Contents

3 Position Analysis

3.1 Absolute Cartesian Method

The position analysis of a kinematic chain requires the determination of the joint positions and/or the position of the center of gravity (CG) of the link. A planar link with the end nodes A and B is considered in Fig. 3.1. Let $(x_A,\, y_A)$ be the coordinates of the joint A with respect to the reference frame xOy, and $(x_B,\, y_B)$ be the coordinates of the joint B with the same reference frame. Using Pythagoras the following relation can be written

$$(x_B - x_A)^2 + (y_B - y_A)^2 = AB^2 = L_{AB}^2, \tag{3.1}$$

where L_{AB} is the length of the link AB.
Let ϕ be the angle of the link AB with the horizontal axis Ox. Then, the slope m of the link AB is defined as

$$m = \tan\phi = \frac{y_B - y_A}{x_B - x_A}. \tag{3.2}$$

Let b be the intercept of AB with the vertical axis Oy. Using the slope m and the y intercept b, the equation of the straight link (line), in the plane, is

$$y = m\,x + b, \tag{3.3}$$

where x and y are the coordinates of any point on this link.

Two lines are perpendicular to each other if and only if the slope of one is the negative reciprocal of the slope of the other. Thus if m and n are the slopes of two perpendicular lines

$$m = -\frac{1}{n} \quad \text{and} \quad m\,n = -1. \tag{3.4}$$

If two distinct points $A(x_A,\, y_A)$ and $B(x_B,\, y_B)$ are on a straight line then the equation of the straight line can be written in the forms

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} \quad \text{and} \quad \begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0. \tag{3.5}$$

Given two points $P(x_P,\, y_P)$ and $Q(x_Q,\, y_Q)$ and a real number k, $k \in \mathbf{R} - \{-1\}$, the coordinates of a point $R(x_R,\, y_R)$ on the line segment PQ, whose distance from P bears to the distance from R to Q the ratio k ($MR = k\,RQ$), are

$$x_R = \frac{x_P + k\,x_Q}{1+k} \quad \text{and} \quad y_R = \frac{y_P + k\,y_Q}{1+k}. \tag{3.6}$$

The symbol $\in$ means "belongs to".

For $k = 1$ the above formulas become

$$x_R = \frac{x_P + x_Q}{2} \quad \text{and} \quad y_R = \frac{y_P + y_Q}{2}. \tag{3.7}$$

These give the coordinates of the midpoint of the interval from P to Q.

For $k > 0$ the point R is interior to the segment PQ and for $k < 0$ the point R is exterior to the segment PQ.

For a link with a translational joint, Fig. 3.2, the sliding direction (Δ) is given by the equation

$$x\,\cos\alpha + y\,\sin\alpha - p = 0, \tag{3.8}$$

where p is the distance from the origin O to the sliding line (Δ). The position function for the joint $A(x_A,\, y_A)$ is

$$x_A\,\cos\alpha + y_A\,\sin\alpha - p = \pm\,d, \tag{3.9}$$

where d is the distance from A to the sliding line. The relation between the joint A and a point B on the sliding direction, $B \in (\Delta)$, is

$$(x_A - x_B)\,\sin\beta + (y_A - y_B)\,\cos\beta = \pm\,d, \tag{3.10}$$

where $\beta = \alpha + \dfrac{\pi}{2}$.

If $A\,x + B\,y + C = 0$ is the linear equation of the line (Δ) then the distance d is, Fig. 3.2

$$d = \frac{|A\,x_A + B\,y_A + C|}{\sqrt{A^2 + B^2}}\ . \tag{3.11}$$

For a driver link in rotational motion, Fig. 3.3(a), the following relations can be written

$$x_B = x_A + L_{AB}\cos\phi \quad \text{and} \quad y_B = y_A + L_{AB}\sin\phi. \tag{3.12}$$

From Fig. 3.3(b), for a driver link in translational motion one can have

$$\begin{aligned} x_B &= x_A + s\cos\phi + L_1\cos(\phi+\alpha),\\ y_B &= y_A + s\sin\phi + L_1\sin(\phi+\alpha). \end{aligned} \tag{3.13}$$

For the RRR dyad, Fig. 3.4, there are two quadratic equations of the form

$$\begin{aligned} (x_A - x_C)^2 + (y_A - y_C)^2 &= AC^2 = L_{AC}^2 = L_2^2,\\ (x_B - x_C)^2 + (y_B - y_C)^2 &= BC^2 = L_{BC}^2 = L_3^2, \end{aligned} \tag{3.14}$$

where the coordinates of the joint C, x_C and y_C, are the unknowns. With x_C and y_C determined, the angles ϕ_1 and ϕ_2 are computed from the relations

$$\tan\phi_1 = \frac{y_C - y_A}{x_C - x_A} \quad \text{and} \quad \tan\phi_2 = \frac{y_C - y_B}{x_C - x_B}. \tag{3.15}$$

