k is given by the assembly condition

$$(N_1 + N_5)/k = INTEGER,$$

and for the planetary gear train $k = 2$ planet gears. The vicinity condition between the sun gear and the planet gear

$$m\,(N_1 + N_2)\,\sin(\pi/k) > d_{a2}$$

is verified.

The group and assembly drawings for this mechanism with planetary gears are given in Appendix 2.

Example 3

The third planetary gear train considered is shown in Fig. 2.13. The system consists of an input sun gear 1 and a planet gear 2 in mesh with 1 at B. Gear $2'$ is fixed on shaft of the gear 2. The system of gears 2 and $2'$ is carried by the arm 3. The gear $2'$ mashes with the fixed frame 0 at E.

There are three moving gears (1, 2, and 3) connected by

- three full joints ($c_5 = 3$): one at A, between the frame 0 and the sun gear 1, one at C, between the arm 3 and the planet gear system 2, and one at D, between the frame 0 and the arm 3;
- two half joints ($c_4 = 2$): one at B, between the sun gear 1 and the planet gear 2, and one at E, between the planet gear $2'$ and the frame 0.

The system possesses one DOF

$$M = 3\,n - 2\,c_5 - c_4 = 3 \cdot 3 - 2 \cdot 3 - 2 = 1. \tag{2.31}$$

The sun gear has the radius of the pitch circle r_1, the planet gear 2 has the radius of the pitch circle r_2, the arm 3 has the length r_3, and the planet gear $2'$ has the radius of the pitch circle r_4 (Fig. 2.13).

The sun gear rotates with the input angular velocity ω_1. Find the speed ratio i_{13} between the sun gear 1 and the arm 3.

Solution

The system shown in Fig. 2.13 has four elements (0, 1, 2, 3) and five kinematic pairs. The number of independent loops is given by

$$n_c = l - p + 1 = 5 - 4 + 1 = 2.$$

This gear system has two independent contours. The graph of the kinematic chain and the independent contours are represented in Fig. 2.14.

The position vectors **AB**, **AC**, **AD**, and **AE** are defined as follow

$$\mathbf{AB} = [x_B, y_B, 0] = [x_B, r_1, 0]\,,$$

$$\mathbf{AC} = [x_C, y_C, 0] = [x_C, r_1 + r_2, 0]\,,$$

$$\mathbf{AD} = [x_D, y_D, 0] = [x_D, r_1 + r_2 - r_3, 0]\,,$$

$$\mathbf{AE} = [x_E, y_E, 0] = [x_E, r_1 + r_2 - r_4, 0]\,. \tag{2.32}$$

First contour

The first closed contour contains the elements 0, 1, 2, and 0 (clockwise path). For the velocity analysis, the following vectorial equations can be written

$$\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{02} = \mathbf{0},$$
$$\mathbf{AB} \times \boldsymbol{\omega}_{21} + \mathbf{AE} \times \boldsymbol{\omega}_{02} = \mathbf{0}, \qquad (2.33)$$

where the input angular velocity is

$$\boldsymbol{\omega}_{10} = [\omega_{10}, 0, 0] = [\omega_1, 0, 0],$$

and the unknown angular velocities are

$$\boldsymbol{\omega}_{21} = [\omega_{21}, 0, 0],$$
$$\boldsymbol{\omega}_{02} = [\omega_{02}, 0, 0].$$

Equation (2.33) becomes

$$\omega_1\, \mathbf{\imath} + \omega_{21}\, \mathbf{\imath} + \omega_{02}\, \mathbf{\imath} = \mathbf{0},$$
$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_B & y_B & 0 \\ \omega_{21} & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_E & y_E & 0 \\ \omega_{02} & 0 & 0 \end{vmatrix} = \mathbf{0}. \qquad (2.34)$$

Equation (2.34) projected on a "fixed" reference frame $xOyz$ is

$$\omega_1 + \omega_{21} + \omega_{02} \quad = 0,$$

$$y_B\,\omega_{21} + y_E\,\omega_{02} \quad = 0. \tag{2.35}$$

Equation (2.35) represents a system of two equations with two unknowns ω_{21} and ω_{02}. Solving the algebraic equations, the following value is obtained the absolute angular velocity of planet gear 2

$$\omega_{20} = -\,\omega_{02} = -\frac{r_1\,\omega_1}{r_2 - r_4}. \tag{2.36}$$

Second contour

The second closed contour contains the elements 0, 3, 2, and 0 (counter-clockwise path). For the velocity analysis, the following vectorial equations can be written

$$\boldsymbol{\omega}_{30} + \boldsymbol{\omega}_{23} + \boldsymbol{\omega}_{02} = \mathbf{0},$$

$$\mathbf{AD} \times \boldsymbol{\omega}_{30} + \mathbf{AC} \times \boldsymbol{\omega}_{23} + \mathbf{AE} \times \boldsymbol{\omega}_{02} = \mathbf{0}, \tag{2.37}$$

The unknown angular velocities are

$$\boldsymbol{\omega}_{30} = [\omega_{21}, 0, 0]\,,$$

$$\boldsymbol{\omega}_{23} = [\omega_{23}, 0, 0]\,.$$

Solving Eq. (2.37) the following value is obtained for the absolute angular velocity of the arm 3

$$\omega_{30} = \frac{r_1\,r_4\,\omega_1}{r_3\,(-r_2 + r_4)}. \tag{2.38}$$

The speed ratio is

$$i_{13} = \frac{\omega_{10}}{\omega_{30}} = \frac{\omega_1}{\omega_{30}} = \frac{r_3\,(-r_2 + r_4)}{r_1\,r_4}. \tag{2.39}$$

2.7 Differential

Figure 2.15 is a schematic drawing of the ordinary bevel-gear automotive differential. The drive shaft pinion 1 and the ring gear 2 are normally hypoid gears. The ring gear 2 acts as the planet carrier for the planet gear 3, and its speed can be calculated as for a simple gear train when the speed of the drive shaft is given. Sun gears 4 and 5 are connected, respectively, to each rear wheel.

When the car is traveling in a straight line, the two sun gears rotate in the same direction with exactly the same speed. Thus for straight-line motion of the car, there is no relative motion between the planet gear 3 and ring 2. The planet gear 3, in effect, serves only as keys to transmit motion from the planet carrier to both wheels.

When the vehicle is making a turn, the wheel on the inside of the turn makes fewer revolutions than the wheel with a larger turning radius. Unless this difference in speed is accommodated in some manner, one or both of the tires would have to slide in order to make the turn. The differential permits the two wheels to rotate at different velocities while at the same time delivering power to both. During a turn, the planet gear 3 rotate about their own axes, thus permitting gears 4 and 5 to revolve at different velocities. The purpose of a differential is to differentiate between the speeds of the two wheels. In the usual passenger-car differential, the torque is divided equally whether the car is traveling in a straight line or on a curve. Sometimes the road conditions are such that the tractive effort developed by the two wheels is unequal. In this case the total tractive effort available will be only twice that at the wheel having the least traction, because the differential divides the torque equally. If one wheel should happen to be resting on snow or ice, the total effort available is very small and only a small torque will be required to cause the wheel to spin. Thus the car sits there with one wheel spinning and the other at rest with no tractive effort. And, if the car is in motion and encounters slippery surfaces, then all traction as well as control is lost.

It is possible to overcome the disadvantages of the simple bevel-gear differential by adding a coupling unit which is sensitive to wheel speeds. The object of such a unit is to cause most of the torque to be directed to the slow-moving wheel. Such a combination is then called a limited-slip differential.

Angular velocities diagram

The velocity analysis is carried out using the contour equation method and the graphical angular velocities diagram.

There are five moving elements (1, 2, 3, 4, and 5) connected by

• five full joints ($c_5 = 5$): one between the frame 0 and the drive shaft pinion gear 1, one between the frame 0 and the ring gear 2, one between the planet carrier arm 2 and the planet gear 3, one between the frame 0 and the sun gear 4, and one between the frame 0 and the sun gear 5;

• three half joints ($c_4 = 3$): one between the drive shaft pinion gear 1 and the ring gear 2, one between the planet gear 3 and the sun gear 4, and one between the planet gear 3 and the sun gear 5.

The system possesses two DOF

$$M = 3\,n - 2\,c_5 - c_4 = 3 \cdot 5 - 2 \cdot 5 - 3 = 2.$$

The input data are the absolute angular velocities of the two wheels ω_{40} and ω_{50}.

The system shown in Fig. 2.15(a) has six elements (0, 1, 2, 3, 4, and 5) and eight joints ($c_4 + c_5$). The number of independent loops is given by

$$n_c = 8 - p + 1 = 8 - 6 + 1 = 3.$$

This gear system has three independent contours. The graph of the kinematic chain and the independent contours are represented in Fig. 2.15(b)

The first closed contour contains the elements 0, 4, 3, 5 and 0 (clockwise path). For the velocity analysis, the following vectorial equations can be written

$$\boldsymbol{\omega}_{40} + \boldsymbol{\omega}_{34} + \boldsymbol{\omega}_{53} + \boldsymbol{\omega}_{05} = \mathbf{0},$$

or

$$\boldsymbol{\omega}_{40} + \boldsymbol{\omega}_{34} = \boldsymbol{\omega}_{50} + \boldsymbol{\omega}_{35}. \tag{2.40}$$

The unknown angular velocities are $\boldsymbol{\omega}_{34}$ and $\boldsymbol{\omega}_{35}$. The relative angular velocity of the planet gear 3 with respect to the sun gear 4 is parallel to Ia line and the relative angular velocity of the planet gear 3 with respect to the sun gear 5 is parallel to Ib. Equation (2.40) can be solved graphically (Fig. 2.16). The vectors OA and OB represent the velocities $\boldsymbol{\omega}_{50}$ and $\boldsymbol{\omega}_{40}$. At A and B two parallels at Ib and Ia are drawn. The intersection between the two lines is the point C. The vector BC represents the relative angular velocity of the planet gear 3 with respect to the sun gear 4, and the vector AC represents the relative angular velocity of the planet gear 3 with respect to the sun gear 5.

The absolute angular velocity of planet gear 3 is

$$\boldsymbol{\omega}_{30} = \boldsymbol{\omega}_{40} + \boldsymbol{\omega}_{34}.$$

The vector $0C$ represents the absolute angular velocity of planet gear.

The second closed contour contains the elements 0, 4, 3, 2 and 0 (counter-clockwise path). For the velocity analysis, the following vectorial equations can be written

$$\boldsymbol{\omega}_{40} + \boldsymbol{\omega}_{34} + \boldsymbol{\omega}_{23} + \boldsymbol{\omega}_{02} = \mathbf{0}. \tag{2.41}$$

Using the velocities diagram (Fig. 2.16) the vector DC represents the relative angular velocity of the planet gear 3 with respect to the ring gear 2, $\boldsymbol{\omega}_{23}$, and the OD represents the absolute angular velocity of the ring gear 2, $\boldsymbol{\omega}_{20}$.

From Fig. 2.16 one can write

$$\begin{aligned} \omega_{20} &= |OD| = \frac{1}{2}(\omega_{40} + \omega_{50}), \\ \omega_{32} &= |DC| = \frac{1}{2}(\omega_{50} - \omega_{40}) \tan \alpha. \end{aligned} \tag{2.42}$$

When the car is traveling in a straight line, the two sun gears rotate in the same direction with exactly the same speed, $\omega_{50} = \omega_{40}$, and there is no relative motion between the planet gear and the ring gear, $\omega_{32} = 0$. When the wheels are jacked up $\omega_{50} = -\omega_{40}$ and the absolute angular velocity of the ring gear 2 is zero.

2.8 Gear force analysis

The force between mating teeth (neglecting the sliding friction) can be resolved at the pith point (P in Fig. 2.17) into two components

- tangential component F_t, which accounts for the power transmitted;
- radial component F_r, which does no work but tends to push the gears apart.

The relationship between these components is

$$F_r = F_t \tan\phi, \tag{2.43}$$

where ϕ is the pressure angle.

The pitch line velocity in feet per minute is equal to

$$V = \pi\, d\, n/12 \ \text{(ft/min)}, \tag{2.44}$$

where d is the pitch diameter in inches of the gear rotating n rpm.

In SI units

$$V = \pi\, d\, n/60,000 \ \text{(m/s)}, \tag{2.45}$$

where d is the pitch diameter in millimeters of the gear rotating n rpm.

The transmitted power in horsepower is

$$H = F_t\, V/33,000 \ \text{(hp)}, \tag{2.46}$$

where F_t is in pounds and V is in feet per minute.

In SI units the transmitted power in watts is

$$H = F_t V \ \ \text{(W)}, \tag{2.47}$$

where F_t is in newtons and V is in meters per second.

The transmitted torque can be express as

$$M_t = 63,000\, H/n \quad \text{(lb in)}, \tag{2.48}$$

where H is in horsepower and n in rpm.

In SI units,

$$M_t = 9549\, H/n \quad \text{(N m)}\ , \tag{2.49}$$

where the power H is in kW and n in rpm.

Example: Forces in ordinary gear trains

Figure 2.18 shows a two stage gear reducer with identical pairs of gears. An electric motor with the power $H = 2$ kW and $n_1 = 900$ rpm is coupled to the shaft a. On this shaft there is rigidly connected the input driver gear 1 with the number of teeth $N_1 = N_p = 17$. The speed reducer uses a countershaft b with two rigidly connected gears 2 and $2'$, having $N_2 = N_g =$ 51 teeth and $N_{2'} = N_p = 17$ teeth. The output gear 3 has $N_3 = N_g = 51$

teeth and is rigidly fixed to the shaft c coupled to the driven machine. The input shaft a and output shaft c are collinear (two identical pairs in each stage), and facilitate machining of the housing. The countershaft b turns freely in bearings A and B. The gears mesh along the pitch diameter and the shafts are parallel. The diametral pitch for each stage is $P_d = 5$, and the pressure angle is $\phi = 20^0$. The distance between the bearings is $s = 100$ mm, and the distance $l = 25$ mm (Fig. 2.18).

Geometry

The pitch diameters of pinions 1 and $2'$ are $d_1 = d_{2'} = d_p = N_p/P_d = 17/5 = 3.4$ in. The pitch diameters of gears 2 and 3 is $d_2 = d_3 = d_g = N_g/P_d = 51/5 = 10.2$ in. The circular pitch is $p = \pi/P_d = 3.14/5 = 0.63$ in.

Angular speeds

The following relation exists for the first stage

$$\frac{n_1}{n_2} = \frac{N_2}{N_1} \quad \Rightarrow \quad n_2 = n_1 \frac{N_1}{N_2} = 900 \frac{17}{51} = 300 \text{ rpm}, \tag{2.50}$$

and for the second stage

$$\frac{n_2}{n_3} = \frac{N_3}{N_{2'}} \quad \Rightarrow \quad n_3 = n_2 \frac{N_{2'}}{N_3} = 300 \frac{17}{51} = 100 \text{ rpm}. \tag{2.51}$$

The angular speed of the countershaft b is $n_b = n_2 = 300$ rpm, and the angular speed of the driven shaft c is $n_c = n_3 = 100$ rpm.

Torque carried by each of the shafts assuming 100% gear efficiency

The relation between the power H_a of the motor and the torque M_a in shaft a is

$$H_a = \frac{M_a\, n_a}{9549}, \tag{2.52}$$

and the torque M_a in shaft a is

$$M_a = \frac{9549\, H_a}{n_a} = \frac{9549\,(2 \text{ kW})}{900 \text{ rpm}} = 21.22 \text{ N m}$$

The torque in shaft b

$$M_b = \frac{9549\, H_a}{n_b} = M_a \frac{N_2}{N_1} = 21.22\, \frac{51}{17} = 63.66 \text{ N m},$$

and the torque in shaft c is

$$M_c = \frac{9549\, H_a}{n_c} = M_b \frac{N_3}{N_{2'}} = 63.66\, \frac{51}{17} = 190.98 \text{ N m}.$$

Torque carried by each of the shafts assuming η=95% efficiency of each gear pair

In this case the torque in shaft b is

$$M_{b'} = \frac{9549\, H_a\, \eta}{n_b} = M_b\, \eta = (63.66)\,(0.95) = 60.47 \text{ N m},$$

and the torque in shaft c is

$$M_{c'} = M_{b'}\, \eta \frac{N_3}{N_{2'}} = (63.66)\,(0.95) \frac{51}{17} = 172.36 \text{ N m}.$$

The effect of power losses in each stage η=95% was to decrease the torque transmitted to the output while keeping the speed ratios the same. Each stage reduces the torque transmitted by a factor.

