Contents

5 DYNAMICS OF THE RIGID BODY

5.1 Introduction

A rigid body is said to move in two dimensions when every particle of it moves parallel to a fixed plane. A rigid lamina moving in its own plane is a particular case of two-dimensional motion. In general the motion of a rigid lamina consists of a translation and a rotation and there is a point in its plane which is instantaneously at restt. Knowing the instantaneous angular velocity of the lamina and the linear velocity of its centroid, one can determine the velocity of any other particle of it at the considered instant. For the planar motion of a rigid two coordinates are required to determine the position of the centroid and an angular coordinate is required to determine the orientation of the lamina. Thus three independent variables are involved in the instantaneous motion of the lamina; two linear velocity components of its centroid and an angular velocity of the distribution.

Knowing the magnitudes and positions of forces acting on a rigid body, one can find 1) the acceleration of its centroid; 2) the angular acceleration.

5.2 Equation of motion for the mass center

Newton stated that the total force on a particle is equal to the rate of change of its linear momentum, which is the product of its mass and velocity. Newton's second law is postulated for a particle, or small element of matter. One may show that the total external force on an arbitrary rigid body is equal to the product of its mass and the acceleration of its mass center. An arbitrary rigid body with the mass m may be divided into N particles. The position vector of the i particle is $\mathbf{r}_i$ and the mass of the i particle is m_i, Fig. 5.1

$$m = \sum_{i=1}^{N} m_i.$$

The position of the mass center of the rigid body is

$$\mathbf{r}_C = \frac{\sum\limits_{i=1}^{N} m_i \mathbf{r}_i}{m}. \tag{5.1}$$

Taking two time derivatives of Eq. (5.1), one may obtain

$$\sum_{i=1}^{N} m_i \frac{d^2\mathbf{r}_i}{dt^2} = m\frac{d^2\mathbf{r}_C}{dt^2} = m\mathbf{a}_C, \tag{5.2}$$

where $\mathbf{a}_C$ is the acceleration of the center of mass of the rigid body.

Let $\mathbf{f}_{ij}$ be the force exerted on the j particle by the i particle. Newton's third law states that the j particle exerts a force on the i particle of equal magnitude and opposite direction Fig. 5.1

$$\mathbf{f}_{ji} = -\mathbf{f}_{ij}.$$

Newton's second law for the i particle is

$$\sum_j \mathbf{f}_{ji} + \mathbf{F}_i^{ext} = m_i \frac{d^2\mathbf{r}_i}{dt^2}, \tag{5.3}$$

where $\mathbf{F}_i^{ext}$ is the external force on the i particle. Equation (5.3) may be written for each particle of the rigid body. Summing the resulting equations from $i = 1$ to N, one may obtain

$$\sum_i \sum_j \mathbf{f}_{ji} + \sum_i \mathbf{F}_i^{ext} = m\mathbf{a}_C, \tag{5.4}$$

The sum of the internal forces on the rigid body, is zero (Newton's third law)

$$\sum_i \sum_j \mathbf{f}_{ji} = \mathbf{0}.$$

The term $\sum_i \mathbf{F}_i^{ext}$ is the sum of the external forces on the rigid body

$$\sum_i \mathbf{F}_i^{ext} = \sum \mathbf{F}.$$

One may conclude that the sum of the external forces equals the product of the mass and the acceleration of the center of mass (Fig. 5.2)

$$m\mathbf{a}_C = \sum \mathbf{F}. \tag{5.5}$$

If the external forces acting on a rigid body in planar motion are known one may use Eqs. (5.5) to determine the acceleration of the center of mass of the rigid body.