The following relations can be written for the RRT dyad, Fig. 3.5(a)

$$\begin{aligned} (x_A - x_C)^2 + (y_A - y_C)^2 &= AC^2 = L_{AC}^2 = L_2^2,\\ (x_C - x_B)\sin\alpha - (y_C - y_B)\cos\alpha &= \pm\, h. \end{aligned} \tag{3.16}$$

From the two above equations the two unknowns x_C and y_C are computed. Figure 3.5(b) depicts the particular case for the RRT dyad when $L_3 = h = 0$ and the position equations are

$$(x_A - x_C)^2 + (y_A - y_C)^2 = L_2^2 \quad \text{and} \quad \tan\alpha = \frac{y_C - y_B}{x_C - x_B}. \tag{3.17}$$

For the RTR dyad, Fig. 3.6(a), the known data are: the positions of the joint A and B, x_A, y_A, x_B, y_B, the angle α and the length L_2 ($h = L_2 \sin\alpha$). There are four unknowns in the position of $C(x_C,\, y_C)$ and in the equation for the sliding line $(\Delta):\ y = m\,x + b$. The unknowns in the sliding line m and b are computed from the relations

$$L_2\sin\alpha = \frac{|m\,x_A - y_A + b|}{\sqrt{m^2+1}} \quad \text{and} \quad y_B = m\,x_B + b. \tag{3.18}$$

The coordinates of the joint C can be obtained using the equations

$$(x_A - x_C)^2 + (y_A - y_C)^2 = L_2^2 \quad \text{and} \quad y_C = m\,x_C + b. \tag{3.19}$$

In Fig. 3.6(b) the particular case when $L_1 = h = 0$ is shown and the position equation is

$$\tan\phi_2 = \tan\phi_3 = \frac{y_A - y_B}{x_A - x_B}. \tag{3.20}$$

To compute the coordinates of the joint C for the TRT dyad, Fig. 3.7, two equations can be written

$$\begin{aligned}(x_C - x_A)\sin\alpha - (y_C - y_A)\cos\alpha &= \pm\, d,\\ (x_C - x_B)\sin\beta - (y_C - y_B)\cos\beta &= \pm\, h.\end{aligned} \tag{3.21}$$

The input data are x_A, y_A, x_B, y_B, α, β, d, h and the output data are x_C, y_C.

Consider the mechanism shown in Fig. 3.8. The angle of the link 1 with the horizontal axis Ax is ϕ, $\phi = \angle(AB, Ax)$, and it is known. The following dimentions are given: $AB = l_1$, $CD = l_3$, $CE = l_4$, $AD = d$, and h is the distance from the slider 5 to the horizontal axis Ax.
The positions of the joints and the angles of the links will be calculated.

The origin of the system is at A, $A \equiv O$, $x_A = y_A = 0$. The coordinates of the rotational joint at B are

$$x_B = l_1 \sin\phi \quad \text{and} \quad y_B = l_1 \cos\phi.$$

The coordinates of the rotational joint at D are

$$x_D = d_1 \quad \text{and} \quad y_D = 0.$$

For the dyad DBC (RTR) the angle $\phi_2 = \phi_3$ of the link 2 or link 3 with the horizontal axis is calculated from the equation

$$\tan\phi_2 = \tan\phi_3 = \frac{y_B - y_D}{x_B - x_D} = \frac{l_1\cos\phi}{l_1\sin\phi - d}. \tag{3.22}$$

The joints $C(x_C, y_C)$ and D are on the link 3 (straight line DBC) and

$$\tan\phi_3 = \frac{y_C - y_D}{x_C - x_D} = \frac{y_C}{x_C - d}. \tag{3.23}$$

Equations (3.22)(3.23) give

$$\frac{y_B - y_D}{x_B - x_D} = \frac{y_C - y_D}{x_C - x_D} \quad \text{or} \quad \frac{l_1\cos\phi}{l_1\sin\phi - d} = \frac{y_C}{x_C - d}. \tag{3.24}$$

The length of the link 3 is $CD = l_3$ (constant) and the distance from C to D is

$$(x_C - x_D)^2 + (y_C - y_D)^2 = l_3^2 \quad \text{or} \quad (x_C - d)^2 + y_C^2 = l_3^2 \tag{3.25}$$

The coordinates x_C and y_C of the joint C result from Eq. (3.24) and Eq. (3.25).