Loads applied to bearings A and B for η=100% gear efficiency

All the gear radial and tangential load is transferred at the pitch point P. The tangential force on the motor pinion is

$$F_t = \frac{M_a}{r_p} = \frac{21.22}{0.0431} = 492.34 \text{ N},$$

where $r_p = d_p/2 = 1.7$ in $= 0.0431$ m. The radial force on the motor pinion is

$$F_r = F_t \tan\phi = 492.34 \tan 20^0 = 179.2 \text{ N}.$$

The force on the motor pinion 1 at P (Fig. 2.19) is

$$\mathbf{F}_{21} = F_{r21}\mathbf{J} + F_{t21}\mathbf{k} = 179.2\mathbf{J} - 492.34\mathbf{k} \text{ N}. \tag{2.53}$$

The force on the countershaft gear 2 at P is

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = F_{r12}\mathbf{J} + F_{t12}\mathbf{k} = -179.2\mathbf{J} + 492.34\mathbf{k} \text{ N}. \tag{2.54}$$

The forces on the countershaft pinion $2'$ at R are three times as large i.e.

$$F_{t'} = \frac{M_b}{r_p} = \frac{63.66}{0.0431} = 1477 \text{ N},$$

$$F_{r'} = F_{t'} \tan\phi = 1477 \tan 20^0 = 537.6 \text{ N},$$

and

$$\mathbf{F}_{32'} = F_{r32'}\mathbf{\jmath} + F_{t32'}\mathbf{k} = -537.6\mathbf{\jmath} - 1477\mathbf{k} \text{ N}. \quad (2.55)$$

The unknown loads applied to bearings A and B can be written as

$$\mathbf{F}_A = F_{Ay}\mathbf{\jmath} + F_{Az}\mathbf{k},$$

$$\mathbf{F}_B = F_{By}\mathbf{\jmath} + F_{Bz}\mathbf{k}.$$

The sketch of the countershaft as a free body in equilibrium is shown in Fig. 2.19. To determine these forces two vectorial equations are used. Sum of moments of all forces that act on the countershaft with respect to A are zero

$$\sum \mathbf{M}_A = \mathbf{AP} \times \mathbf{F}_{12} + \mathbf{AR} \times \mathbf{F}_{32'} + \mathbf{AB} \times \mathbf{F}_B =$$
$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ -l & r_2 & 0 \\ 0 & F_{r12} & F_{t12} \end{vmatrix} + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ s+l & r_{2'} & 0 \\ 0 & F_{r32'} & F_{t32'} \end{vmatrix} +$$
$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ s & 0 & 0 \\ 0 & F_{By} & F_{Bz} \end{vmatrix} = \mathbf{0}, \quad (2.56)$$

or

$$\sum \mathbf{M}_A \cdot \mathbf{\jmath} = lF_{t12} - (s+l)F_{t32'} - sF_{Bz} = 0,$$

$$\sum \mathbf{M}_A \cdot \mathbf{k} = -lF_{r12} + (s+l)F_{r32'} + sF_{By} = 0. \qquad (2.57)$$

From the above equations $F_{By} = 627.2$ N, and $F_{Bz} = 1969.33$ N. The radial load at B is

$$F_B = \sqrt{{F_{By}}^2 + {F_{Bz}}^2} = 2066.8 \text{ N}.$$

Sum of all forces that act on the countershaft are zero

$$\sum \mathbf{F} = \mathbf{F}_{12} + \mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_{32'} = \mathbf{0}, \qquad (2.58)$$

or

$$-F_{r12} + F_{Ay} + F_{By} - F_{r32'} = 0,$$

$$F_{t12} + F_{Az} + F_{Bz} - F_{t32'} = 0. \qquad (2.59)$$

From Eq. (2.59) $F_{Ay} = 89.6$ N, and $F_{Az} = -984.67$ N. The radial load at A is

$$F_A = \sqrt{{F_{Ay}}^2 + {F_{Az}}^2} = 988.73 \text{ N}.$$

Example: Joint reactions for planetary gear trains

The planetary gear train considered is shown in Fig. 2.20. The sun gear has $N_1 = 19$-tooth external gear, the planet gear has $N_2 = N_{2'} = 28$-tooth external gear, and the ring gear has $N_5 = 75$-tooth internal gear. The gear 3 has $N_3 = 18$-tooth external gear, and the output gear has $N_4 = 36$-tooth external gear. The module of the gears is $m = 5$ mm, and the pressure angle is $\phi = 20^0$. The resistant or technological torque is $\mathbf{M}_4 = M_4\,\mathbf{\imath}$, where $M_4 = 500$ Nm, and is opposed to the angular velocity of the output gear, $\boldsymbol{\omega}_{40} = \omega_4\,\mathbf{\imath}$, $\omega_4 < 0$ (Fig. 2.21). The joints are frictionless.

The position vectors of the joints are defined as follow (Fig. 2.20)

$$\begin{aligned}
&\mathbf{r}_A = [0, 0, 0]\,,\\
&\mathbf{r}_B = \mathbf{AB} = [0, r_1, 0] = [0, m\,N_1/2, 0]\,,\\
&\mathbf{r}_C = \mathbf{AB} = [0, r_1 + r_2, 0] = [0, m\,(N_1 + N_2)/2, 0]\,,\\
&\mathbf{r}_{C'} = \mathbf{AC}' = [0, -r_1 - r_2, 0] = [0, -m\,(N_1 + N_2)/2, 0]\,,\\
&\mathbf{r}_D = \mathbf{AD} = [0, r_1 + 2\,r_2, 0] = [0, m\,(N_1 + 2\,N_2)/2, 0]\,,\\
&\mathbf{r}_E = [\#, 0, 0]\,,\\
&\mathbf{r}_F = \mathbf{AF} = [\#, r_3, 0] = [\#, m\,N_3/2, 0]\,,\\
&\mathbf{r}_G = \mathbf{AG} = [\#, r_3 + r_4, 0] = [\#, m\,(N_3 + N_4)/2, 0]\,. \qquad (2.60)
\end{aligned}$$

The x parameter # is not important for the calculation.

Gear 4

The force of gear 3 that acts on gear 4 at the pitch point F is denoted with F_{34}. The force between mating teeth can be resolved at the pith point into two components, a tangential component $F_{t34} = F_{34}\cos\phi$, and a radial component $F_{r34} = F_{34}\sin\phi$ or

$$\mathbf{F}_{34} = [0, F_{r34}, F_{t34}] = F_{34}\sin\phi\,\boldsymbol{\jmath} + F_{34}\cos\phi\,\mathbf{k}. \tag{2.61}$$

The equilibrium of moments for the gear 4 with respect to its center G can be written as

$$\sum \mathbf{M}_G^{(\text{gear }4)} = \mathbf{M}_4 + \mathbf{GF} \times \mathbf{F}_{34} = \mathbf{0}, \tag{2.62}$$

where $\mathbf{GF} = \mathbf{r}_F - \mathbf{r}_G = -r_4\,\boldsymbol{\jmath}$. Equation (2.62) can be written as

$$M_4\boldsymbol{\imath} + \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & -r_4 & 0 \\ 0 & F_{34}\sin\phi & F_{34}\cos\phi \end{vmatrix} = \mathbf{0}. \tag{2.63}$$

Solving Eq. (2.63) the reaction F_{34} is obtained

$$F_{34} = \frac{2\,M_4}{m\,N_4\,\cos\phi} = \frac{2\,\cdot\,500}{0.005\,\cdot\,36\,\cdot\,\cos 20^0} = 5912.1 \text{ N}. \tag{2.64}$$

The reaction of the ground 0 on gear 4 at G is

$$\mathbf{F}_{04} = -\mathbf{F}_{34}.$$

Link 3

The link 3 is composed by the gear 3 and the planetary arm. The reaction of the gear 4 on gear 3 at F is known

$$\mathbf{F}_{43} = -\mathbf{F}_{34} = -F_{34}\sin\phi\mathbf{\jmath} - F_{34}\cos\phi\mathbf{k}.$$

The unknowns are the reactions of the planets gears 2 and $2'$ on planet arm at C and C'

$$\mathbf{F}_{23} = F_{23r}\mathbf{\jmath} + F_{23t}\mathbf{k},$$
$$\mathbf{F}_{2'3} = -F_{23r}\mathbf{\jmath} - F_{23t}\mathbf{k}. \quad (2.65)$$

The reaction of the ground 0 on gear 3 at E is

$$\mathbf{F}_{03} = -\mathbf{F}_{43}.$$

From free-body diagram of link 3 (Fig. 2.21), the tangential component of the force F_{23t} can be computed writing a moment equation with respect to the center of gear 3, E

$$\sum \mathbf{M}_E^{(\text{link}\,3)} = (\mathbf{r}_F - \mathbf{r}_E)\times\mathbf{F}_{43} + (\mathbf{r}_C - \mathbf{r}_E)\times\mathbf{F}_{23} + (\mathbf{r}_{C'} - \mathbf{r}_E)\times\mathbf{F}_{2'3} =$$
$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & r_3 & 0 \\ 0 & -F_{34}\sin\phi & -F_{34}\cos\phi \end{vmatrix} + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ \# & r_1 + r_2 & 0 \\ 0 & F_{23r} & F_{23t} \end{vmatrix} +$$
$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ \# & -r_1 - r_2 & 0 \\ 0 & -F_{23r} & -F_{23t} \end{vmatrix} = -F_{34}r_3\cos\phi\mathbf{\imath} + 2F_{23t}(r_1 + r_2)\mathbf{\imath} = \mathbf{0}. \quad (2.66)$$

The force F_{23t} is

$$F_{23t} = \frac{M_4\, r_3}{2\,(r_1 + r_2)\, r_4} = 1063.83 \text{ N}. \tag{2.67}$$

Gear 2

The forces that act on gear 2 are

$\mathbf{F}_{32} = -F_{23r}\mathbf{\jmath} - F_{23t}\mathbf{k}$ the reaction of the arm on the planet 2 at C, the tangential component $F_{32t} = -F_{23t}$ is known;

$\mathbf{F}_{12} = F_{12}\sin\phi\mathbf{\jmath} + F_{12}\cos\phi\mathbf{k}$ the reaction of the sun gear 1 on the planet 2 at B, unknown;

$\mathbf{F}_{02} = -F_{02}\sin\phi\mathbf{\jmath} + F_{02}\cos\phi\mathbf{k}$ the reaction of the ring gear 0 on the planet 2 at D, unknown.

Two vectorial equilibrium equations can be written. The sum of moments that act on gear 2 with respect to the center C is zero

$$\sum \mathbf{M}_C^{(\text{gear}\,2)} = (\mathbf{r}_D - \mathbf{r}_C) \times \mathbf{F}_{02} + (\mathbf{r}_B - \mathbf{r}_C) \times \mathbf{F}_{12} =$$

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & r_2 & 0 \\ 0 & -F_{02}\sin\phi & F_{02}\cos\phi \end{vmatrix} + \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & -r_2 & 0 \\ 0 & F_{12}\sin\phi & F_{12}\cos\phi \end{vmatrix} = \mathbf{0}, \tag{2.68}$$

and the sum of all the forces that act on gear 2 is zero

$$\sum \mathbf{F}^{(\text{gear}\,2)} = \mathbf{F}_{02} + \mathbf{F}_{12} + \mathbf{F}_{32} =$$

$$(-F_{02}\sin\phi\mathbf{\jmath} + F_{02}\cos\phi\mathbf{k}) + (F_{12}\sin\phi\mathbf{\jmath} + F_{12}\cos\phi\mathbf{k}) +$$

$$(F_{23r}\mathbf{J} + F_{23t}\mathbf{k}) = \mathbf{0}. \tag{2.69}$$

Solving the system of Eqs. (2.68) and (2.69) result

$$F_{32r} = 0,\ \ F_{12} = F_{02} = \frac{M_4\, r_3\, \sec\phi}{4\,(r_1 + r_2)\, r_4} = 566.052 \text{ N}. \tag{2.70}$$

Gear 1

The equilibrium torque $\mathbf{M}_e = M_e\,\boldsymbol{\imath}$ that acts on the input sun gear 1 is computed from the moment equation with respect to the center A

$$\sum \mathbf{M}_A^{(\text{gear 1})} = \mathbf{M}_e + 2\mathbf{r}_B \times \mathbf{F}_{21}, \tag{2.71}$$

and

$$M_e = \frac{M_4\, r_1\, r_3}{2\,(r_1\, r_4\ +\ r_2\, r_4)} = 50.531 \text{ Nm}. \tag{2.72}$$

The equilibrium torque $\mathbf{M}_e$ has the same direction and orientation as the angular velocity $\boldsymbol{\omega}_{10}$.

Example: Forces diagrams

Figures 2.22, 2.23, 2.24, and 2.25 show free body diagrams for different types of planetary gear trains. The torque on the sun gear is M_1 and the torque on the planet arm is M_3. The tangential force that acts on the sun gear 1 at the pitch point is

$$F_{t21} = -F_{t12} = \frac{M_1}{r_1}, \tag{2.73}$$

and the radial force is

$$F_{r21} = -F_{r12} = F_{t21} \tan \phi, \tag{2.74}$$

where ϕ is the pressure angle. The reactions of the ground on the sun gear are

$$F_{01r} = -F_{r21} \text{ and } F_{01t} = -F_{t21}. \tag{2.75}$$

Figure 2.22 shows a planetary gear train with a single planet. For the planetary gear trains with double planet (Figs. 2.24, and 2.25) the tangential force of the planet system that act on the arm is

$$F_{23t} = -F_{32t} = \frac{M_3}{r_1 + r_2}. \tag{2.76}$$

The output torque on the ring gear is

$$M_4 = F_{t2'4} \, r_4. \tag{2.77}$$

2.9 Strength of gear teeth

The flank of the driver tooth makes contact with the tip of the driven tooth at the beginning of action between a pair of gear teeth. The total load F is assumed to be carried by one tooth, and is normal to the tooth profile (see Fig. 2.26). The bending stress at the base of the tooth is produced by the tangential load component F_t which is perpendicular to the centerline of the tooth. The friction and the radial component F_r are neglected. The parabola in Fig. 2.26 outlines a beam of uniform strength. The weakest section of the gear tooth is at section $A - A$ where the parabola is tangent to the tooth outline.

The bending stress σ is

$$\sigma = \frac{6M}{Bt^2} = \frac{6F_t h}{Bt^2}, \tag{2.78}$$

and

$$F_t = \sigma B(t^2/6h) = \sigma B(t^2/6hp)p, \tag{2.79}$$

where $M = F_t\, h$ is the bending moment, h is the distance between the section $A - A$ and the point where the load is applied, and t is the tooth thickness. In the above equations B is the face width and is limited to a maximum of 4 times the circular pitch, i.e. $B = k\, p$, where $k \leq 4$.

The form factor $\gamma = t^2/6hp$ is a dimensionless quantity tabulated in Table 2.1.

Substituting γ in the above equation, the usual form of the Lewis equation is

$$F_t = \sigma B p \gamma, \tag{2.80}$$

or

$$F_t = \sigma p^2 k\gamma = \sigma\pi^2 k\gamma/P_d^2. \tag{2.81}$$

If the pitch diameter P_d is known, then the following form of the Lewis equation may be used

$$P_d^2/\gamma = \sigma k\pi^2/F_t, \tag{2.82}$$

where σ is the allowable stress, $k = 4$ (upper limit), $F_t = 2M_t/d$ is the transmitted force, and M_t is the torque on the weaker gear.

If the pitch diameter is unknown, the following form of the Lewis equation may be used

$$\sigma = \frac{2M_t P_d^3}{k\pi^2\gamma N}, \tag{2.83}$$

where σ = stress $\leq$ than the allowable stress, and N is the number of teeth on weaker gear. The minimum number of teeth, N, is usually limited to 15.

2.9.1 Allowable tooth stresses

The allowable stress for gear tooth design is

$$\begin{aligned} \text{Allowable } \sigma &= \sigma_0\left(\frac{600}{600+V}\right) \text{ for } V \text{ less than 2000 ft/min} \\ &= \sigma_0\left(\frac{1200}{1200+V}\right) \text{ for } V \text{ 2000 to 4000 ft/min} \\ &= \sigma_0\left(\frac{78}{78+\sqrt{V}}\right) \text{ for } V \text{ greater than 4000 ft/min,} \end{aligned}$$

where σ_0 is the endurance strength for released loading corrected for average stress concentration values of the gear material, measured in psi, and V is the pitch line velocity, measured in ft/min. The endurance strength is $\sigma_0 = 8000$ psi for cast iron, and $\sigma_0 = 12,000$ psi for bronze. For carbon steels the endurance strength range is from 10,000 psi to 50,000 psi.

2.9.2 Dynamic tooth loads

The dynamic forces on the teeth are produced by the transmitted force, and also by the velocity changes due to inaccuracies of the tooth profiles, spacing, misalignments in mounting, and tooth deflection under load.

The dynamic load F_d proposed by Buckingham is

$$F_d = \frac{0.05V(BC + F_t)}{0.05V + \sqrt{BC + F_t}} + F_t,$$

where F_d is the dynamic load (lb), F_t is the transmitted force (tangential load), and C is a constant which depends on the tooth material, form, and the accuracy of the tooth cutting process. The constant C is tabulated in Table 2.2. The dynamic force F_d must be less than the allowable endurance load F_0. The allowable endurance load is $F_0 = \sigma_0 \, B \, \gamma \, p$, where σ_0 is based on average stress concentration values.

2.9.3 Wear Tooth loads

The wear load F_w is

$$F_w = d_p B K Q, \tag{2.84}$$

where d_p is the pitch diameter of smaller gear (pinion), K is the stress factor for fatigue, $Q = 2N_g/(N_p + N_g)$, N_g is the number of teeth on gear, N_p is the number of teeth on pinion.