Resolving the sum of the external forces into cartesian rectangular components one can write

$$\sum \mathbf{F} = \sum F_x \,\imath + \sum F_y \,\jmath + \sum F_z \,\mathbf{k}. \tag{5.6}$$

The position vector of the mass center in terms of the cartesian components is

$$\mathbf{r}_C = x_C(t)\,\imath + y_C(t)\,\jmath + z_C(t)\,\mathbf{k}. \tag{5.7}$$

Newton's second law for the rigid body is

$$m\ddot{\mathbf{r}}_C = \sum \mathbf{F},$$

or

$$m\ddot{x}_C = \sum F_x, \;\; m\ddot{y}_C = \sum F_y, \;\; m\ddot{z}_C = \sum F_z. \tag{5.8}$$

5.3 Angular momentum principle for a system of particles

An arbitrary system with the mass m may be divided into N particles P_1, $P_2, ..., P_N$. The position vector of the i particle is $\mathbf{r}_i = \mathbf{OP}_i$ and the mass of the i particle is m_i, Fig. 5.3. The position of the mass center, C, of the system is $\mathbf{r}_C = \sum_{i=1}^{N} m_i\mathbf{r}_i/m$. The position of the the particle P_i of the system relative to O is

$$\mathbf{r}_i = \mathbf{r}_C + \mathbf{CP}_i. \tag{5.9}$$

Multiplying Eq. (5.9) by m_i, summing from 1 to N, one may find that

$$\sum_{i=1}^{N} m_i\mathbf{CP}_i = \mathbf{0}. \tag{5.10}$$

The total angular momentum of the system about its mass center C, is the sum of the angular momenta of the particles about C

$$\mathbf{H}_C = \sum_{i=1}^{N} \mathbf{CP}_i \times m_i\mathbf{v}_i, \tag{5.11}$$

where $\mathbf{v}_i = \dfrac{d\mathbf{r}_i}{dt}$ is the velocity of the particle P_i.
The total angular momentum of the system about O is the sum of the angular momenta of the particles

$$\mathbf{H}_O = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_{i=1}^{N} (\mathbf{r}_C + \mathbf{CP}_i) \times m_i \mathbf{v}_i = \mathbf{r}_C \times m \mathbf{v}_C + \mathbf{H}_C, \quad (5.12)$$

or the total angular momentum about O is the sum of the angular momentum about O due to the velocity $\mathbf{v}_C$ of the mass center of the system and the total angular momentum about the center of mass, Fig. 5.4.

Newton's second law for the i particle is

$$\sum_j \mathbf{f}_{ji} + \mathbf{F}_i^{ext} = m_i \frac{d\mathbf{v}_i}{dt},$$

and the cross product with the position vector $\mathbf{r}_i$, and sum from $i = 1$ to N gives

$$\sum_i \sum_j \mathbf{r}_i \times \mathbf{f}_{ji} + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \sum_i \mathbf{r}_i \times \frac{d}{dt}(m_i \mathbf{v}_i). \quad (5.13)$$

The first term on the left side of Eq. (5.13) is the sum of the moments about O due to internal forces, and

$$\mathbf{r}_i \times \mathbf{f}_{ji} + \mathbf{r}_j \times \mathbf{f}_{ij} = \mathbf{r}_i \times \mathbf{f}_{ji} + \mathbf{r}_j \times (-\mathbf{f}_{ji}) = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ji} = \mathbf{0}.$$

The term vanishes if the internal forces between each pair of particles are equal, opposite, and directed along the straight line between the two particles, Fig. 5.1.
The second term on the left side of Eq. (5.13)

$$\sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \sum \mathbf{M}_O,$$

represents the sum of the moments about O due to external forces and couples. The term on the right side of Eq. (5.13) is

$$\sum_i \mathbf{r}_i \times \frac{d}{dt}(m_i \mathbf{v}_i) = \sum_i \left[\frac{d}{dt}(\mathbf{r}_i \times m_i \mathbf{v}_i) - \mathbf{v}_i \times m_i \mathbf{v}_i \right] = \frac{d\mathbf{H}_O}{dt}, \quad (5.14)$$

which represents the rate of change of the total angular momentum of the system about the point O.
Equation. (5.13) may be rewritten as