Because of the quadratic equation, two solutions are obtained for x_C and y_C. For continuous motion of the mechanism there are constraint relations for choosing the correct solution: $x_C < x_B < x_D$ and $y_C > 0$.

For the last dyad CEE (RRT) a position function can be written for the joint E ($CE = l_4$ =constant)

$$(x_C - x_E)^2 + (y_C - h)^2 = l_4^2.$$

It results in values x_{E1} and x_{E2}, and it will be selected the solution $x_E > x_C$ for continuous motion of the mechanism.

The angle ϕ_4 of the link 4 with the horizontal axis is obtain from

$$\tan \phi_4 = \frac{y_C - y_E}{x_C - x_E} = \frac{y_C - h}{x_C - x_E}. \tag{3.26}$$

3.2 Vector Loop Method

First the independent closed loops are identified. A vector equation corresponding to each independent loop is established. The vector equation gives rise to two scalar equations, one for the horizontal axis x, and one for the vertical axis y.

For an open kinematic chain, Fig. 3.9, with general joints (pin joints, slider joints, etc.), a vector loop equation can be considered

$$\mathbf{r}_A + \mathbf{r}_1 + ...\mathbf{r}_n = \mathbf{r}_B, \tag{3.27}$$

or

$$\sum_{k=1}^{n} \mathbf{r}_k = \mathbf{r}_B - \mathbf{r}_A. \tag{3.28}$$

The vectorial Eq. (3.28) can be projected on the reference frame xOy

$$\sum_{k=1}^{n} r_k \cos \phi_k = x_B - x_A \quad \text{and} \quad \sum_{k=1}^{n} r_k \sin \phi_k = y_B - y_A. \tag{3.29}$$

RRR Dyad

The input data are: the position of A is (x_A, y_A), the position of B is (x_B, y_B), the length of AC is $L_{AC} = L_2$, and the length of BC is $L_{BC} = L_3$, Fig. 3.4.
The unknown data are: the position of $C(x_C, y_C)$, the angles ϕ_2 and ϕ_3.
The position equation for the RRR dyad is $\mathbf{r}_{AC} + \mathbf{r}_{CB} = \mathbf{r}_B - \mathbf{r}_A$, or

$$
\begin{aligned}
L_2 \cos \phi_2 + L_3 \cos(\phi_3 + \pi) &= x_B - x_A, \\
L_2 \sin \phi_2 + L_3 \sin(\phi_3 + \pi) &= y_B - y_A.
\end{aligned} \tag{3.30}
$$

The angles ϕ_2 and ϕ_3 can be computed from Eq. (3.30). The position of C can be computed using the known angle ϕ_2

$$
x_C = x_A + L_2 \cos \phi_2 \quad \text{and} \quad y_C = y_A + L_2 \sin \phi_2. \tag{3.31}
$$

RRT Dyad

The input data are: the position of A is (x_A, y_A), the position of B is (x_B, y_B), the length of AC is L_2,the length of CD is L_3, the angles α and β are constants, Fig. 3.5(a). The unknown data are: the position of $C(x_C, y_C)$, the angle ϕ_2, and the distance $r = DB$.
The vectorial equation for this kinematic chain is $\mathbf{r}_{AC} + \mathbf{r}_{CD} + \mathbf{r}_{DB} = \mathbf{r}_B - \mathbf{r}_A$, or

$$
\begin{aligned}
L_2 \cos \phi_2 + L_3 \cos(\alpha + \beta + \pi) + r \cos(\alpha + \pi) &= x_B - x_A, \\
L_2 \sin \phi_2 + L_3 \sin(\alpha + \beta + \pi) + r \sin(\alpha + \pi) &= y_B - y_A.
\end{aligned} \tag{3.32}
$$

One can compute r and ϕ_2 from Eq. (3.32). The position of C can be found with Eq. (3.31).

Particular Case $L_3 = 0$, Fig. 3.5(b).

In this case Eq. (3.32) can be written as

$$
\begin{aligned}
L_2 \cos \phi_2 + r \cos(\alpha + \pi) &= x_B - x_A, \\
L_2 \sin \phi_2 + r \sin(\alpha + \pi) &= y_B - y_A.
\end{aligned} \tag{3.33}
$$

RTR Dyad

The input data are: the position of A is (x_A, y_A), the position of B is (x_B, y_B),

the length of AC is L_2, and the angle α is constant, Fig. 3.6(a).
The unknown data are: the distance $r = CB$ and the angles ϕ_2 and ϕ_3.
The vectorial loop equation can be written as $\mathbf{r}_{AC} + \mathbf{r}_{CB} = \mathbf{r}_B - \mathbf{r}_A$, or

$$L_2 \cos\phi_2 + r\cos(\alpha + \phi_2 + \pi) = x_B - x_A,$$
$$L_2 \sin\phi_2 + r\sin(\alpha + \phi_2 + \pi) = y_B - y_A. \tag{3.34}$$