The stress factor for fatigue has the following expression

$$K = \frac{s_{es}^2 (\sin\phi)(1/Ep + 1/Eg)}{1.4},$$

where s_{es} is the surface endurance limit of a gear pair (psi), E_p is the modulus of elasticity of the pinion material (psi), E_g is the modulus of elasticity of the gear material (psi), and ϕ is the pressure angle. An estimated value for surface endurance is

$$s_{es} = (400)(\text{BHN}) - 10,000\,\text{psi},$$

where BHN may be approximated by the average brinell hardness number of the gear and pinion. The wear load F_w is an allowable load and must be greater than the dynamic load F_d. Table 2.3 presents several values of K for various materials and tooth forms.

Example 1

A driver steel pinion with $\sigma_0 = 20,000$ rotates at $n_1 = 1500$ rpm and transmits 13.6 hp. The transmission ratio is $i = -4$ (external gearing). The gear is made of mild steel with $\sigma_0 = 15,000$ psi. Both gears have 20° pressure angle, and are full depth involute gears teeth. Design a gear with the smallest diameter that can be used. No less than 15 teeth are to be used on either gear.

Solution

In order to determine the smallest diameter gears that can be used, the minimum number of teeth for the pinion will be selected $N_p = 15$. Then $N_g = N_p i = 15(4) = 60$. It is first necessary to determine which is weaker, the gear or the pinion. The load carrying capacity of the tooth is a function of the $\sigma_0 \gamma$ product. For the pinion $\sigma_0 \gamma = 20,000(0.092) = 1840$ psi, where $\gamma = 0.092$ was selected from Table 1 for a 20° full-depth involute gear with 15 teeth. For the gear $\sigma_0 \gamma = 15,000(0.134) = 2010$ psi, where $\gamma = 0.134$ correspond to a 20° full-depth involute gear with 60 teeth. Hence, the pinion is weaker. The torque transmitted by the pinion is

$$M_t = 63,000H/n_1 = 63,000(13.6)/1500 = 571.2 \text{ lb in.} \qquad (2.85)$$

Since the diameter is unknown the induced stress is

$$\sigma = \frac{2M_t P_d^3}{k\pi^2 \gamma N_p} = \frac{2(571.2)P_d^3}{4\pi^2(0.092)(15)} = 20.97 P_d^3, \tag{2.86}$$

where a maximum value of $k = 4$ was considered. Assume allowable stress $\sigma \approx \sigma_0/2 = 20,000/2 = 10,000$ psi. This assumption permits the determination of an approximate P_d. Equation (2.86) yields $P_d^3 \approx 10,000/20.97 = 476.87$. Hence, $P_d \approx 8$. Try $P_d = 8$. Then $d_p = 15/8 = 1.875$ in. The pitch line velocity is $V = d_p \pi n_1/12 = 1.875\pi(1500)/12 = 736.31$ ft/min. Because the pitch line velocity is less than 2000 ft/min, the allowable stress will be

$$\sigma = 20,000 \left(\frac{600}{600 + 736.31} \right) = 8979.95 \, \text{psi}.$$

Using Eq. (2.86) the induced stress will be $\sigma = 20.97(8^3) = 10736.64$ psi. The pinion is weak because the induced stress is larger than the allowable stress ($10736.64 > 8979.95$). Try a stronger tooth, $P_d = 7$. Then $d_p = 15/7 = 2.14$ in. The pitch line velocity is $V = d_p \pi n_1/12 = 2.14\pi(1500)/12 = 841.5$ ft/min. Because the pitch line velocity is less than 2000 ft/min, the allowable stress is

$$\sigma = 20,000 \left(\frac{600}{600 + 736.31} \right) = 8324.66 \, \text{psi}.$$

Using Eq. (2.86) the induced stress will be $\sigma = 20.97(7^3) = 7192.71$ psi. Now the pinion is stronger because the induced stress is smaller than the allowable stress. Then the parameter k can be reduced from the maximum value of $k = 4$ to $k = 4(7192.71/8324.66) = 3.45$. Hence, the face width $B = kp = 3.45(\pi/7) = 1.55$ in. Then $P_d = 7$, $B = 1.55$ in, $d_p = 2.14$ in, and $d_g = d_p(4) = 2.14(4) = 8.57$ in. The circular pitch for gears is $p = \pi d_p/N_p = \pi d_g/N_g = 0.448$ in, and the center distance is $c = (d_p + d_g)/2 = 5.35$ in. The addendum of the gears is $a = 1/P_d = 1/7 = 0.14$ in, while the minimum dedendum for 20° full-depth involute gears is $b = 1.157/P_d = 1.157/7 = 0.165$ in. The base circle diameter for pinion and gear are $d_{bp} = d_p \cos\phi = 2.14 \cos 20^\circ = 2.01$ in, and $d_{bg} = d_g \cos\phi = 8.56 \cos 20^\circ = 8.05$ in, respectively. The maximum possible addendum circle radius of pinion or gear without interference can be computed as

$$r_{a(max)} = \sqrt{r_b^2 + c^2 \sin^2 \phi},$$

where $r_b = d_b/2$. Hence, for pinion $r_{a(max)} = \sqrt{1 + 5.35^2 \sin 20^\circ} = 3.29$ in, while for the gear $r_{a(max)} = \sqrt{4^2 + 5.35^2 \sin 20^\circ} = 5.1$ in. The contact ratio CR is calculated from the equation

$$CR = \frac{\sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c \sin\phi}{p_b},$$

where r_{ap}, r_{ag} are addendum radii of the mating pinion and gear, and r_{bp}, r_{bg} are base circle radii of the mating pinion and gear. Here, $r_{ap} = r_p + a =$

$d_p/2 + a = 1.21$ in, $r_{ag} = r_g + a = 4.42$ in, $r_{bp} = d_{bp}/2 = 1.0$ in., and $r_{bg} = d_{bg}/2 = 4.02$ in. The base pitch is computed as $p_b = \pi d_b/N = p\cos 20° = 0.42$ in. Finally the contact ratio will be $CR = 1.63$, which should be a suitable value (> 1.2).

Example 2

A steel pinion ($\sigma_0 = 137.9\times10^6$ N/m^2) rotates an iron gear ($\sigma_0 = 102.88\times 10^6$ N/m^2), and transmits a power of 20 kW. The pinion operates at $n_1 = 2000$ rpm, and the transmission ratio is 4 to 1 (external gearing). Both gears are full depth involute gears and have a pressure angle of 20°. Design a gear with the smallest diameter that can be used. No less than 15 teeth are to be used on either gear.

Solution

To find the smallest diameter gears that can be used, the number of teeth for the pinion will be $N_p = 15$. Hence, $N_g = N_p 4 = 15(4) = 60$.

It is first necessary to determine which is weaker, the gear or the pinion. For pinion, the product $\sigma_0\gamma = 137.9(0.092) = 12.686 \times 10^6$ N/m^2, where $\gamma = 0.092$ was selected from Table 1 for a 20° full-depth involute gear with 15 teeth. For gear $\sigma_0\gamma = 102.88(0.134) = 13.785\times10^6$ N/m^2, where $\gamma = 0.134$

corresponds to a 20° full-depth involute gear with 60 teeth. Hence, the pinion is weaker.

The torque transmitted by the pinion is

$$M_t = 9549H/n_1 = 9549(20)/2000 = 95.49\,\text{Nm}. \tag{2.87}$$

Since the diameter is unknown the induced stress is

$$\sigma = \frac{2M_t}{k\pi^2\gamma N_p m^3} = \frac{2(95.49)}{4\pi^2(0.092)(15)m^3} = \frac{3.5}{m^3}, \tag{2.88}$$

where P_d was replaced by $1/m$, and a maximum value of $k = 4$ was considered. Assume allowable stress $\sigma \approx \sigma_0/2 = 137.9/2 = 68.95 \times 10^6$ N/m^2. This assumption permits the determination of an approximate m. Equation (2.88) yields $m^3 \approx 3.5/68.95 = 3.7$ mm. Try $m = 3$ mm. Then $d_p = N_p m = 15(3) = 45$ mm. The pitch line velocity is $V = d_p\pi n_1/60,000 = 45\pi(2000)/60,000 = 4.71$ m/s. The allowable stress will be

$$\sigma = 137.9\left(\frac{600}{600 + 4.71}\right) = 136.85 \times 10^6\,\text{N/m}^2.$$

Using Eq. (2.88) the induced stress will be $\sigma = 3.5/(3 \times 10^{-3})^3 = 129.83 \times 10^6$ N/m^2. The pinion is stronger. Because the smallest diameter is required, will determine the smallest m such that the induced stress to remain lower than allowable stress. Try $m = 2.75$ mm. Then $d_p = N_p m =$

$15(2.75) = 41.25$ mm. The pitch line velocity is $V = d_p\pi n_1/60,000 = 41.25\pi(2000)/60,000 = 4.32$ m/s. The allowable stress will be

$$\sigma = 137.9\left(\frac{600}{600+4.32}\right) = 136.91 \times 10^6\,\mathrm{N/m^2}.$$

The induced stress will be $\sigma = 3.5/(2.75 \times 10^{-3})^3 = 168.56 \times 10^6$ N/m^2. Now the pinion is weak. Hence, the minimum m which satisfies the stress constraints is $m = 3$ mm. Then the parameter k can be reduced from the maximum value of $k = 4$ to $k = 4(129.83/136.85) = 3.79$. Hence, the face width $B = kp = 3.79(\pi m) = 35.77$ mm, and $d_p = 45$ mm. Then $d_g = d_p(4) = 45(4) = 180$ mm. The circular pitch for gears is $p = \pi d_p/N_p = \pi d_g/N_g = 9.42$ mm, and the center distance is $c = (d_p + d_g)/2 = 112.5$ mm. The addendum of the gears is $a = m = 3$ mm, while the minimum dedendum for 20° full-depth involute gears is $b = 1.25m = 3.75$ mm. The base circle diameter for pinion and gear are $d_{bp} = d_p \cos\phi = 45\cos 20° = 42.28$ mm, and $d_{bg} = d_g \cos\phi = 180\cos 20° = 169.14$ mm, respectively. The maximum possible addendum circle radius without interference for the pinion is $r_{a(max)} = \sqrt{21.14^2 + 112.5^2 \sin 20°} = 69.1$ mm, and for the gear is $r_{a(max)} = \sqrt{84.57^2 + 112.5^2 \sin 20°} = 107.15$ mm. The contact ratio CR is

$$CR = \frac{\sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c\sin\phi}{p_b}.$$

Here, r_{ap}, r_{ag} are addendum radii of the pinion and the gear, and r_{bp}, r_{bg} are base circle radii of the pinion and the gear. Here, $r_{ap} = r_p + a = d_p/2 + a =$ 25.5 mm, $r_{ag} = r_g + a = 93$ mm, $r_{bp} = d_{bp}/2 = 21.14$ mm, and $r_{bg} = d_{bg}/2 =$ 84.57 mm. The base pitch is computed as $p_b = \pi d_b/N = p \cos 20° = 8.85$ mm. Hence, $CR = 1.63 > 1.2$ should be a suitable value.

TABLE 2.1 - Form Factors γ - for use in Lewis strength equation

Number of Teeth	$14\frac{1}{2}^\circ$ Full-Depth Involute or Composite	20° Full-Depth Involute	20° Stub Involute
12	0.067	0.078	0.099
13	0.071	0.083	0.103
14	0.075	0.088	0.108
15	0.078	0.092	0.111
16	0.081	0.094	0.115
17	0.084	0.096	0.117
18	0.086	0.098	0.120
19	0.088	0.100	0.123
20	0.090	0.102	0.125
21	0.092	0.104	0.127
23	0.094	0.106	0.130
25	0.097	0.108	0.133
27	0.099	0.111	0.136
30	0.101	0.114	0.139
34	0.104	0.118	0.142
38	0.106	0.122	0.145
43	0.108	0.126	0.147
50	0.110	0.130	0.151
60	0.113	0.134	0.154
75	0.115	0.138	0.158
100	0.117	0.142	0.161
150	0.119	0.146	0.165
300	0.122	0.150	0.170
Rack	0.124	0.154	0.175

Source: A. S. Hall, A. R. Holowenko, and H. G. Laughlin, "Theory and Problems of Machine Design", Schaum's Outline Series McGraw-Hill, 1961.

TABLE 2.2 - Values of Deformation Factor C - for dynamic load check

Materials		Involute	Tooth Error inches			
Pinion	Gear	tooth form	0.0005	0.001	0.002	0.003
cast iron	cast iron	$14\frac{1}{2}^\circ$	400	800	1600	2400
steel	cast iron	$14\frac{1}{2}^\circ$	550	1100	2200	3300
steel	steel	$14\frac{1}{2}^\circ$	800	1600	3200	4800
cast iron	cast iron	20° full depth	415	830	1660	2490
steel	cast iron	20° full depth	570	1140	2280	3420
steel	steel	20° full depth	830	1660	3320	4980
cast iron	cast iron	20° stub	430	860	1720	2580
steel	cast iron	20° stub	590	1180	2360	3540
steel	steel	20° stub	860	1720	3440	5160

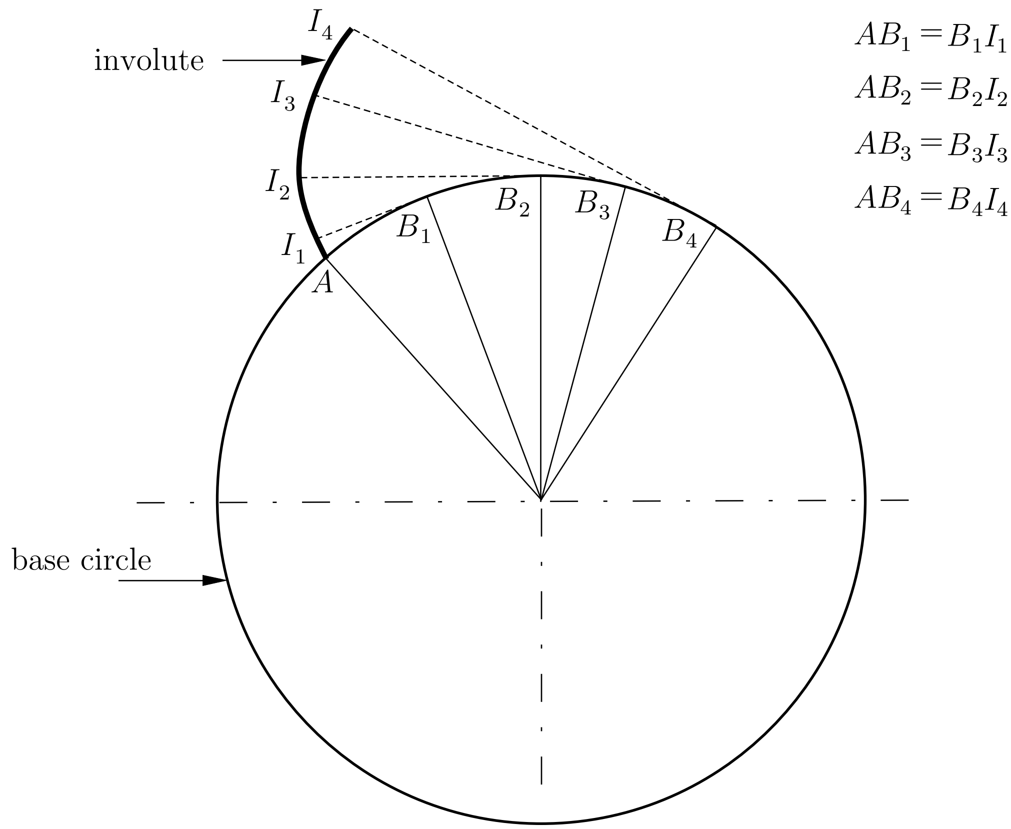
Source: A. S. Hall, A. R. Holowenko, and H. G. Laughlin, "Theory and Problems of Machine Design", Schaum's Outline Series McGraw-Hill, 1961.