$$\frac{d\mathbf{H}_O}{dt} = \sum \mathbf{M}_O. \tag{5.15}$$

The rate of change of the angular momentum about O equals the sum of the moments about O due to external forces and couples.
Using Eqs. (5.12) and (5.15) the following result is obtained

$$\sum \mathbf{M}_O = \frac{d}{dt}(\mathbf{r}_C \times m\mathbf{v}_C + \mathbf{H}_C) = \mathbf{r}_C \times m\mathbf{a}_C + \frac{d\mathbf{H}_C}{dt}, \tag{5.16}$$

where $\mathbf{a}_C$ is the acceleration of the mass center.
If the point O is coincident with the mass center at the present instant $C = O$, then $\mathbf{r}_C = \mathbf{0}$ and Eq. (5.16) becomes

$$\frac{d\mathbf{H}_C}{dt} = \sum \mathbf{M}_C. \tag{5.17}$$

The rate of change of the angular momentum about the mass center equals the sum of the moments about the mass center.

5.4 Equations of motion for general planar motion

An arbitrary rigid body with the mass m may is divided into N particles P_i, $i = 1, 2, ..., N$. The position vector of the P_i particle is $\mathbf{r}_i = \mathbf{OP}_i$ and the mass of the particle is m_i. Figure 5.5 represents the rigid body moving with general planar motion in the (X, Y) plane. The origin of the cartesian reference frame is O. The mass center C, of the rigid body is located in the plane of the motion, $C \in (X, Y)$.

Let $d_O = OZ$ be the axis through the fixed origin point O that is perpendicular to the plane of the motion of the rigid body X, Y, $d_O \perp (X, Y)$. Let $d_C = Czz$ be the parallel axis through the mass center C, $d_C || d_O$. The rigid body has a general planar motion and one may express the angular velocity vector as $\boldsymbol{\omega} = \omega\mathbf{k}$. The unit vector of the $d_C = Czz$ axis is $\mathbf{k}$.

The velocity of the P_i particle relative to the mass center is

$$\frac{d\mathbf{R}_i}{dt} = \omega\mathbf{k} \times \mathbf{R}_i,$$

where $\mathbf{R}_i = \mathbf{CP}_i$. The sum of the moments about O due to external forces and couples is

$$\sum \mathbf{M}_O = \frac{d\mathbf{H}_O}{dt} = \frac{d}{dt}[(\mathbf{r}_C \times m\mathbf{v}_C) + \mathbf{H}_C], \tag{5.18}$$

where

$$\mathbf{H}_C = \sum_i [\mathbf{R}_i \times m_i(\omega\mathbf{k} \times \mathbf{R}_i)],$$

is the angular momentum about d_C. The magnitude of the angular momentum about d_C is

$$\begin{aligned} &H_C = \mathbf{H}_C \cdot \mathbf{k} = \sum_i [\mathbf{R}_i \times m_i(\omega\mathbf{k} \times \mathbf{R}_i)] \cdot \mathbf{k} = \\ &\sum_i m_i\,[(\mathbf{R}_i \times \mathbf{k}) \times \mathbf{R}_i)] \cdot \mathbf{k}\,\omega = \sum_i m_i\,[(\mathbf{R}_i \times \mathbf{k}) \cdot (\mathbf{R}_i \times \mathbf{k})]\,\omega = \\ &\sum_i m_i\,|\mathbf{R}_i \times \mathbf{k}|^2\,\omega = \sum_i m_i r_i \omega, \end{aligned} \tag{5.19}$$

where the term $|\mathbf{k} \times \mathbf{R}_i| = r_i$ is the perpendicular distance from d_C to the P_i particle. The identity

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

has been used.
The summation $\sum_i m_i r_i^2$, which also may be written $\int mr^2$, is defined as mass moment of inertia I_{Czz} of the body about the z-axis through C

$$I_{Czz} = I_{zz} = \sum_i m_i r_i^2.$$

The mass moment of inertia $I_{Czz} = I_{zz}$ is a constant property of the body and is a measure of the rotational inertia or resistance to change in angular velocity due to the radial distribution of the rigid body mass around z-axis through C.