One can compute r and ϕ_2 from Eq. (3.34). The angle ϕ_3 can be written

$$\phi_3 = \phi_2 + \alpha. \tag{3.35}$$

Particular Case $L_2 = 0$, Fig. 3.6(b).
In this case from Eqs. (3.34) and (3.35) one can obtain

$$r\cos\phi_3 = x_B - x_A \quad \text{and} \quad r\sin\phi_3 = y_B - y_A. \tag{3.36}$$

The method is illustrated through the following examples. Figure 3.10(a) shows a four-bar mechanism (R-RRR mechanism) with link lengths r_0, r_1, r_2 and r_3. Find the angles ϕ_2 and ϕ_3 as functions of the driver link angle $\phi = \phi_1$.

The links are denoted as vectors $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, ($|\mathbf{r}_i| = r_i$, $i = 0, 1, 2, 3$), and the angles are measured counterclockwise from the x axis, Fig. 3.10(b). For the closed loop $ABCD$, a vectorial equation can be written

$$\mathbf{r}_0 + \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = \mathbf{0}. \tag{3.37}$$

By projecting the above vectorial equation onto, x and y, two scalar equations are obtained

$$r_0 + r_1\cos\phi_1 + r_2\cos\phi_2 - r_3\cos\phi_3 = 0, \tag{3.38}$$

and

$$r_1\sin\phi_1 + r_2\sin\phi_2 - r_3\sin\phi_3 = 0. \tag{3.39}$$

Equations (3.38) and (3.39) represent a set of nonlinear equations in two unknowns, ϕ_2 and ϕ_3. The solution of these two equations solves the position analysis.
Rearranging Eqs. (3.38) and (3.39)

$$r_2\cos\phi_2 = (r_3\cos\phi_3 - r_0) - r_1\cos\phi_1, \tag{3.40}$$

and

$$r_2 \sin\phi_2 = r_3 \sin\phi_3 - r_1 \sin\phi_1. \tag{3.41}$$

Squaring both sides of the above equations and adding

$$\begin{aligned} r_2^2 &= r_0^2 + r_1^2 + r_3^2 - 2r_3 \cos\phi_3 (r_0 + r_1 \cos\phi_1) \\ &\quad -2r_1 r_3 \sin\phi_1 \sin\phi_3 + 2r_0 r_1 \cos\phi_1, \end{aligned}$$

or

$$a \sin\phi_3 + b \cos\phi_3 = c, \tag{3.42}$$

where

$$a = \sin\phi_1, \quad b = \cos\phi_1 + (r_0/r_1), \quad \text{and}$$

$$c = (r_0/r_3)\cos\phi_1 + [(r_0^2 + r_1^2 + r_3^2 - r_2^2)/(2r_1 r_3)]. \tag{3.43}$$

Using the relations

$$\sin\phi_3 = 2\tan(\phi_3/2)[1 + \tan^2(\phi_3/2)],$$

and

$$\cos\phi_3 = [1 - \tan^2(\phi_3/2)]/[1 + \tan^2(\phi_3/2)], \tag{3.44}$$

in Eq. (3.42), the following relation is obtained

$$(b + c)\tan^2(\phi_3/2) - 2a\tan(\phi_3/2) + (c - b) = 0,$$

which gives

$$\tan(\phi_3/2) = (a \pm \sqrt{a^2 + b^2 - c^2})/(b + c). \tag{3.45}$$

Thus, for each given value of ϕ_1 and the length of the links, two distinct values of the angle ϕ_3 are obtained

$$\begin{aligned} \phi_{3(1)} &= 2\tan^{-1}[(a + \sqrt{a^2 + b^2 - c^2})/(b + c)], \\ \phi_{3(2)} &= 2\tan^{-1}[(a - \sqrt{a^2 + b^2 - c^2})/(b + c)]. \end{aligned} \tag{3.46}$$

The two values of ϕ_3 correspond to the two different positions of the mechanism.

The angle ϕ_2 can be eliminated from Eqs. (3.38) and (3.39) to give ϕ_1 in a similar way to that just described.

3.3 Examples

Example 3.1. Figure 3.11(a) shows a quick-return shaper mechanism. Given the lengths $AB = 0.20$ m, $AD = 0.40$ m, $CD = 0.70$ m, $CE = 0.30$ m, and the input angle $\phi = \phi_1$=45°, obtain the positions of all the other joints. The distance from the slider 5 to the horizontal axis Ax is $y_E = 0.35$ m.