TABLE 2.3

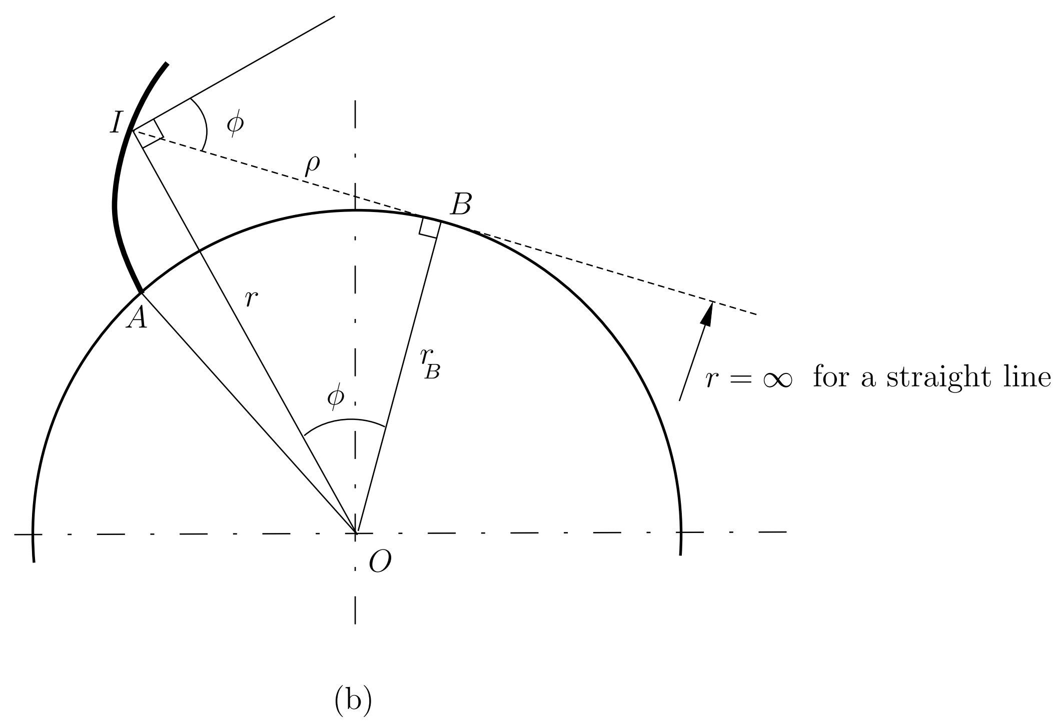
Values for Surface Endurance Limit s_{es} and Stress Fatigue Factor K

Average Brinell Hardness Number of steel pinion and steel gear		Surface Endurance Limit s_{es}	Stress Fatigue Factor K	
			$14\frac{1}{2}°$	$20°$
150		50,000	30	41
200		70,000	58	79
250		90,000	96	131
300		110,000	144	196
400		150,000	268	366
Brinell Hardness Number, BHN				
Steel pinion	Gear			
150	C.I.	50,000	44	60
200	C.I.	70,000	87	119
250	C.I.	90,000	144	196
150	Phosphor Bronze	50,000	46	62
200	Phosphor Bronze	65,000	73	100
C.I. Pinion	C.I. Gear	80,000	152	208
C.I. Pinion	C.I. Gear	90,000	193	284

Source: A. S. Hall, A. R. Holowenko, and H. G. Laughlin, "Theory and Problems of Machine Design", Schaum's Outline Series McGraw-Hill, 1961.

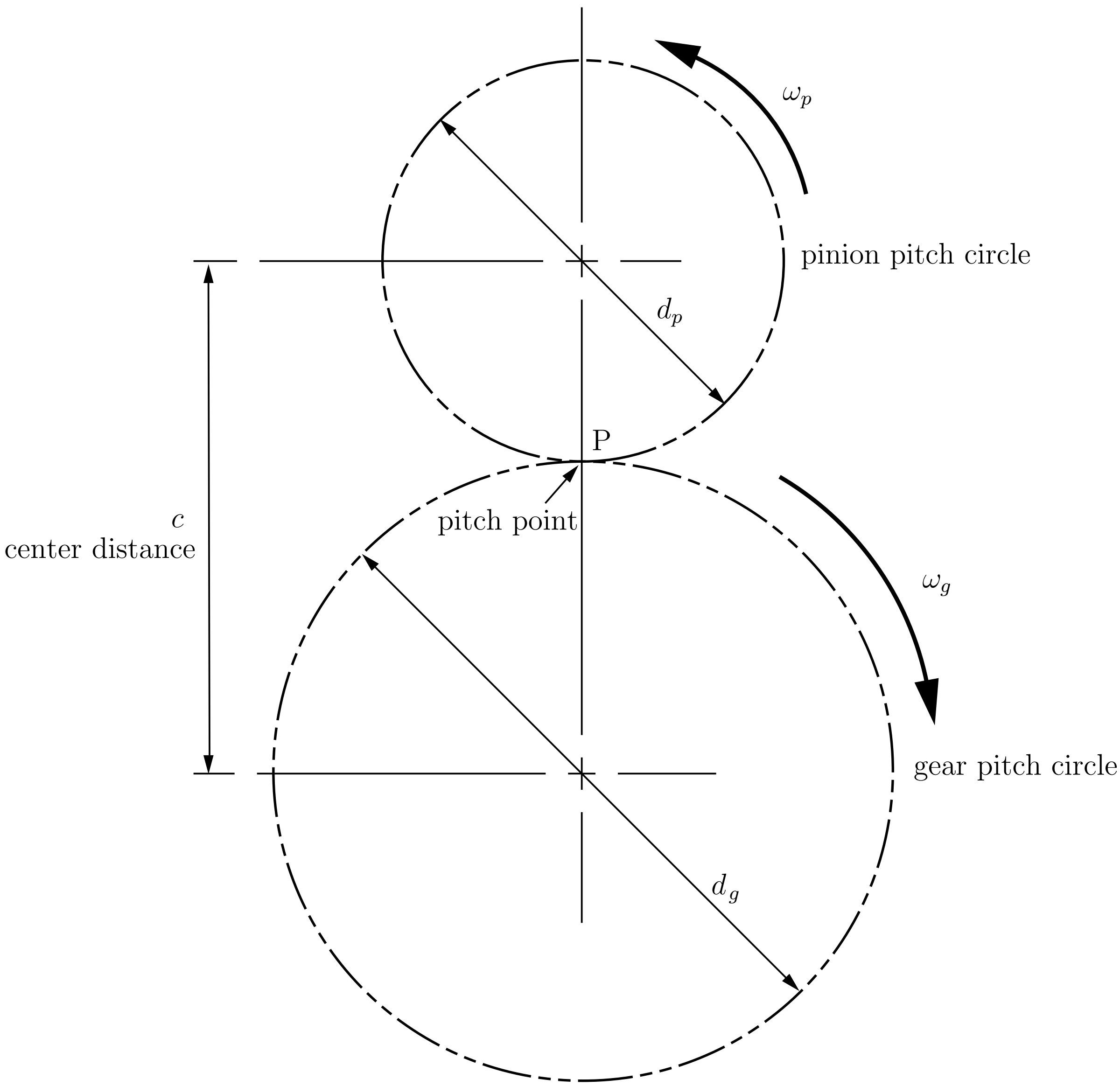

involute
base circle
I_4
I_3
I_2
I_1
A
B_1
B_2
B_3
B_4
$AB_1 = B_1I_1$
$AB_2 = B_2I_2$
$AB_3 = B_3I_3$
$AB_4 = B_4I_4$

(a)


I
ϕ
ρ
B
A
r
r_B
ϕ
O
$r = \infty$ for a straight line

(b)

Figure 2.1



ω_p
pinion pitch circle
d_p
P
pitch point
c
center distance
ω_g
gear pitch circle
d_g

Figure 2.2

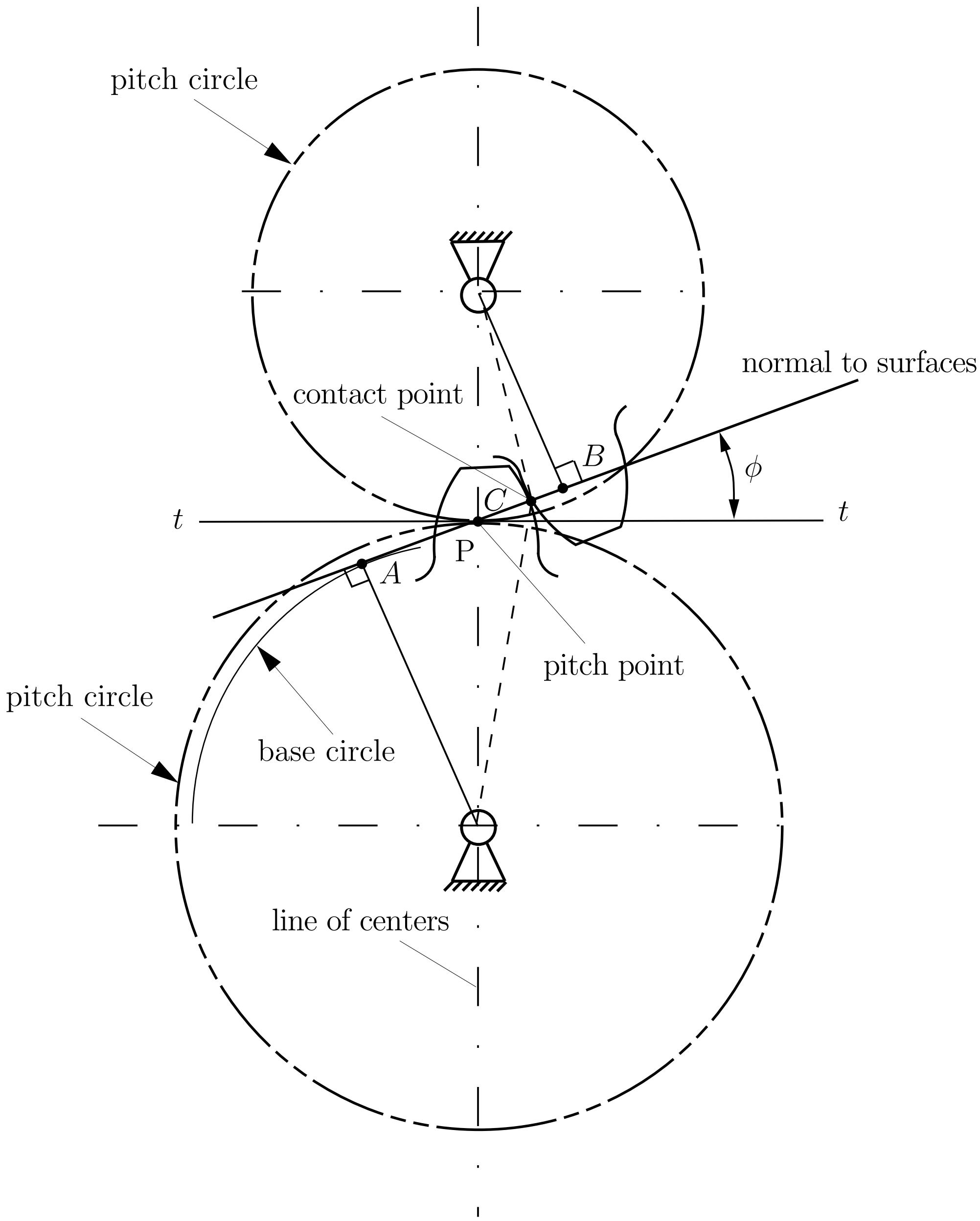

pitch circle
normal to surfaces
contact point
B
C
t
t
ϕ
P
A
pitch point
pitch circle
base circle
line of centers

Figure 2.3

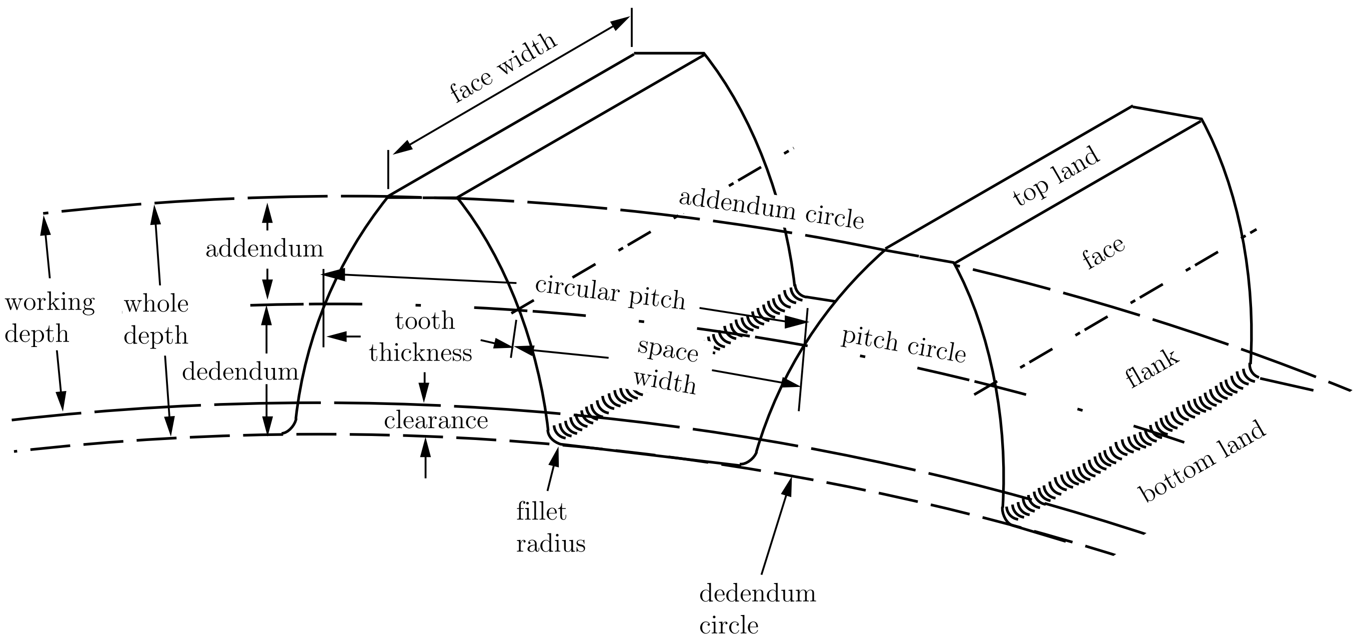

face width
top land
addendum circle
face
addendum
working depth
whole depth
circular pitch
tooth thickness
space width
pitch circle
flank
dedendum
clearance
bottom land
fillet radius
dedendum circle

Figure 2.4

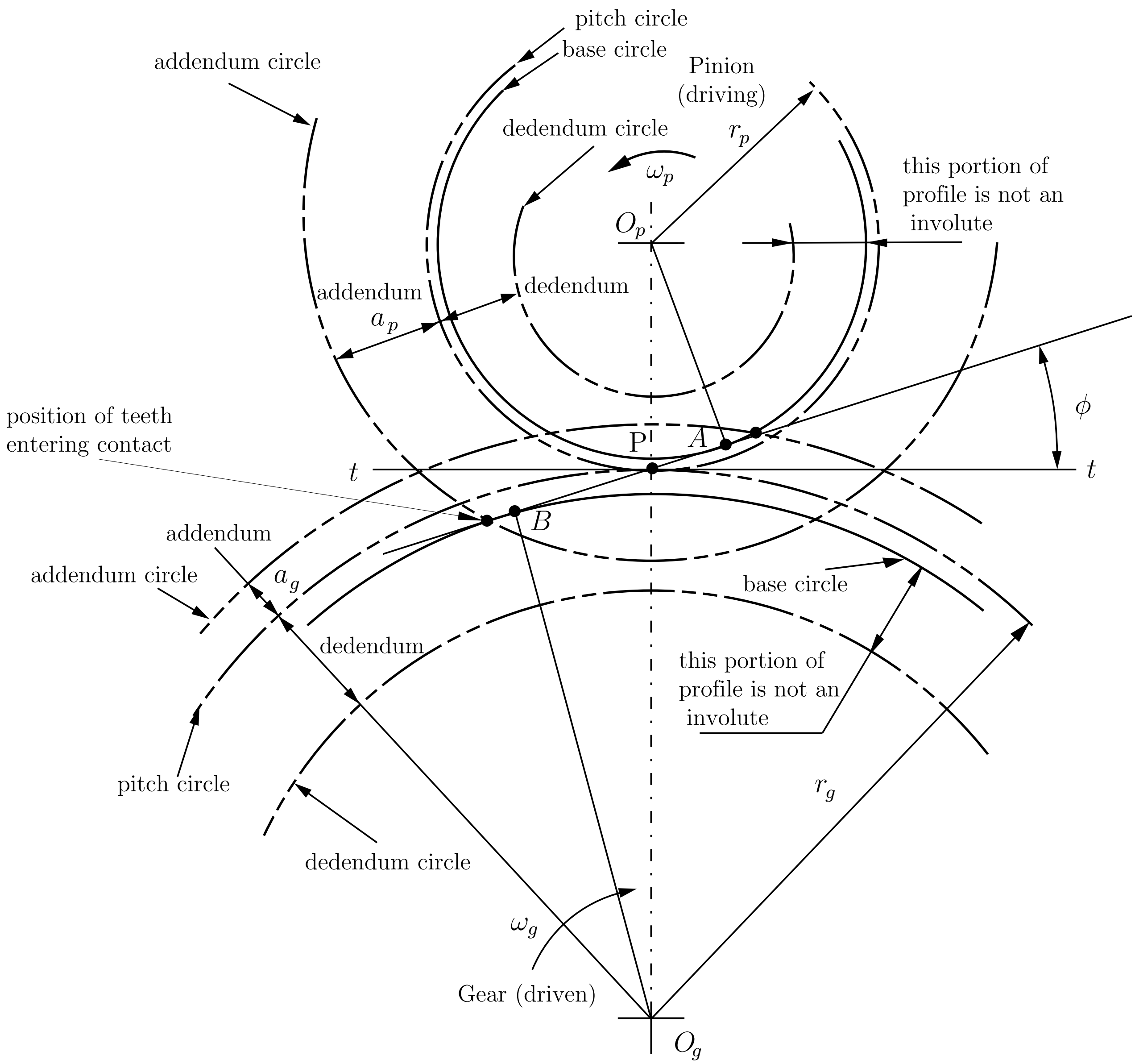
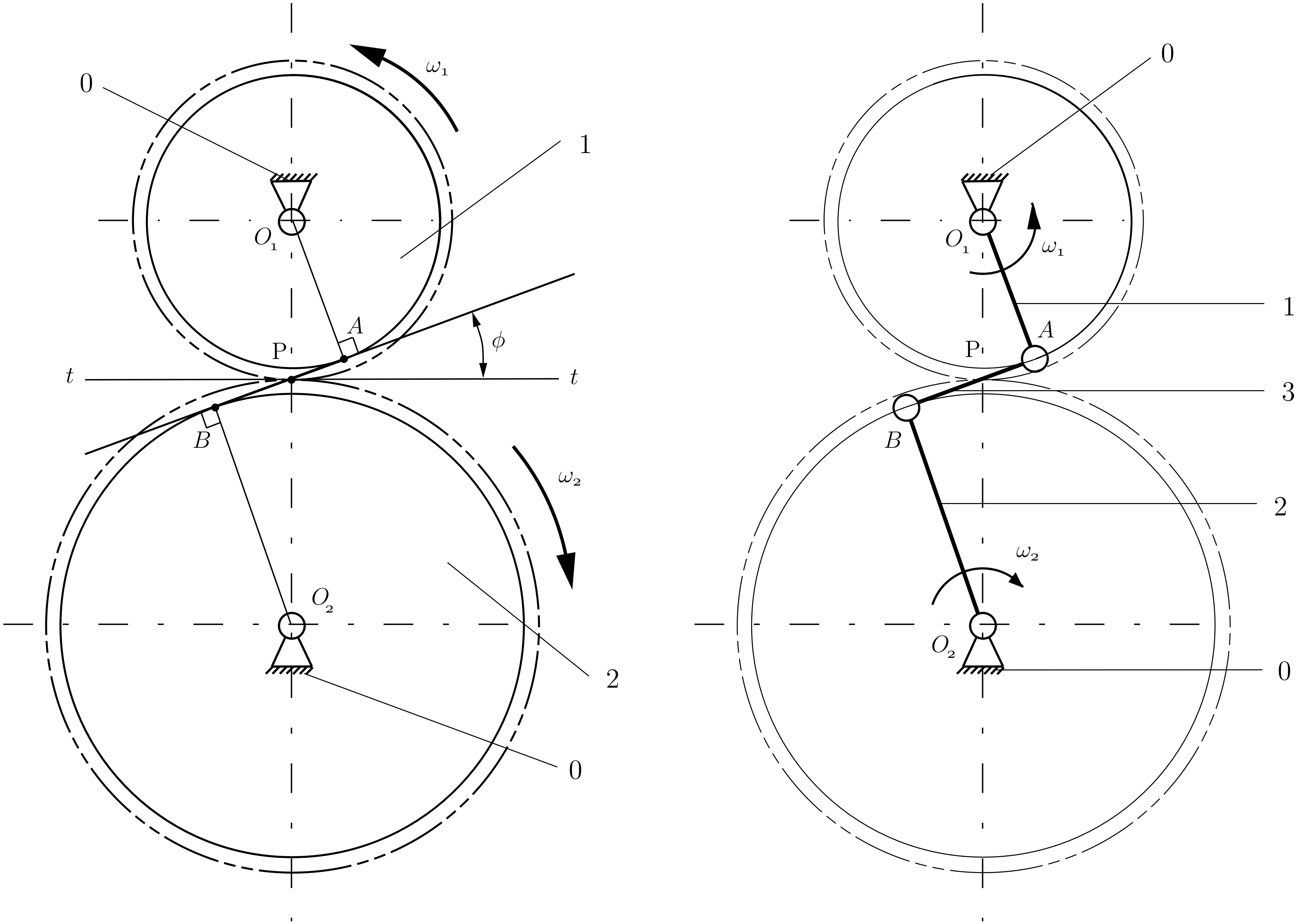
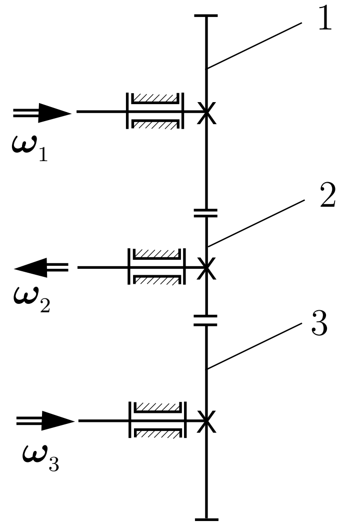