Equation (5.19) defines the angular momentum of the rigid body about d_C (z-axis through C)

$$H_C = I_{zz}\,\omega \ \text{ or } \ \mathbf{H}_C = I_{zz}\,\omega\,\mathbf{k} = I_{zz}\,\boldsymbol{\omega}.$$

Substituting this expression into Eq. (5.18), one may obtain

$$\sum \mathbf{M}_O = \frac{d}{dt}[(\mathbf{r}_C \times m\mathbf{v}_C) + I_{zz}\boldsymbol{\omega}] = (\mathbf{r}_C \times m\mathbf{a}_C) + I_{zz}\boldsymbol{\alpha}. \qquad (5.20)$$

If the fixed axis d_O is coincident with d_C at the present instant, $\mathbf{r}_C = \mathbf{0}$, and from Eq. (5.20) one may obtain

$$I_{zz}\,\boldsymbol{\alpha} = \sum \mathbf{M}_C, \qquad (5.21)$$

or

$$I_{zz}\,\alpha\,\mathbf{k} = \sum M_C\,\mathbf{k}, \qquad (5.22)$$

The product of the moment of inertia about z-axis through C and the angular acceleration equals the sum of the moments about z-axis through C.
For general planar motion the angular acceleration is

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \ddot{\theta}\mathbf{k}, \qquad (5.23)$$

where the angle θ describes the position, or orientation, of the rigid body about a fixed axis.

The relation between the sum of the moments of the external forces about the fixed point O and the product $I_{zz}\,\boldsymbol{\alpha}$ is given by Eq. (5.20)

$$\sum \mathbf{M}_O = \mathbf{r}_C \times m\mathbf{a}_C + I_{zz}\boldsymbol{\alpha}. \qquad (5.24)$$

The following expression can be written for the acceleration of C

$$\mathbf{a}_C = \boldsymbol{\alpha} \times \mathbf{r}_C - \omega^2\mathbf{r}_C. \qquad (5.25)$$

Equations (5.24) and Eq. (5.25) give

$$\begin{aligned}\sum \mathbf{M}_O &= \mathbf{r}_C \times m\,(\boldsymbol{\alpha} \times \mathbf{r}_C - \omega^2\mathbf{r}_C) + I_{zz}\boldsymbol{\alpha} = \\ &\quad m\,\mathbf{r}_C \times (\boldsymbol{\alpha} \times \mathbf{r}_C) + I_{zz}\boldsymbol{\alpha} = \\ &\quad m\,[(\mathbf{r}_C \cdot \mathbf{r}_C)\boldsymbol{\alpha} - (\mathbf{r}_C \cdot \boldsymbol{\alpha})\mathbf{r}_C] + I_{zz}\boldsymbol{\alpha} = \\ &\quad m\,\mathbf{r}_C^2\boldsymbol{\alpha} + I_{zz}\boldsymbol{\alpha} = (m\,r_C^2 + I_{zz})\boldsymbol{\alpha}. \qquad (5.26)\end{aligned}$$

According to parallel-axis theorem

$$I_{Ozz} = m\,r_C^2 + I_{zz},$$

where I_{Ozz} denotes the mass moment of inertia of the rigid body about z-axis through O. Therefore one can write the formula

$$I_{Ozz}\boldsymbol{\alpha} = \sum \mathbf{M}_O. \tag{5.27}$$

If the rigid body is a plate moving in the plane of motion (X, Y) the mass moment of inertia of the rigid body about z-axis through C becomes the polar mass moment of inertia of the rigid body about C, $I_{Czz} = I_{zz} = I_C$, and the mass moment of inertia of the rigid body about z-axis through O becomes the polar mass moment of inertia of the rigid body about O, $I_{Ozz} = I_O$. For this case the Eqs. (5.21) and (5.27) give

$$I_C\,\boldsymbol{\alpha} = \sum \mathbf{M}_C, \;\; I_O\,\boldsymbol{\alpha} = \sum \mathbf{M}_O. \tag{5.28}$$