Solution

The coordinates of the joint B are

$$x_B = AB\ \sin\phi = 0.20\ \sin 45^\circ = 0.141\,\text{m},$$
$$y_B = AB\ \cos\phi = 0.20\ \cos 45^\circ = 0.141\,\text{m}.$$

The vector diagram Fig. 3.11(b) is drawn by representing the RTR (BBD) dyad. The vector equation, corresponding to this loop, is written as

$$\mathbf{r}_B + \mathbf{r} - \mathbf{r}_D = \mathbf{0} \quad \text{or} \quad \mathbf{r}- = \mathbf{r}_D - \mathbf{r}_B,$$

where $\mathbf{r} = \mathbf{r}_{BD}$ and $|\mathbf{r}| = r$. Projecting the above vectorial equation on x and y axis two scalar equations are obtained

$$r\ \cos(\pi + \phi_3) = x_D - x_B = -0.141\,\text{m},$$
$$r\ \sin(\pi + \phi_3) = y_D - y_B = -0.541\,\text{m.},$$

The angle ϕ_3 is obtained by solving the system equations

$$\tan\phi_3 = \frac{y_D - y_B}{x_D - x_B} = \frac{0.541}{0.141} \implies \phi_3 = 75.36^\circ.$$

The distance r is

$$r = \frac{x_D - x_B}{\cos(\pi + \phi_3)} = 0.56\,\text{m}.$$

The coordinates of the joint C are

$$x_C = CD\ \sin\phi_3 = 0.17\,\text{m},$$
$$y_C = AB\ \cos\phi_3 - AD = 0.26\,\text{m}.$$

For the next dyad RRT (CEE), Fig. 3.11(c), one can write

$$CE\ \cos(\pi - \phi_4) = x_E - x_C,$$
$$CE\ \sin(\pi - \phi_4) = y_E - y_C.$$

Solving this system, the unknowns ϕ_4 and x_C are obtained

$$\phi_4 = 165.9^\circ \quad \text{and} \quad x_C = -0.114 \text{ m}.$$

Example 3.2. R-RTR-RRT Mechanism.
The planar R-RTR-RRT mechanism is considered in Fig. 3.12. The driver is the rigid link 1 (the element AB) and makes an angle $\phi = \phi_1 = \pi/6$ with the horizontal. The length of the links are AB=0.02 m, BC=0.03 m, and CD=0.06 m. The following dimensions are given: AE=0.05 m and L_a=0.02 m. Find the positions of the joints and the angles of the links.

Solution

Position of joint A

A cartesian reference frame $xOyz$ with the unit vectors $[\mathbf{\imath}, \mathbf{\jmath}, \mathbf{k}]$ is selected, as shown in Fig. 3.12. Since the joint A is in the origin of the reference system $A \equiv O$ then

$$x_A = y_A = 0.$$

Position of joint E

The the coordinates of the joint E are

$$x_E = -AE = -0.05 \text{ m} \quad \text{and} \quad y_E = 0.$$

Position of joint B

Because the joint A is fixed and the angle ϕ is known, the coordinates of the joint B are computed with

$$x_B = AB \cos\phi = 0.02 \cos\pi/6 = 0.017 \,\text{m},$$
$$y_B = AB \sin\phi = 0.02 \cos\pi/6 = 0.010 \,\text{m}.$$

Position of joint C

The joints E, B, and C are located on the same straight line EBC. The slope of this straight line is

$$m = \frac{y_B - y_E}{x_B - x_E} = \frac{y_C - y_E}{x_C - x_E} \quad \text{or} \quad \frac{0.010}{0.017 - (-0.05)} = \frac{y_C}{x_C - (-0.05)}. \tag{3.47}$$

The lengths of the link BC is constant and a quadratic equation can be written

$$(x_C - x_B)^2 + (y_C - y_B)^2 = BC^2 \quad \text{or}$$
$$(x_C - 0.017)^2 + (y_C - 0.01)^2 = 0.03^2. \tag{3.48}$$

Solving Eq. (3.47) and Eq. (3.48) two sets of solutions are found for the position of the joint C. These solutions are

$$x_{C_1} = -0.012 \text{ m}, \quad y_{C_1} = 0.005 \text{ m},$$
$$x_{C_2} = 0.046 \text{ m}, \quad y_{C_2} = 0.014 \text{ m}.$$