pitch circle
base circle
addendum circle
Pinion
(driving)
dedendum circle
r_p
ω_p
this portion of
profile is not an
involute
O_p
addendum
a_p
dedendum
position of teeth
entering contact
P
A
ϕ
t
t
B
addendum
addendum circle
a_g
base circle
dedendum
this portion of
profile is not an
involute
pitch circle
r_g
dedendum circle
ω_g
Gear (driven)
O_g

Figure 2.5

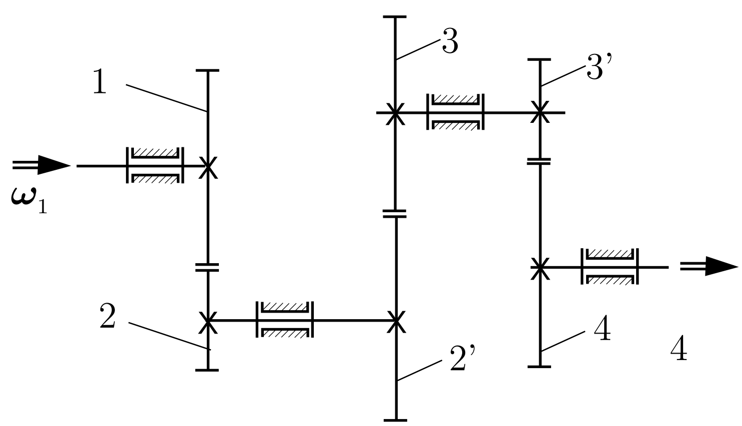


0
1
2
0
ω_1
ω_2
O_1
O_2
A
B
P
t
t
ϕ
0
1
3
2
0
ω_1
ω_2
O_1
O_2
A
B
P

Figure 2.6



(a)



(b)

Figure 2.7

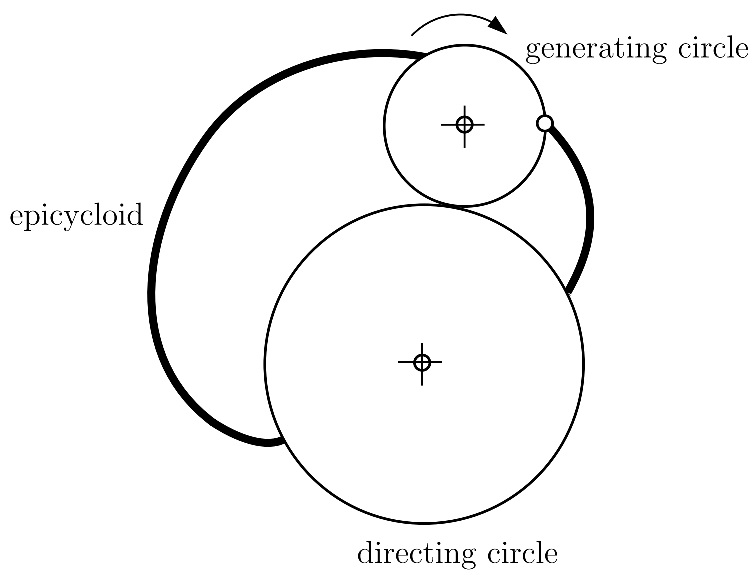
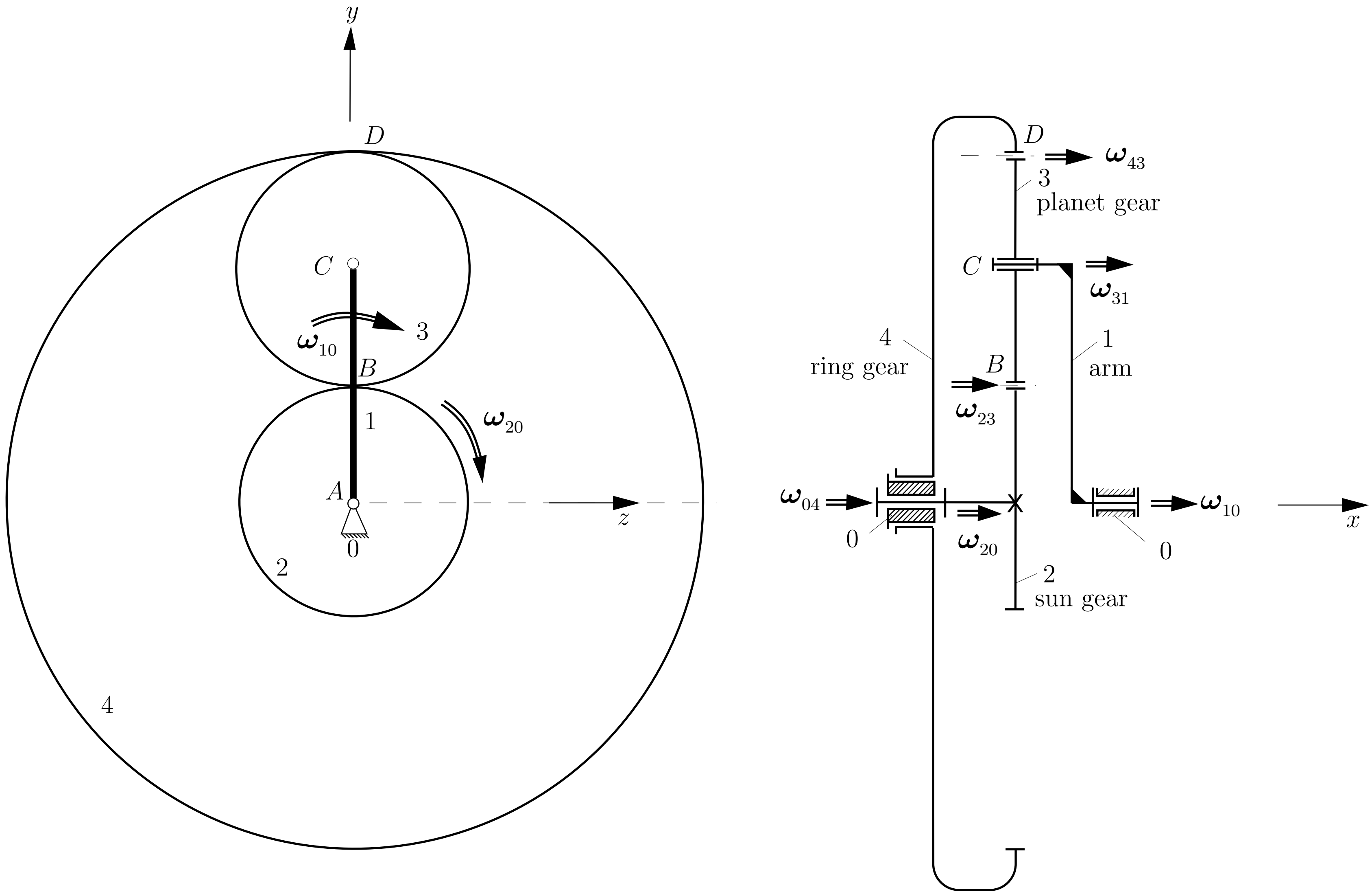


Figure 2.8



y
D
C
ω_{10}
3
B
1
ω_{20}
A
z
0
2
4
D
ω_{43}
3
planet gear
C
ω_{31}
4
ring gear
1
arm
B
ω_{23}
ω_{04}
ω_{10}
x
0
ω_{20}
0
2
sun gear

Figure 2.9

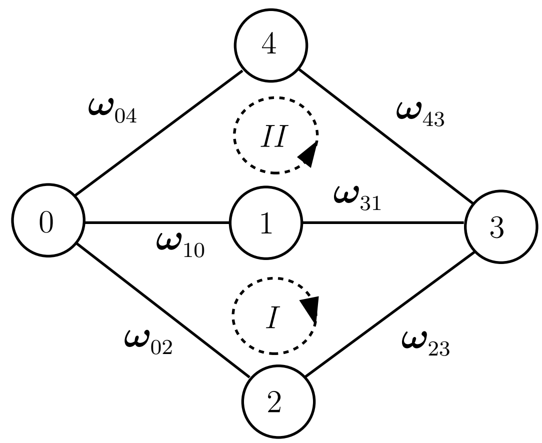

4
$\boldsymbol{\omega}_{04}$
$\boldsymbol{\omega}_{43}$
II
$\boldsymbol{\omega}_{31}$
0
1
3
$\boldsymbol{\omega}_{10}$
I
$\boldsymbol{\omega}_{02}$
$\boldsymbol{\omega}_{23}$
2

Figure 2.10

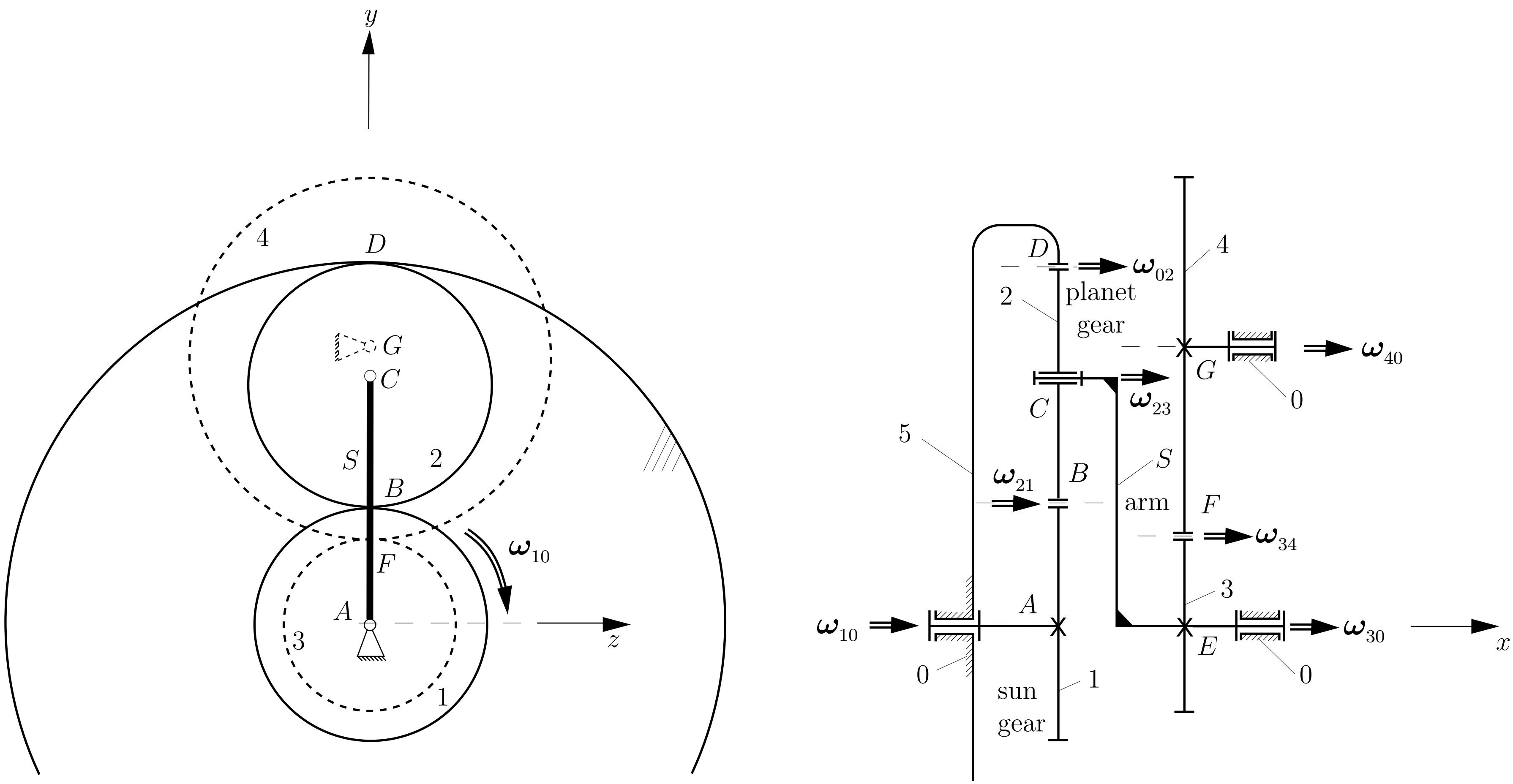

y
z
x
4
D
G
C
S
2
B
F
A
3
1
ω_{10}
D
ω_{02}
planet
gear
ω_{40}
G
0
C
ω_{23}
5
S
ω_{21}
B
arm
F
ω_{34}
3
A
E
ω_{30}
0
0
sun
gear
1