The general equations of motion for a rigid body in plane motion are now written (Fig. 5.6)

$$m\ddot{\mathbf{r}}_C = \sum \mathbf{F}, \;\; I_{zz}\boldsymbol{\alpha} = \sum \mathbf{M}_C, \tag{5.29}$$

or using the cartesian components

$$m\ddot{x}_C = \sum F_x, \;\; m\ddot{y}_C = \sum F_y, \;\; I_{zz}\ddot{\theta} = \sum M_C. \tag{5.30}$$

5.5 D'Alembert's principle

Newton's second law may be writen as

$$\mathbf{F} + (-m\mathbf{a}_C) = \mathbf{0}, \;\text{ or }\; \mathbf{F} + \mathbf{F}_{in} = \mathbf{0}, \tag{5.31}$$

where the term $\mathbf{F}_{in} = -m\mathbf{a}_C$ is the *inertial force*. Newton's second law may be regarded it as an "equilibrium" equation.

Equation (5.20) relates the total moment about a fixed point O to the acceleration of the mass center and the angular acceleration

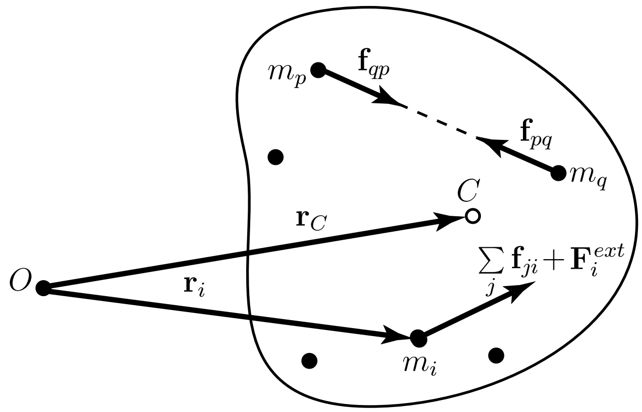
$$\sum \mathbf{M}_O = (\mathbf{r}_C \times m\mathbf{a}_C) + I_{zz}\boldsymbol{\alpha},$$

or

$$\sum \mathbf{M}_O + [\mathbf{r}_C \times (-m\mathbf{a}_C)] + (-I_{zz}\boldsymbol{\alpha}) = \mathbf{0}. \tag{5.32}$$

The term $\mathbf{M}_{in} = -I_{zz}\boldsymbol{\alpha}$ is the *inertial moment.* The sum of the moments about any point, including the moment due to the inertial force $-m\mathbf{a}$ acting at mass center and the inertial moment, equals zero.

The equations of motion for a rigid body are analogous to the equations for static equilibrium:
The sum of the forces equals zero and the sum of the moments about any point equals zero when the inertial forces and moments are taking into account This is called *D'Alembert's principle.*



m_p
$\mathbf{f}_{qp}$
$\mathbf{f}_{pq}$
m_q
C
$\mathbf{r}_C$
O
$\mathbf{r}_i$
$\sum_j \mathbf{f}_{ji} + \mathbf{F}_i^{ext}$
m_i