The points C_1 and C_2 are the intersections of the circle of radius BC (with its center at B) with the straight line EC, as shown in Fig. 3.13. To determine the position of the joint C for this position of the mechanism ($\phi = \pi/6$), an additional constraint condition is needed: $x_C > x_B$. With this constraint the coordinates of joint C have the following numerical values

$$x_C = x_{C_2} = 0.046 \text{ m} \quad \text{and} \quad y_C = y_{C_2} = 0.014 \text{ m}.$$

Position of joint D

The $x-$coordinate of D is $x_D = L_a = 0.02$ m. The lengths of the link CD is constant and a quadratic equation can be written

$$(x_D - x_C)^2 + (y_D - y_C)^2 = CD^2 \quad \text{or}$$
$$(0.02 - 0.046)^2 + (y_C - 0.014)^2 = 0.06^2. \tag{3.49}$$

Solving Eq. (3.52) two sets of solutions are found for the position of the joint D. These solutions are

$$y_{D_1} = -0.039 \text{ m} \quad \text{and} \quad y_{D_2} = 0.067 \text{ m}.$$

The points D_1 and D_2 are the intersections of the circle of radius CD (with its center at C) with the vertical line $x = L_a$, as shown in Fig. 3.14. To determine the correct position of the joint D for the angle $\phi = \pi/6$, an additional constraint condition is needed: $y_D < y_C$. With this constraint the coordinates of joint D are

$$x_D = 0.02 \text{ m} \quad \text{and} \quad y_D = y_{D_1} = -0.039 \text{ m}.$$

Angle ϕ_2

The angle of the link 2 (or link 3) with the horizontal is calculated from the slope of the straight line EB

$$\phi_2 = \phi_3 = \arctan \frac{y_B - y_E}{x_B - x_E} = \frac{0.010}{0.017 - (-0.050)} = 0.147 \text{ rad} = 8.449°.$$

Angle ϕ_4

The angle of the link 4 with the horizontal is obtained from the slope of the straight line CD

$$\phi_4 = \arctan \frac{y_C - y_D}{x_C - x_D} = \frac{0.014 + 0.039}{0.046 - 0.020} = 1.104 \text{ rad} = 63.261^\circ.$$

Example 3.3. R-TRR-RRT Mechanism.
The mechanism is shown in Fig. 3.15. The following data are given: $AC = 0.100$ m, $BC = 0.300$ m, $BD = 0.900$ m, and $L_a = 0.100$ m. If the angle of the link 1 with the horizontal axis is $\phi = 45^\circ$ find the positions of the joint D.

Solution

Position of joint A

A cartesian reference frame with the origin at A is selected. The coordinates of the joint A are

$$x_A = y_A = 0.$$

Position of joint C

The coordinates of the joint C are

$$x_C = AC = 0.100 \text{ m} \quad \text{and} \quad y_C = 0.$$

Position of joint B

The slope of the line AB is

$$\tan\phi = \frac{y_B}{x_B} \quad \text{or} \quad \tan 45^\circ = \frac{y_B}{x_B}. \tag{3.50}$$

The lengths of the link BC is constant the following equation can be written

$$(x_B - x_C)^2 + (y_B - y_C)^2 = BC^2 \quad \text{or} \quad (x_B - 0.1)^2 + y_B^2 = 0.3^2. \tag{3.51}$$

Equations (3.50) and (3.51) form a system of two equations with the unknowns x_B and y_B. The following numerical results are obtained

$$x_{B_1} = -0.156 \text{ m}, \quad y_{B_1} = -0.156 \text{ m},$$
$$x_{B_2} = 0.256 \text{ m}, \quad y_{B_2} = 0.256 \text{ m}.$$

To determine the correct position of the joint B for the angle $\phi = 45°$, an additional constraint condition is needed: $x_B > x_C$. With this constraint the coordinates of joint B are

$$x_B = x_{B_2} = 0.256 \text{ m} \quad \text{and} \quad y_B = y_{B_2} = 0.256 \text{ m}.$$

Position of joint D

The slider 5 has a translational motion in the horizontal direction and $y_D = L_a$. There is only one unknown, x_D, for the joint D. The following expression can be written

$$(x_B - x_D)^2 + (y_B - y_D)^2 = BD^2 \quad \text{or}$$
$$(0.256 - x_D)^2 + (0.256 - 0.1)^2 = 0.9^2 \tag{3.52}$$

Solving Eq. (3.52), two numerical values are obtained

$$x_{D_1} = -0.630 \text{ m}, \quad x_{D_2} = 1.142 \text{ m}. \tag{3.53}$$

For continuous motion of the mechanism, a geometric constraint $x_D > x_B$ has to be selected. Using this relation the coordinates of the joint D are

$$x_D = 1.142 \text{ m} \quad \text{and} \quad y_D = 0.100 \text{ m}.$$

3.4 Problems

3.1 The following data are given for the four-bar mechanism shown in Fig. 3.16: $AB = CD = 0.04$ m and $AD = BC = 0.09$ m. Find the trajectory of the point M located on the link BC, for the case a) $BM = MC$, and b) $MC = 2\,BM$.