Figure 2.11

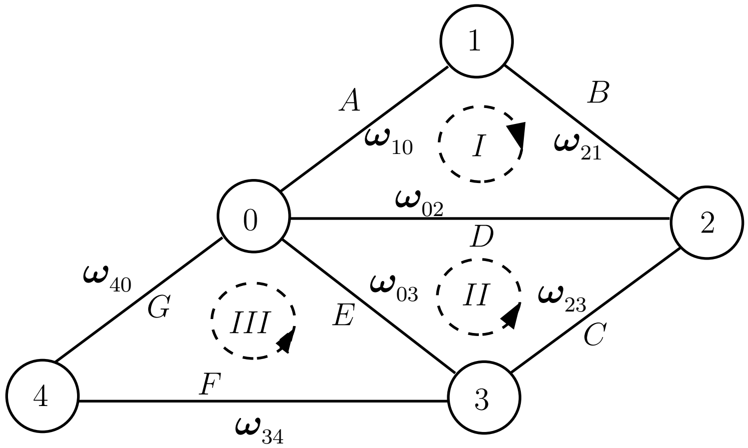
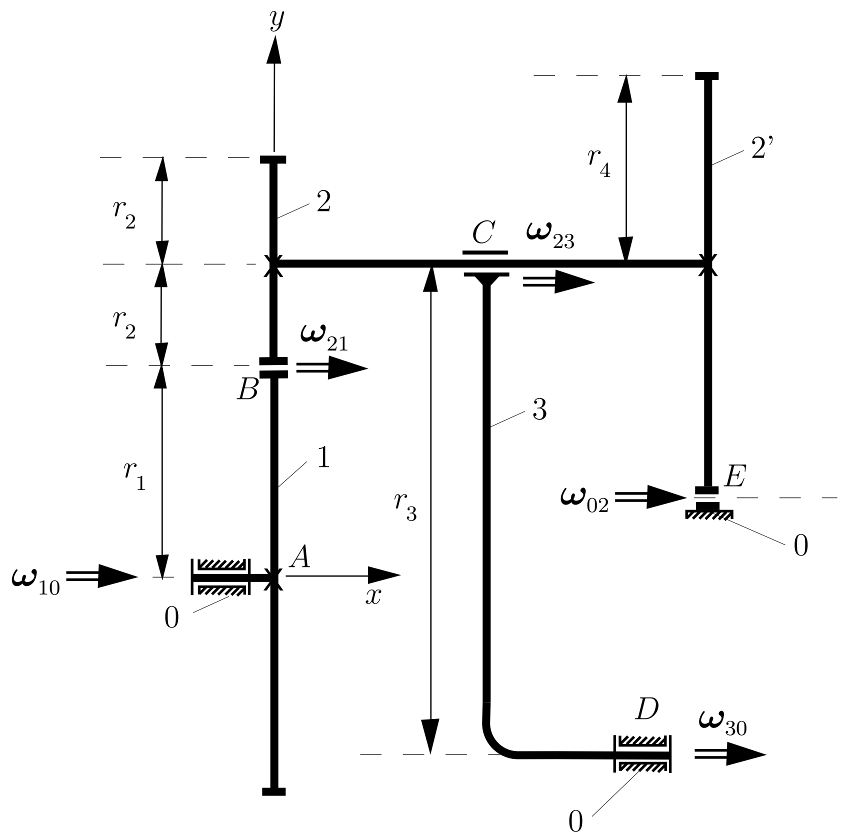

1
A
B
ω_{10}
I
ω_{21}
0
ω_{02}
2
D
ω_{40}
G
III
E
ω_{03}
II
ω_{23}
C
4
F
3
ω_{34}

Figure 2.12



y
r_2
r_2
r_1
2
C
ω_{23}
r_4
2'
ω_{21}
B
3
1
r_3
E
ω_{02}
0
ω_{10}
A
x
0
D
ω_{30}
0

Figure 2.13

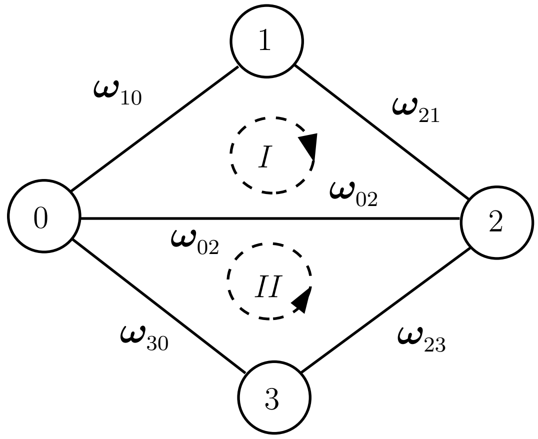
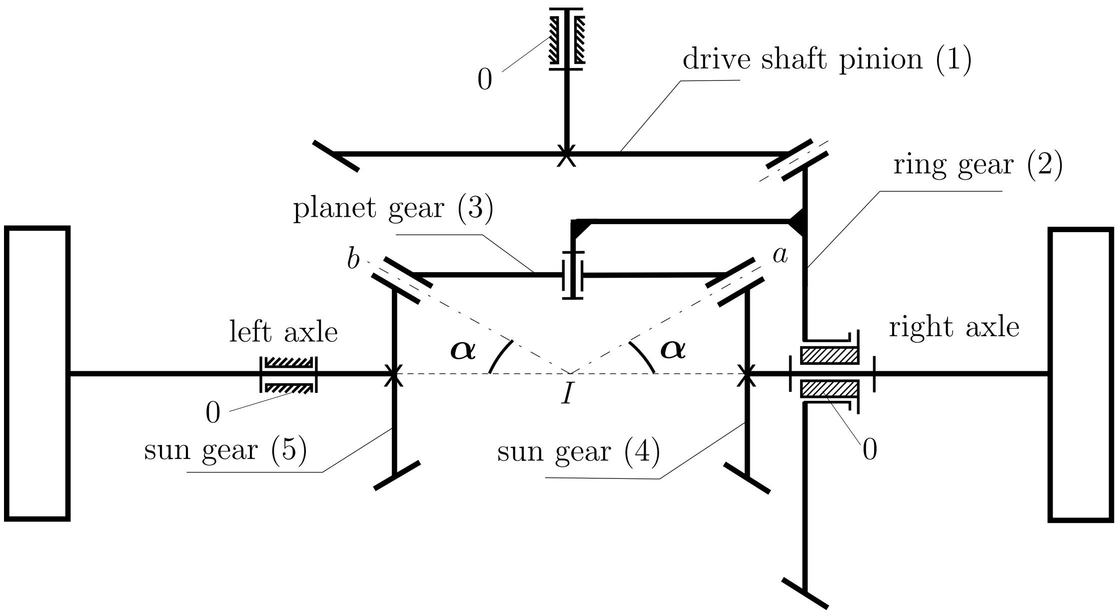
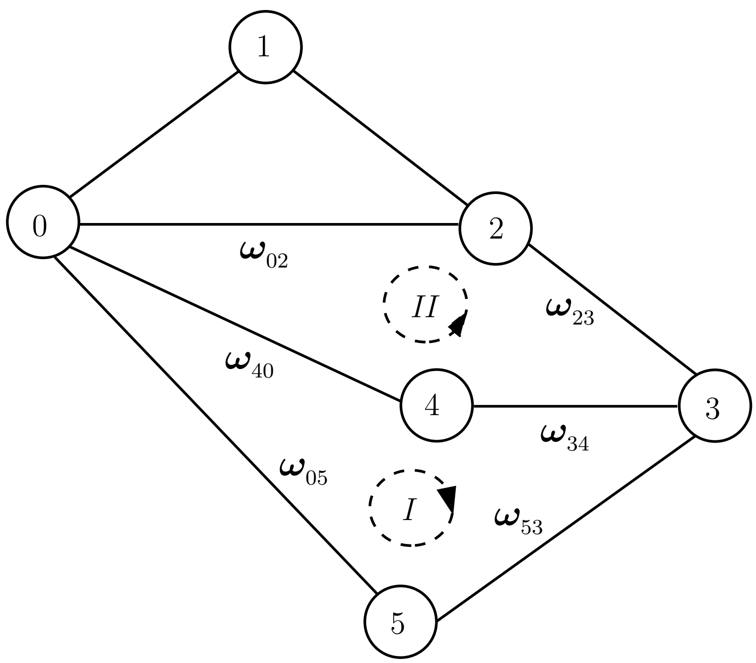

1
$\boldsymbol{\omega}_{10}$
$\boldsymbol{\omega}_{21}$
I
$\boldsymbol{\omega}_{02}$
0
2
$\boldsymbol{\omega}_{02}$
II
$\boldsymbol{\omega}_{30}$
$\boldsymbol{\omega}_{23}$
3

Figure 2.14

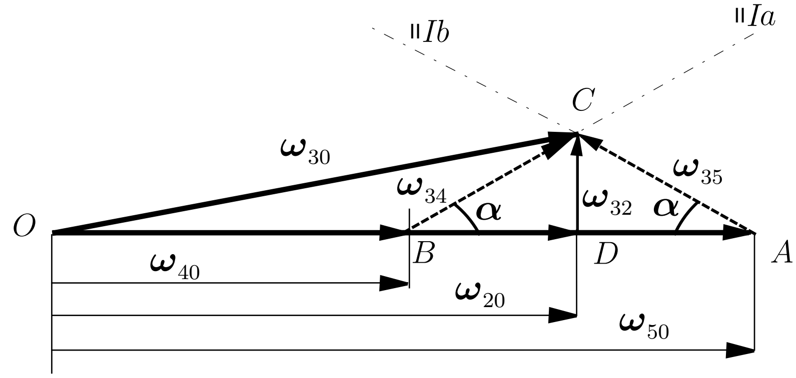


(a)



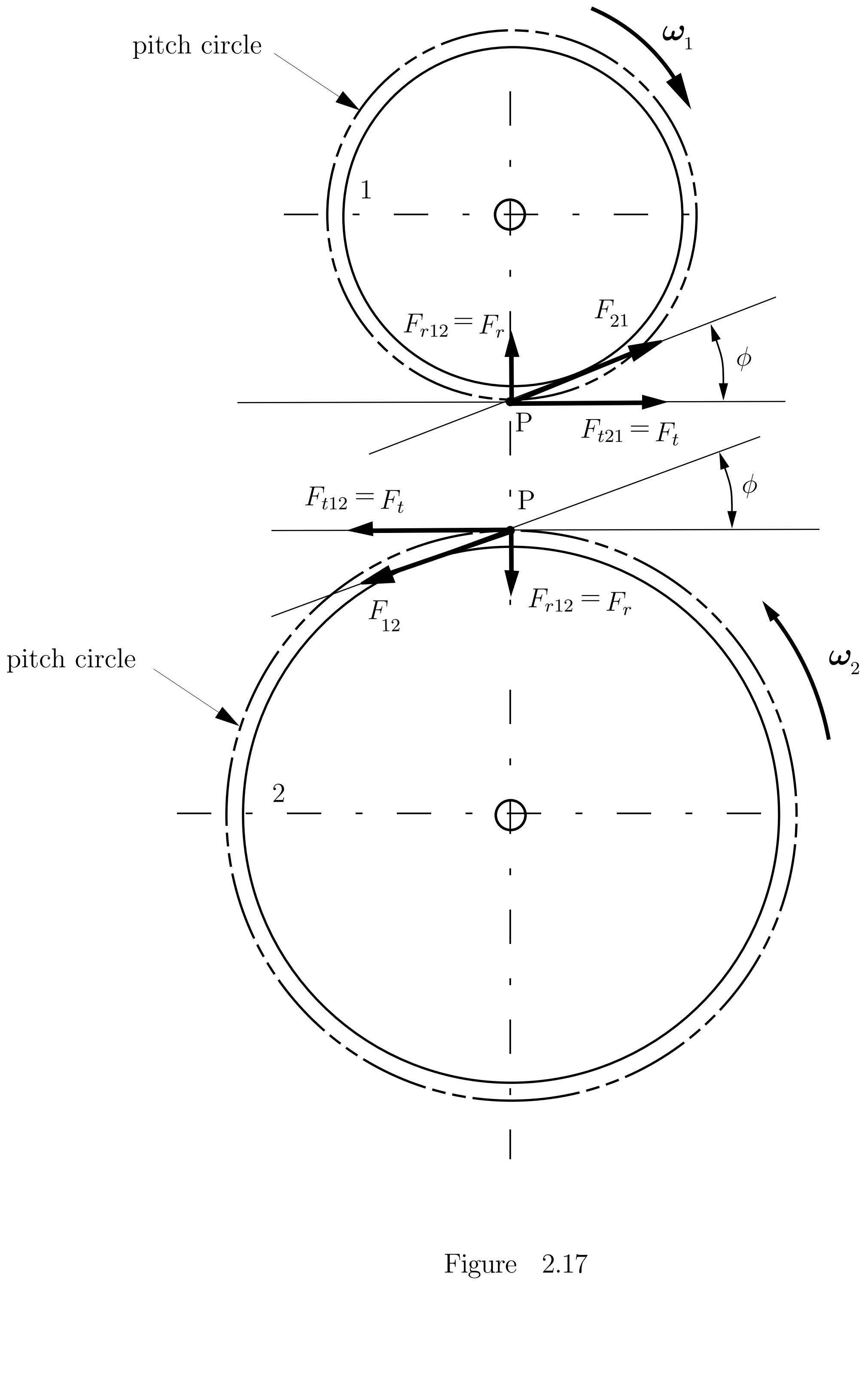
(b)

Figure 2.15



∥Ib
∥Ia
C
$\boldsymbol{\omega}_{30}$
$\boldsymbol{\omega}_{34}$
$\boldsymbol{\omega}_{35}$
$\boldsymbol{\omega}_{32}$
$\boldsymbol{\alpha}$
$\boldsymbol{\alpha}$
O
B
D
A
$\boldsymbol{\omega}_{40}$
$\boldsymbol{\omega}_{20}$
$\boldsymbol{\omega}_{50}$

Figure 2.16



pitch circle
ω_1
1
$F_{r12} = F_r$
F_{21}
ϕ
P
$F_{t21} = F_t$
$F_{t12} = F_t$
P
ϕ
$F_{r12} = F_r$
F_{12}
pitch circle
ω_2
2