Figure 1

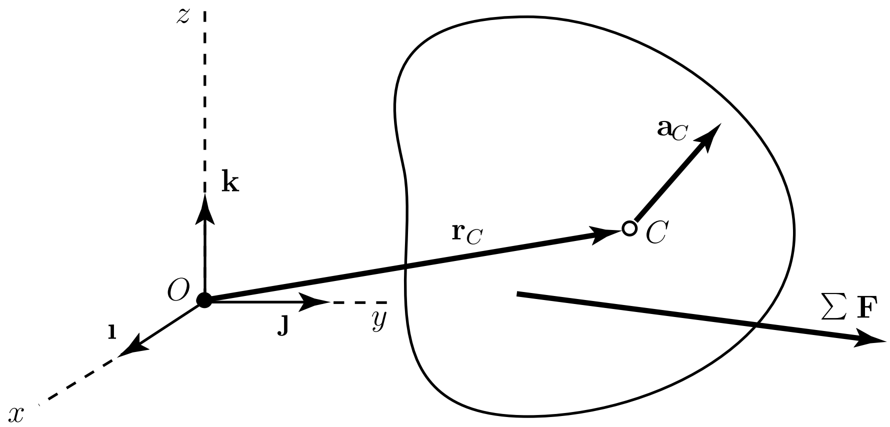
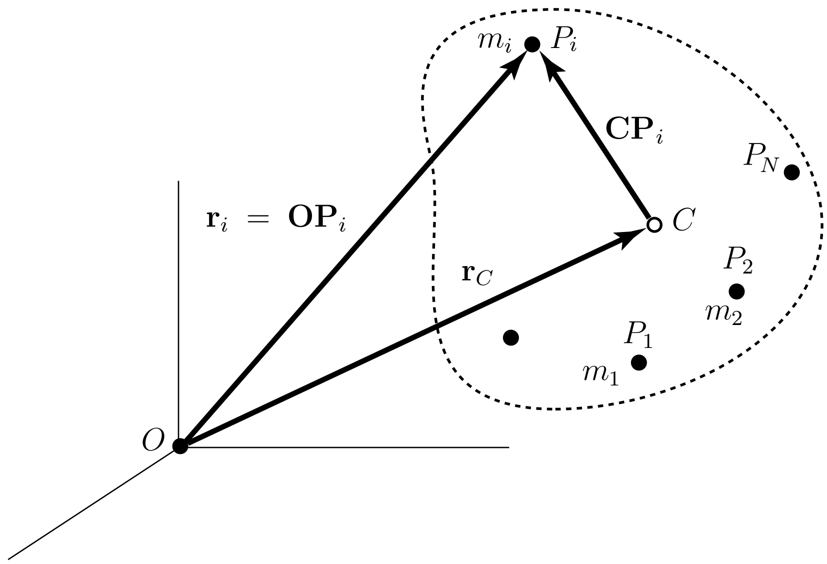


Figure 2


m_i
P_i
$\mathbf{CP}_i$
P_N
$\mathbf{r}_i \;=\; \mathbf{OP}_i$
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$\mathbf{r}_C$
P_2
m_2
P_1
m_1
O