3.2 The planar four-bar mechanism depicted in Fig. 3.17 has dimensions $AB = 0.03$ m, $BC = 0.065$ m, $CD = 0.05$ m, $BM = 0.09$ m, and $CM = 0.12$ m. Find the trajectory described by the point M.

3.3 The mechanism shown in Fig. 3.18 has dimensions $AB = 0.03$ m, $BC = 0.12$ m, $CD = 0.12$ m, $DE = 0.07$ m, $CF = 0.17$ m, $R_1 = 0.04$ m, $R_4 = 0.08$ m, $L_a = 0.025$ m, and $L_b = 0.105$ m. Find the trajectory of the joint C.

3.4 The planar R-RRR-RRT mechanism considered is depicted in Fig. 3.19. The driver link is the rigid link 1 (the element AB). The following data are given: AB=0.150 m, BC=0.400 m, CD=0.370 m, CE=0.230 m, EF=CE, L_a=0.300 m, L_b=0.450 m, and L_c=CD. The angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 45°$. Find the positions of the joints and the angles of the links.

3.5 The R-RRR-RTT mechanism is shown in Fig. 3.20. The following data are given: AB=0.080 m, BC=0.350 m, CD=0.150 m, CE=0.200 m, L_a=0.200 m, L_b=0.350 m, L_c=0.040 m. The angle of the driver element (link AB) with the horizontal axis is $\phi = 135°$. Determine the positions of the joints and the angles of the links.

3.6 The mechanism shown in Fig. 3.21 has the following dimensions: $AB =$ 40 mm, $AD = 150$ mm, $BC = 100$ mm, $CE = 30$ mm, $EF = 120$ mm, and $a = 90$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 30°$. Find the positions of the joints and the angles of the links.

3.7 The dimensions for the mechanism shown in Fig. 3.22 are: $AB =$ 250 mm, $BD = 670$ mm, $DE = 420$ mm, $AE = 640$ mm, $BC =$ 240 mm, $CD = 660$ mm, $CF = 850$ mm, and $b = 170$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 30°$. Find the positions of the joints and the angles of the links.

3.8 The mechanism in Fig. 3.23 has the dimensions: $AB = 120$ mm, $AC = 60$ mm, $BD = 240$ mm, $DE = 330$ mm, $EF = 190$ mm, $L_a = 300$ mm, and $L_b = 70$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 150°$. Find the positions of the joints and the angles of the links.

3.9 The dimensions for the mechanism shown in Fig. 3.24 are: $AB = 100$ mm, $BC = 260$ mm, $AD = 240$ mm, $CD = 140$ mm, $DE = 80$ mm, $EF = 250$ mm, and $L_a = 20$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 45°$. Find the positions of the joints and the angles of the links.

3.10 The mechanism in Fig. 3.25 has the dimensions: $AB = 150$ mm, $AC = 450$ mm, $BD = 700$ mm, $L_a = 100$ mm, and $L_b = 200$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 120°$. Find the positions of the joints and the angles of the links.

3.11 Figure 3.26 shows a mechanism with the following dimensions: $AB = 180$ mm, $BD = 700$ mm, and $L_a = 210$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 135°$. Find the positions of the joints and the angles of the links.

3.12 The mechanism in Fig. 3.27 has the dimensions: $AB = 100$ mm, $AC = 240$ mm, $BD = 400$ mm, $DE = 200$ mm, $EF = 135$ mm, $L_a = 35$ mm, and $L_b = 170$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 150°$. Find the positions of the joints and the angles of the links.

3.13 Figure 3.28 shows a mechanism with the following dimensions: $AB = 120$ mm, $BC = 450$ mm, $CD = DE = 180$ mm, $EF = 300$ mm, $L_a = 450$ mm, $L_b = 150$ mm, and $L_c = 140$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 120°$. Find the positions of the joints and the angles of the links.

3.14 Figure 3.29 shows a mechanism with the following dimensions: $AB = 140$ mm, $BC = 650$ mm, $CE = 250$ mm, $CD = 400$ mm, $EF = 350$ mm, $L_a = 370$ mm, $L_b = 550$ mm, and $L_c = 700$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 150°$. Find the positions of the joints and the angles of the links.