Figure 2.17

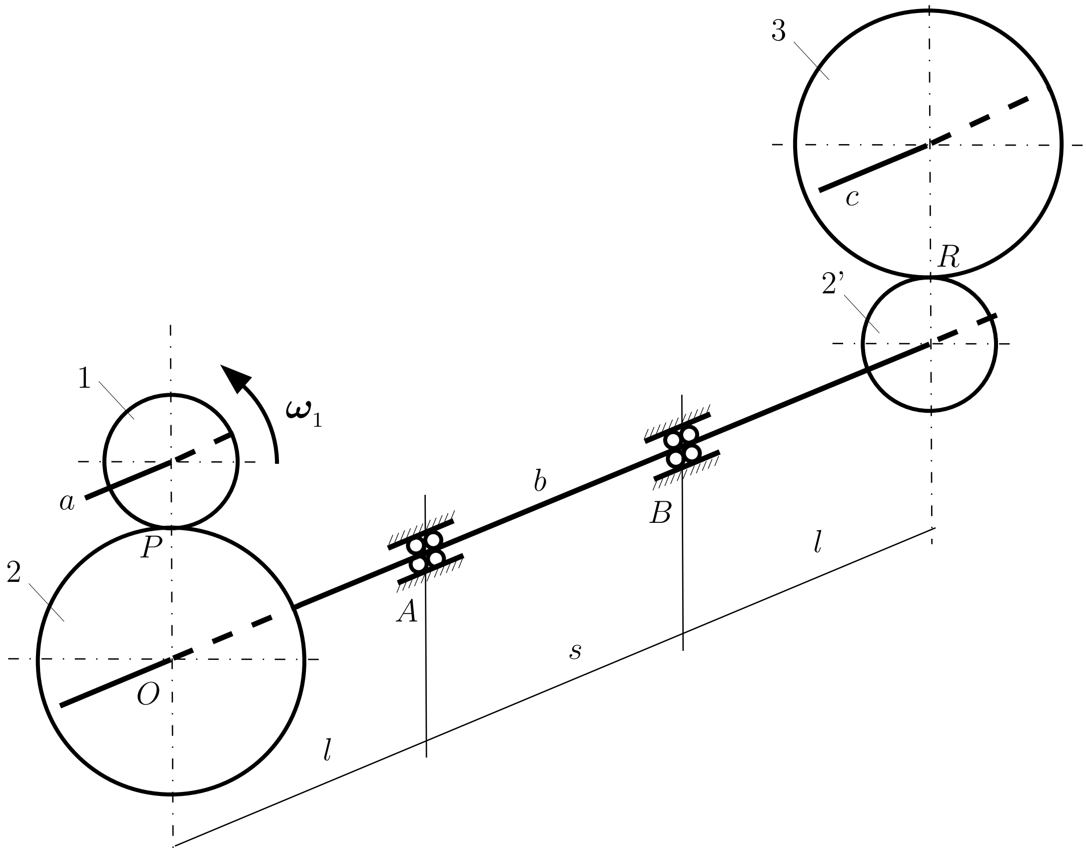


Figure 2.18

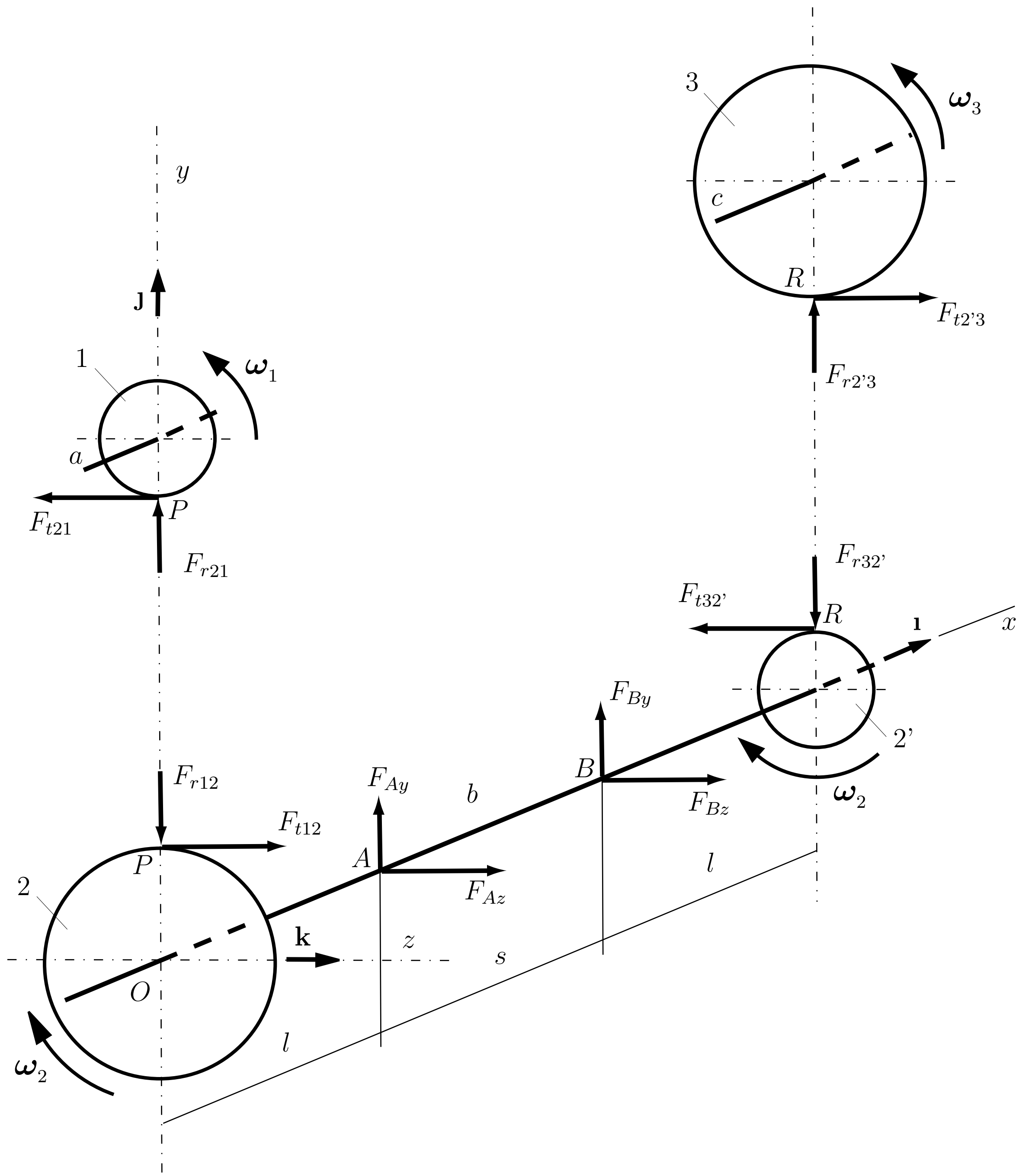
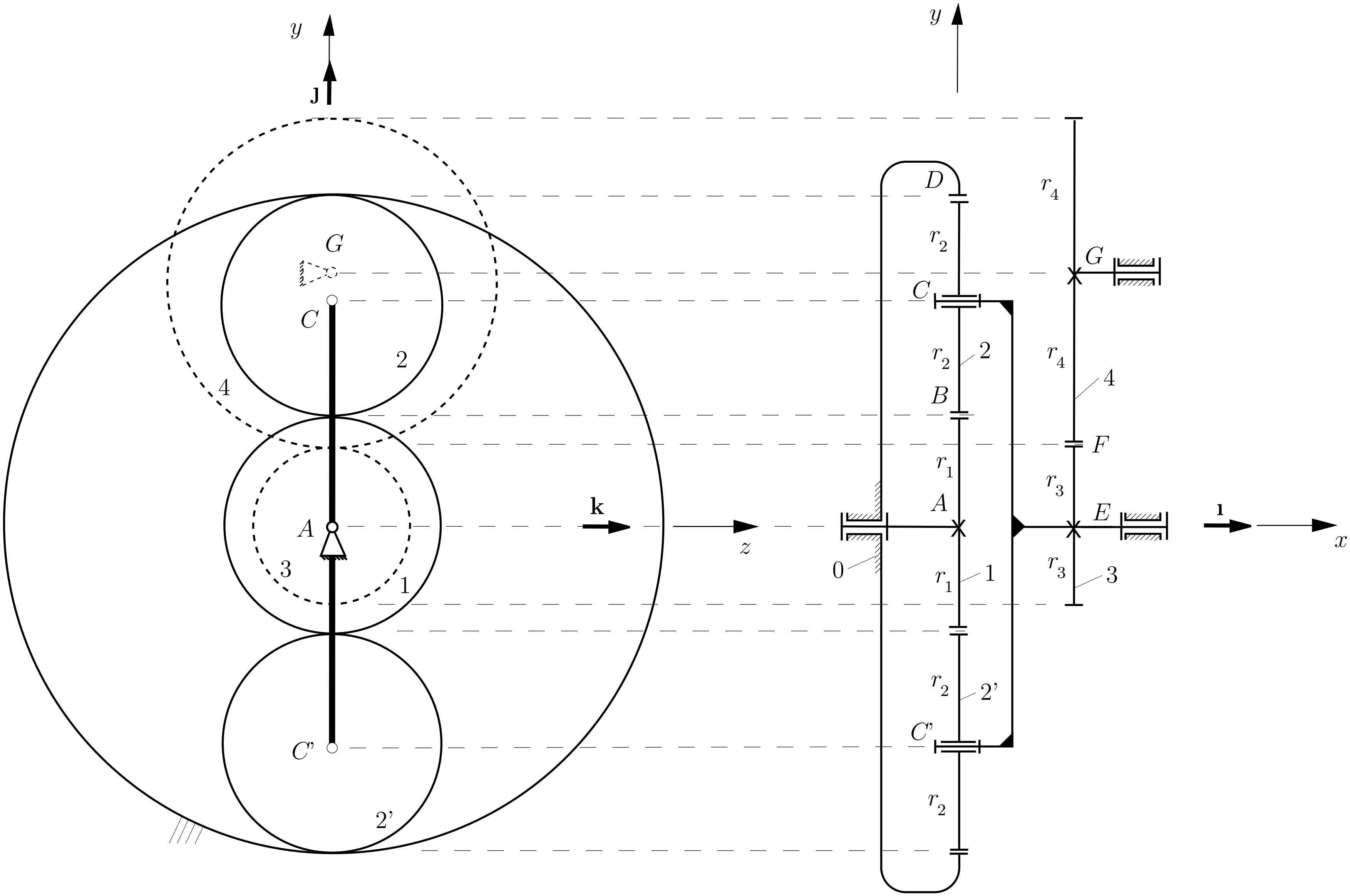
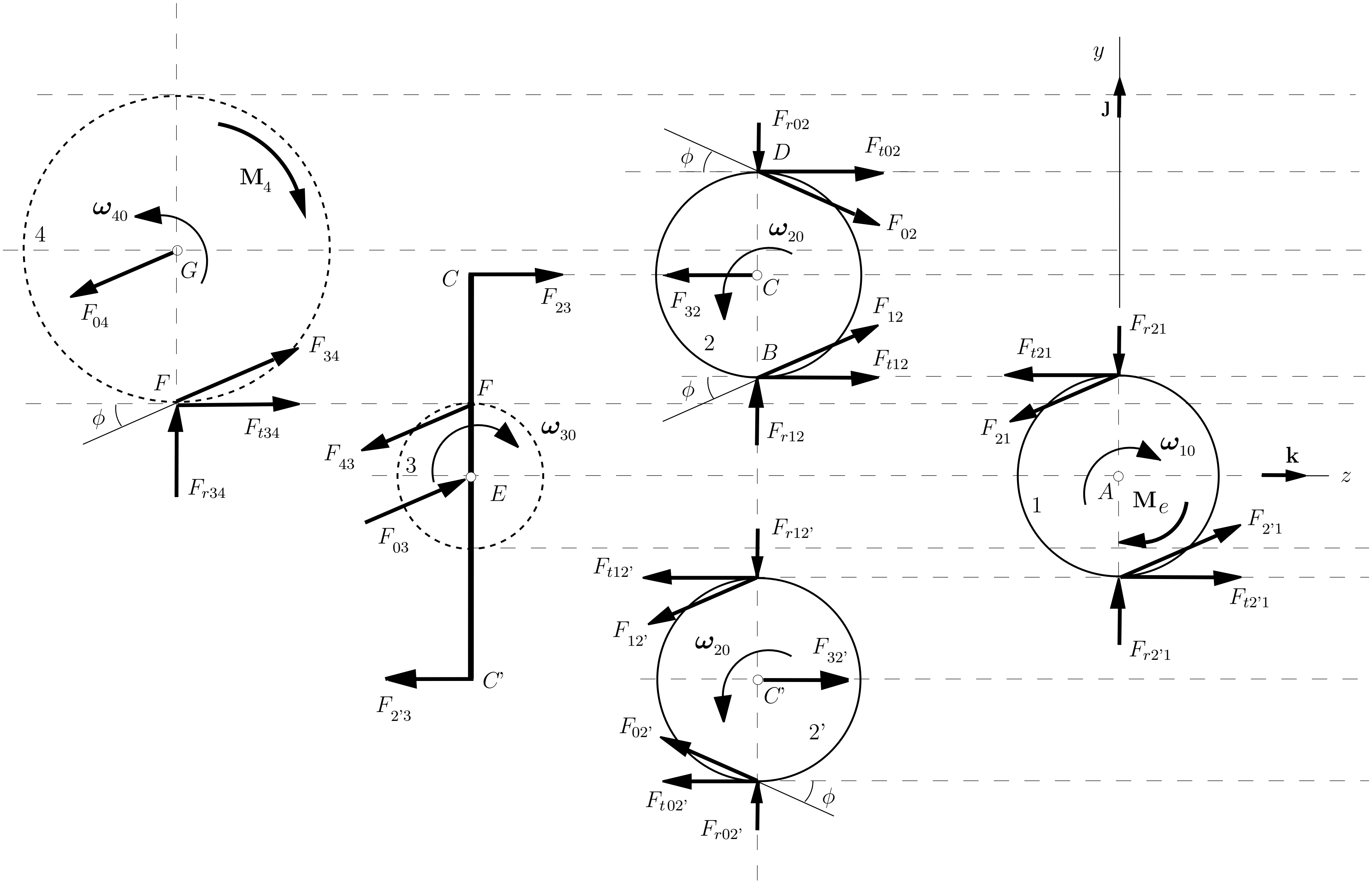

y
J
1
ω_1
a
P
F_{t21}
F_{r21}
3
ω_3
c
R
$F_{t2'3}$
$F_{r2'3}$
$F_{r32'}$
$F_{t32'}$
R
x
ı
F_{By}
B
2'
ω_2
F_{Bz}
F_{r12}
F_{t12}
F_{Ay}
b
A
P
2
F_{Az}
l
k
z
s
O
l
ω_2

Figure 2.19



y
J
G
C
2
4
A
3
1
k
z
C'
2'
y
D
r2
C
r2
2
B
r1
A
0
r1
1
r2
2'
C'
r2
r4
G
r4
4
F
r3
E
r3
3
ı
x

Figure 2.20



$\mathbf{M}_4$
$\boldsymbol{\omega}_{40}$
4
G
F_{04}
F
F_{34}
ϕ
F_{t34}
F_{r34}
C
F_{23}
F
$\boldsymbol{\omega}_{30}$
F_{43}
3
E
F_{03}
C'
$F_{2'3}$
F_{r02}
D
F_{t02}
ϕ
F_{02}
$\boldsymbol{\omega}_{20}$
C
F_{32}
F_{12}
2
B
F_{t12}
ϕ
F_{r12}
$F_{r12'}$
$F_{t12'}$
$F_{12'}$
$\boldsymbol{\omega}_{20}$
$F_{32'}$
C'
$F_{02'}$
2'
$F_{t02'}$
ϕ
$F_{r02'}$
y
$\mathbf{J}$
F_{r21}
F_{t21}
F_{21}
$\boldsymbol{\omega}_{10}$
$\mathbf{k}$
z
1
A
$\mathbf{M}_e$
$F_{2'1}$
$F_{t2'1}$
$F_{r2'1}$

Figure 2.21

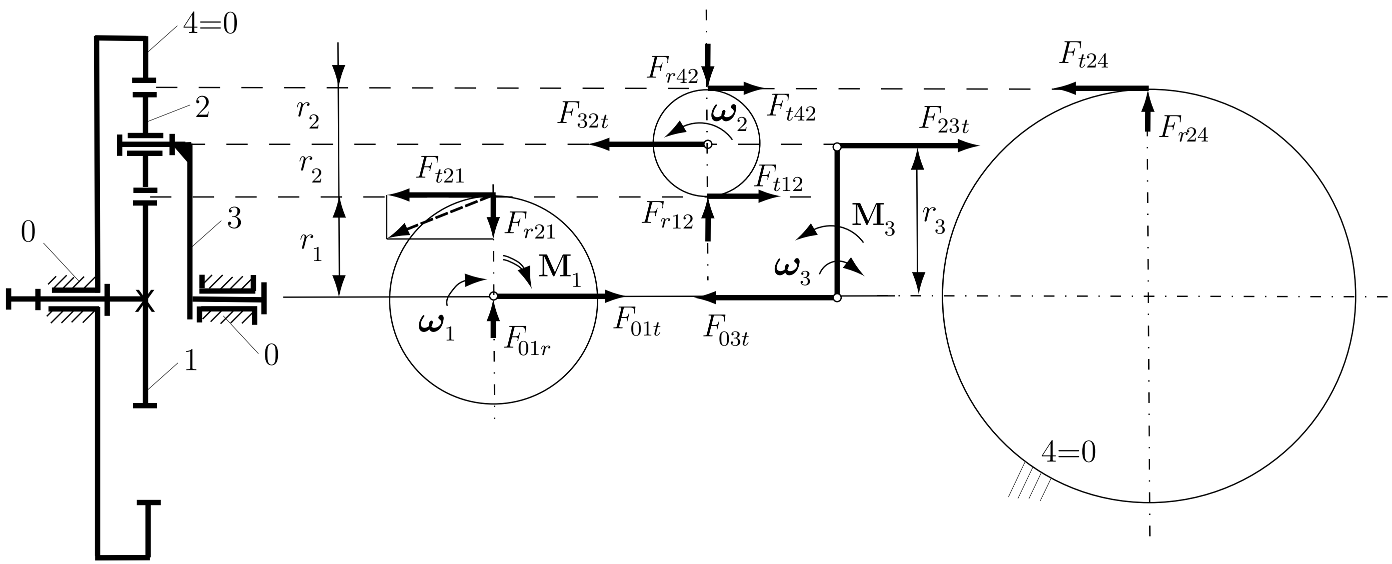
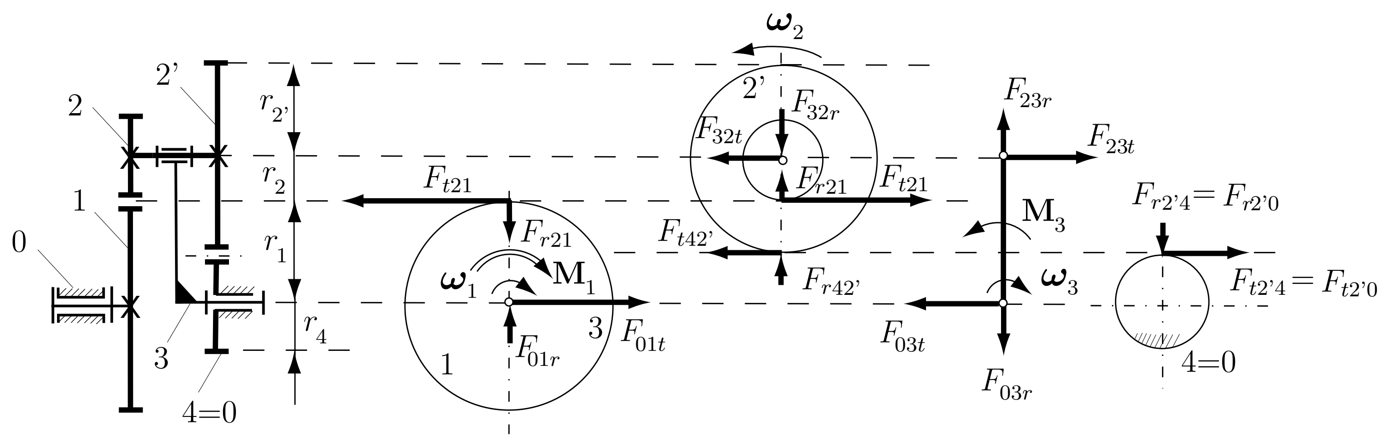
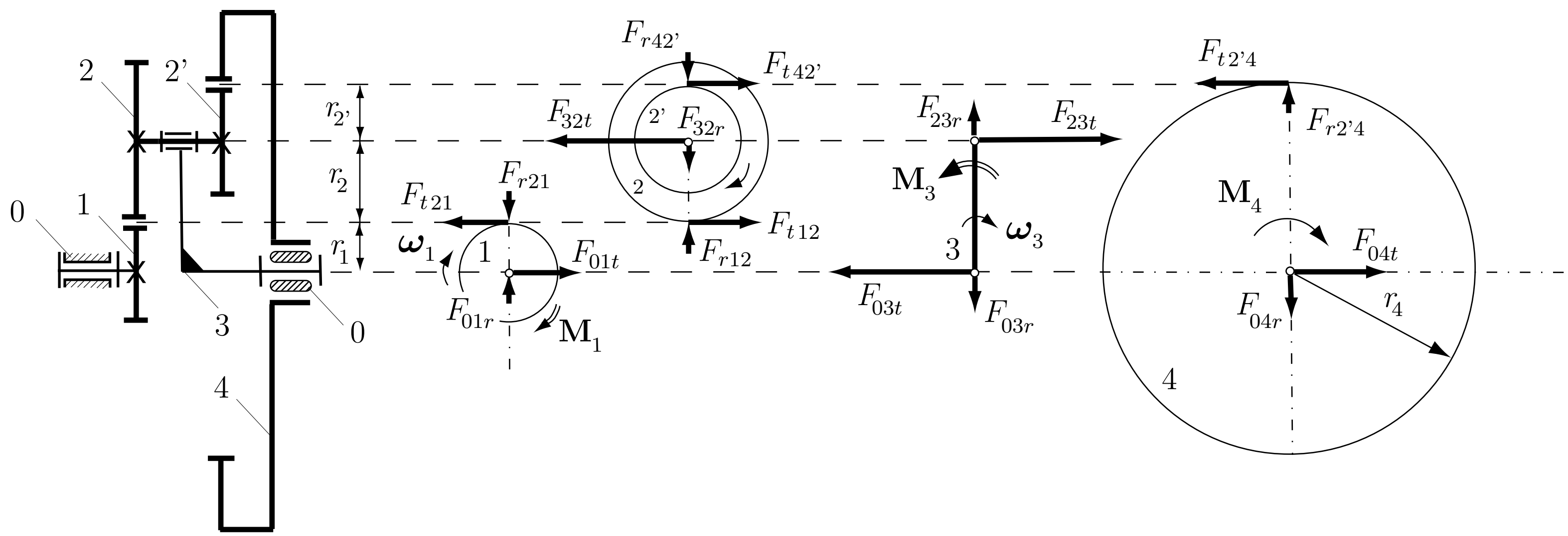