Figure 3

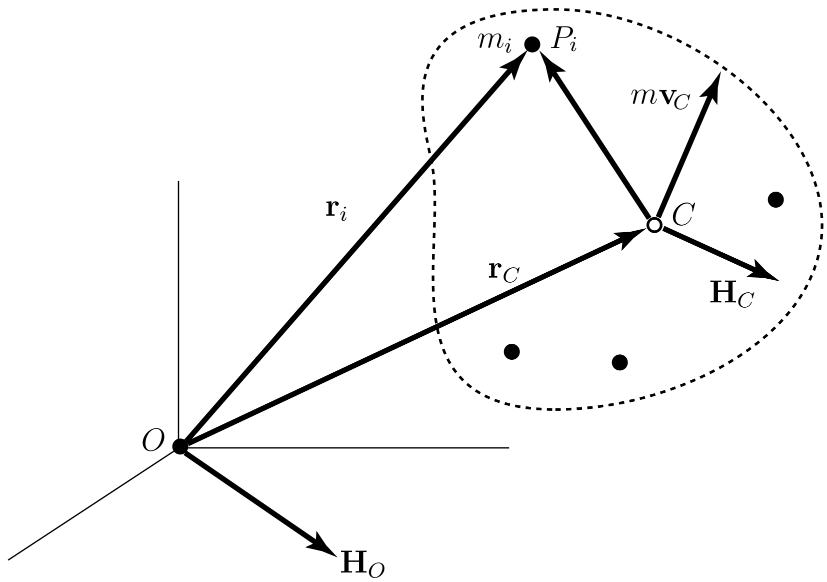
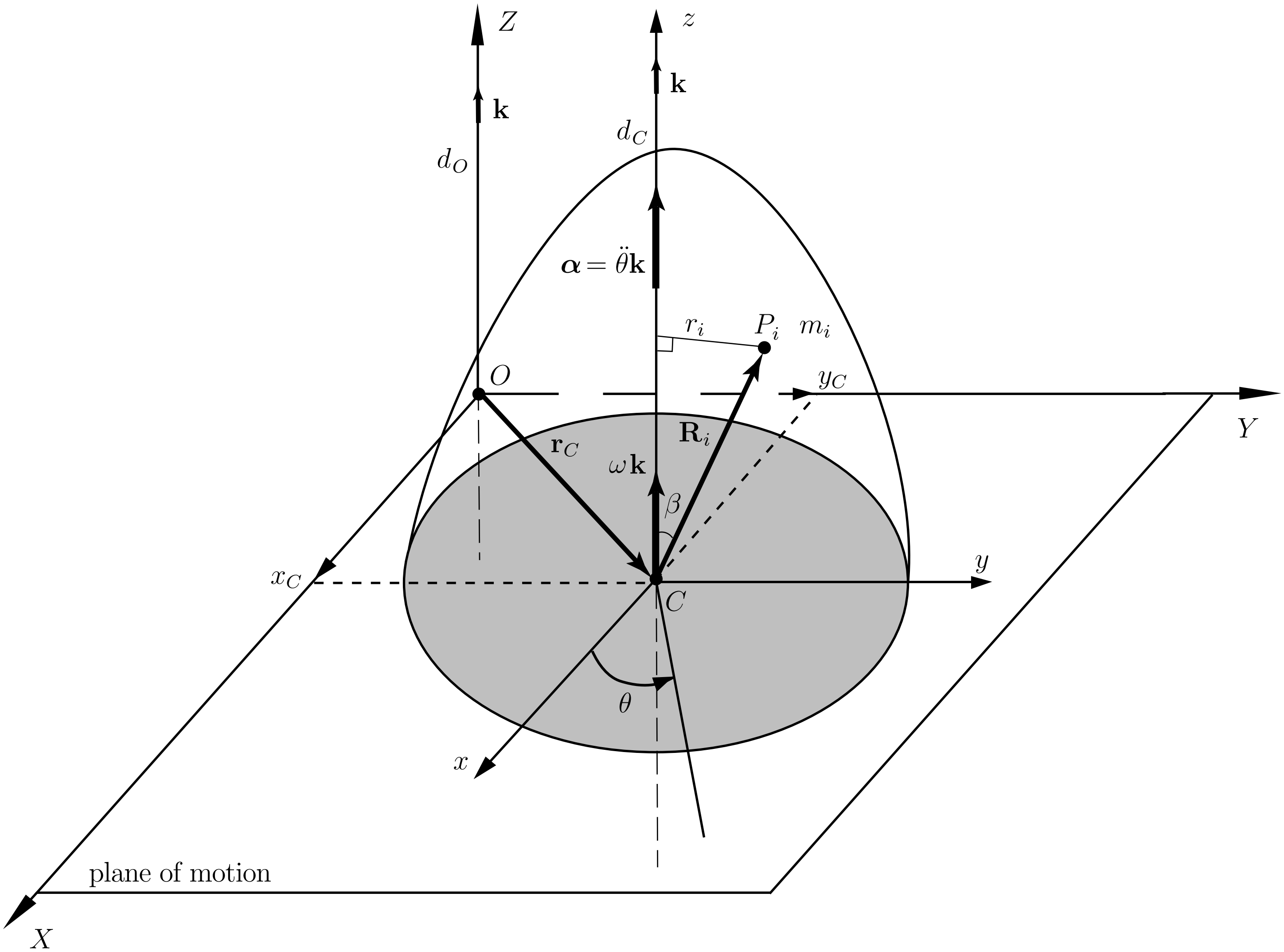