3.15 Figure 3.30 shows a mechanism with the following dimensions: $AB = 60$ mm, $BC = 160$ mm, $CF = 150$ mm, $CD = 60$ mm, $DE = 180$ mm, $L_a = 210$ mm, $L_b = 120$ mm, and $L_c = 65$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 30°$. Find the positions of the joints and the angles of the links.

3.16 Figure 3.31 shows a mechanism with the following dimensions: $AB = 20$ mm, $BC = 50$ mm, $AD = 25$ mm, and $BE = 60$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 60°$. Find the positions of the joints and the angles of the links.

3.17 The dimensions of the mechanism shown in Fig. 3.32 are: $AB = 150$ mm, $BC = 300$ mm, $BE = 600$ mm, $CE = 850$ mm, $CD = 330$ mm, $EF = 1200$ mm, $L_a = 350$ mm, $L_b = 200$ mm, and $L_c = 100$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 120°$. Find the positions of the joints and the angles of the links.

3.18 The dimensions of the mechanism shown in Fig. 3.33 are: $AB = 150$ mm, $AC = 220$ mm, $CD = 280$ mm, $DE = 200$ mm, and $L_a = 230$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 60°$. Find the positions of the joints and the angles of the links.

3.19 The dimensions of the mechanism shown in Fig. 3.34 are: $AB = 200$ mm, $AC = 60$ mm, $CD = 200$ mm, and $DE = 500$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 45°$. Find the positions of the joints and the angles of the links.

3.20 The dimensions of the mechanism shown in Fig. 3.35 are: $AB = 120$ mm, $AC = 200$ mm, $CD = 380$ mm, and $b = 450$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 30°$. Find the positions of the joints and the angles of the links.

3.21 The dimensions of the mechanism shown in Fig. 3.36 are: $AB = 160$ mm, $AC = 90$ mm, and $CD = 160$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 30°$. Find the positions of the joints and the angles of the links.

3.22 The dimensions of the mechanism shown in Fig. 3.37 are: $AB = 100$ mm, $AC = 280$ mm, $BD = L_a = 470$ mm, and $DE = 220$ mm.

The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 30°$. Find the positions of the joints and the angles of the links.

3.23 The dimensions of the mechanism shown in Fig. 3.38 are: $AB = 250$ mm, $AD = 700$ mm, $BC = 300$ mm, and $a = 650$ mm. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 145°$. Find the positions of the joints and the angles of the links.

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Figure captions

Figure 3.1 Planar link with two end nodes A and B

Figure 3.2 Link with a translational joint

Figure 3.3 Driver link: (a) in rotational motion, and (b) in translational motion

Figure 3.4 RRR dyad

Figure 3.5. (a) RRT dyad; (b) RRT dyad, particular case, $L_3 = h = 0$

Figure 3.6. (a) RTR dyad; (b) RTR dyad, particular case, $L_2 = h = 0$

Figure 3.7 TRT dyad

Figure 3.8 Planar mechanism

Figure 3.9 Kinematic chain

Figure 3.10 (a) Four-bar mechanism, and (b) closed loop $ABCD$

Figure 3.11 (a) Quick-return shaper mechanism, (b) vector diagram representing the RTR (BBD) dyad, and (c) vector diagram representing the RRT (CEE) dyad

Figure 3.12 R-RTR-RRT mechanism

Figure 3.13 Position of joint C

Figure 3.14 Position of joint D

Figure 3.15 R-TRR-RRT mechanism

Figure 3.16 Four-bar mechanism for Problem 3.1

Figure 3.17 Four-bar mechanism for Problem 3.2

Figure 3.18 Mechanism for Problem 3.3

Figure 3.19 R-RRR-RRT mechanism for Problem 3.4

Figure 3.20 R-RRR-RTT mechanism for Problem 3.5

Figure 3.21 Mechanism for Problem 3.6

Figure 3.22 Mechanism for Problem 3.7

Figure 3.23 Mechanism for Problem 3.8

Figure 3.24 Mechanism for Problem 3.9

Figure 3.25 Mechanism for Problem 3.10

Figure 3.26 Mechanism for Problem 3.11

Figure 3.27 Mechanism for Problem 3.12

Figure 3.28 Mechanism for Problem 3.13

Figure 3.29 Mechanism for Problem 3.14

Figure 3.30 Mechanism for Problem 3.15

Figure 3.31 Mechanism for Problem 3.16

Figure 3.32 Mechanism for Problem 3.17

Figure 3.33 Mechanism for Problem 3.18

Figure 3.34 Mechanism for Problem 3.19
Figure 3.35 Mechanism for Problem 3.20
Figure 3.36 Mechanism for Problem 3.21
Figure 3.37 Mechanism for Problem 3.22
Figure 3.38 Mechanism for Problem 3.23