4=0
2
3
0
0
1
r_2
r_2
r_1
F_{t21}
F_{r21}
$\mathbf{M}_1$
$\boldsymbol{\omega}_1$
F_{01r}
F_{01t}
F_{r42}
F_{t42}
F_{32t}
$\boldsymbol{\omega}_2$
F_{t12}
F_{r12}
F_{03t}
F_{23t}
$\mathbf{M}_3$
$\boldsymbol{\omega}_3$
r_3
F_{t24}
F_{r24}
4=0

Figure 2.22



2'
2
1
0
3
4=0
$r_{2'}$
r_2
r_1
r_4
F_{t21}
F_{r21}
ω_1
M_1
F_{01r}
F_{01t}
1
3
ω_2
2'
F_{32r}
F_{32t}
F_{r21}
F_{t21}
$F_{t42'}$
$F_{r42'}$
F_{23r}
F_{23t}
M_3
ω_3
F_{03t}
F_{03r}
$F_{r2'4} = F_{r2'0}$
$F_{t2'4} = F_{t2'0}$
4=0

Figure 2.23



2
2'
0
1
3
0
4
$r_{2'}$
r_2
r_1
$F_{r42'}$
$F_{t42'}$
F_{32t}
2'
F_{32r}
2
F_{t12}
F_{r12}
F_{r21}
F_{t21}
ω_1
1
F_{01t}
F_{01r}
$\mathbf{M}_1$
F_{23r}
F_{23t}
$\mathbf{M}_3$
3
ω_3
F_{03t}
F_{03r}
$F_{t2'4}$
$F_{r2'4}$
$\mathbf{M}_4$
F_{04t}
F_{04r}
r_4
4

Figure 2.24

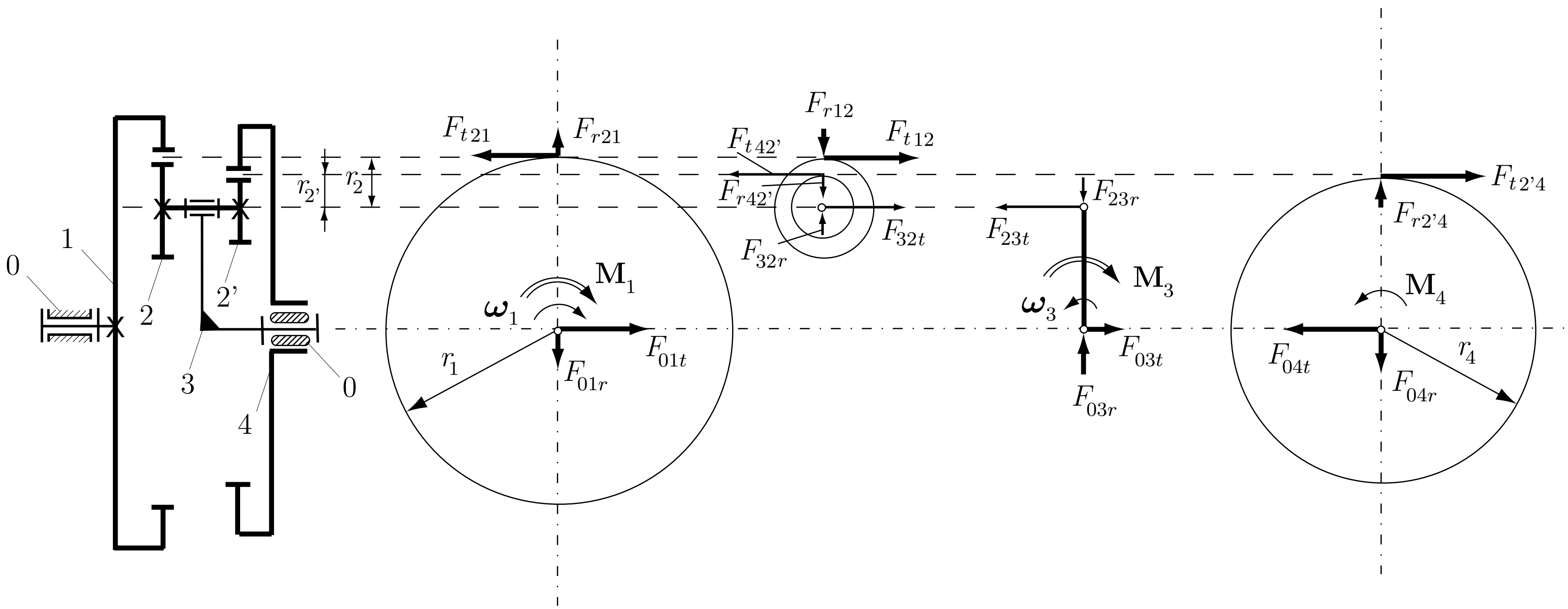

0
1
2
2'
3
4
0
$r_{2'}$
r_2
F_{t21}
F_{r21}
$\mathbf{M}_1$
$\boldsymbol{\omega}_1$
r_1
F_{01r}
F_{01t}
F_{r12}
F_{t12}
$F_{t42'}$
$F_{r42'}$
F_{32t}
F_{32r}
F_{23r}
F_{23t}
$\mathbf{M}_3$
$\boldsymbol{\omega}_3$
F_{03t}
F_{03r}
$F_{t2'4}$
$F_{r2'4}$
$\mathbf{M}_4$
F_{04t}
F_{04r}
r_4

Figure 2.25

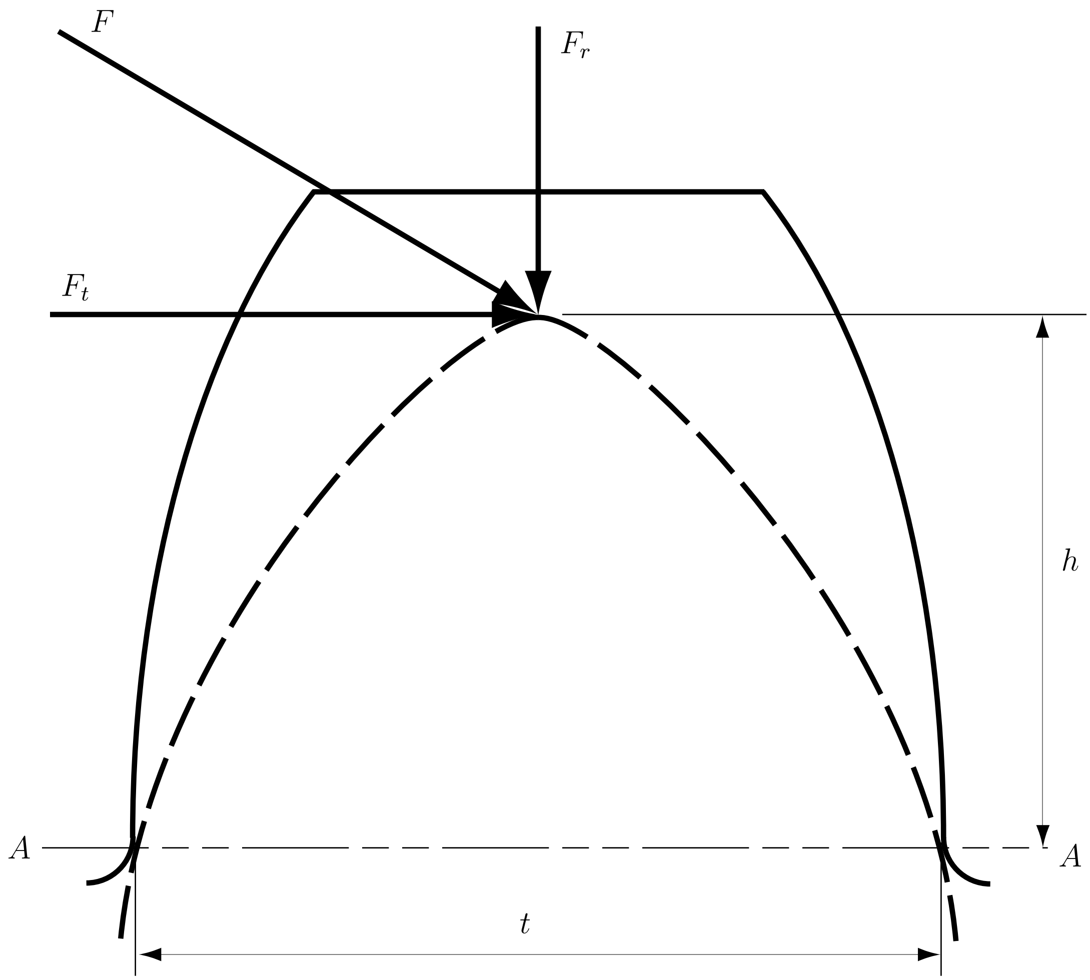


Figure 2.26

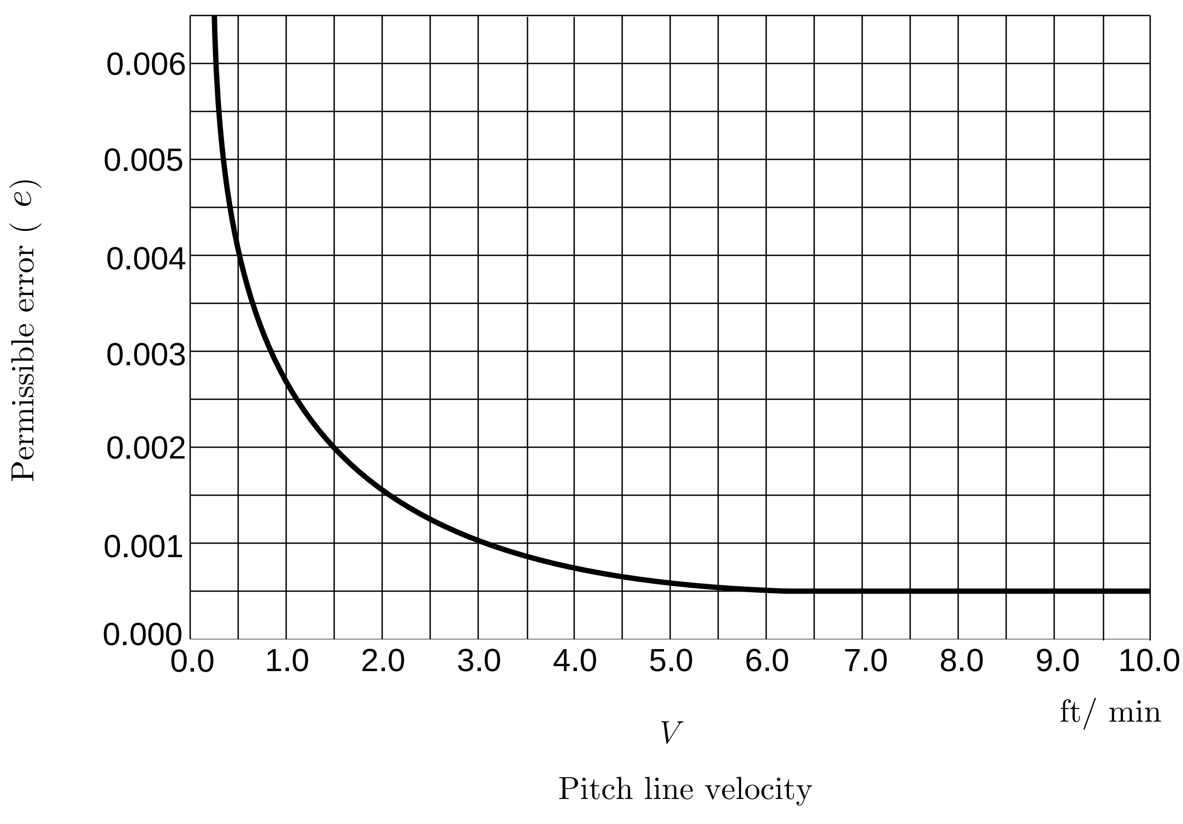

Permissible error (e)
0.006
0.005
0.004
0.003
0.002
0.001
0.000
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
ft/ min
V
Pitch line velocity

Figure 2.27

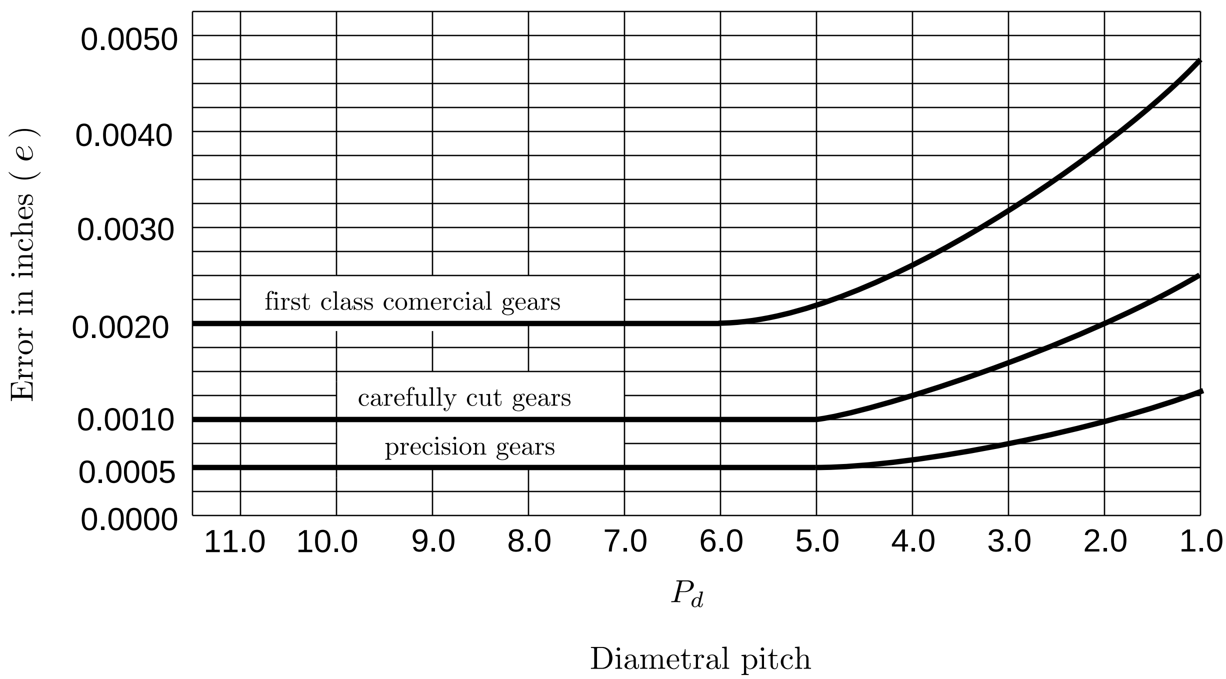

0.0050
0.0040
0.0030
0.0020
0.0010
0.0005
0.0000
Error in inches (e)
first class comercial gears
carefully cut gears
precision gears
11.0
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
P_d
Diametral pitch

Figure 2.28