Figure 4


Z
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d_O
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$\boldsymbol{\alpha} = \ddot{\theta}\mathbf{k}$
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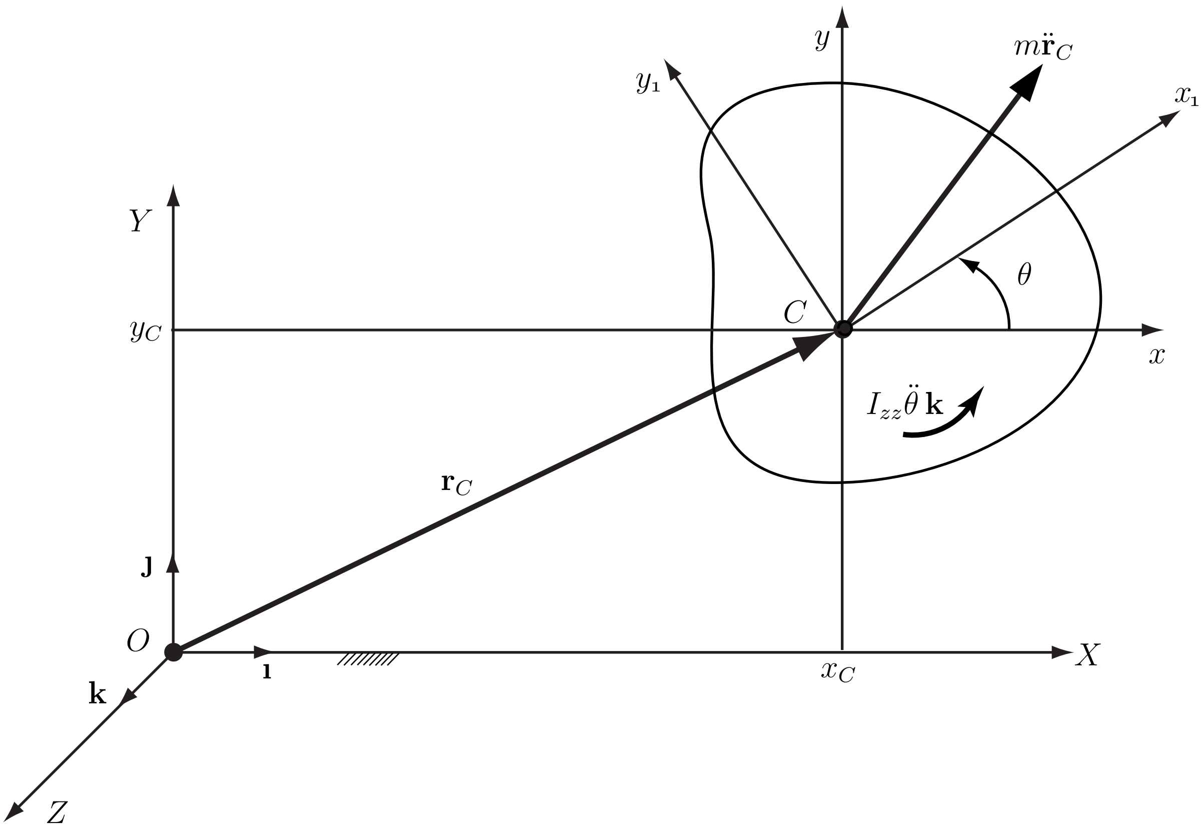
Figure 5



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$I_{zz}\ddot{\theta}\,\mathbf{k}$
$\mathbf{r}_C$
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O
X
$\mathbf{i}$
x_C
$\mathbf{k}$
Z

Figure 6