8/2 Free Vibration of Particles

When a spring-mounted body is disturbed from its equilibrium position, its ensuing motion in the absence of any imposed external forces is termed *free vibration*. In every actual case of free vibration, there exists some retarding or damping force which tends to diminish the motion. Common damping forces are those due to mechanical and fluid friction. In this article we first consider the ideal case where the damping forces are small enough to be neglected. Then we treat the case where the damping is appreciable and must be accounted for.

Equation of Motion for Undamped Free Vibration

We begin by considering the horizontal vibration of the simple frictionless spring-mass system of Fig. 8/1*a*. Note that the variable *x* denotes the displacement of the mass from the equilibrium position, which, for this system, is also the position of zero spring deflection. Figure 8/1*b* shows a plot of the force F_s necessary to deflect the spring versus the corresponding spring deflection for three types of springs. Although nonlinear hard and soft springs are useful in some applications, we will restrict our attention to the linear spring. Such a spring exerts a restoring force -*kx* on the mass—that is, when the mass is displaced to the right, the spring force is to the left, and vice versa. We must be careful to distinguish between the forces of magnitude F_s which must be applied to both ends of the massless spring to cause tension or compression and the force $F = -kx$ of equal magnitude which the spring exerts on the mass. The constant of proportionality *k* is called the *spring constant, modulus*, or *stiffness* and has the units N/m or lb/ft.

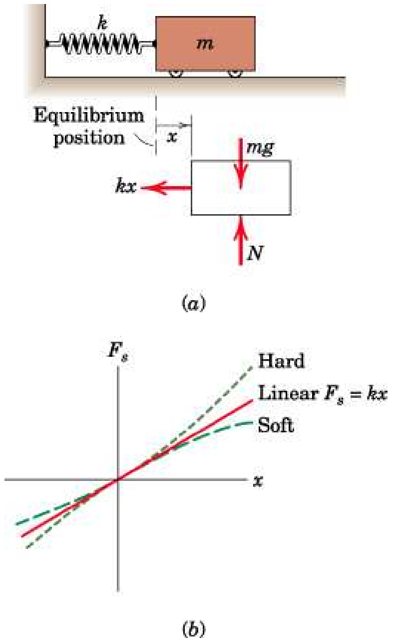


Figure 8/1

The equation of motion for the body of Fig. 8/1*a* is obtained by first drawing its free-body diagram. Applying Newton's second law in the form $\Sigma F_x = m\ddot{x}$ gives

$$-kx = m\ddot{x} \qquad \text{or} \qquad m\ddot{x} + kx = 0 \qquad (8/1)$$

The oscillation of a mass subjected to a linear restoring force as described by this equation is called *simple harmonic motion* and is characterized by acceleration which is proportional to the displacement but of opposite sign. Equation 8/1 is normally written as

$$\ddot{x} + \omega_n{}^2 x = 0 \qquad (8/2)$$

where

$$\omega_n = \sqrt{k/m} \qquad (8/3)$$

is a convenient substitution whose physical significance will be clarified shortly.

Solution for Undamped Free Vibration

Because we anticipate an oscillatory motion, we look for a solution which gives *x* as a periodic function of time. Thus, a logical choice is

$$x = A \cos \omega_n t + B \sin \omega_n t \qquad (8/4)$$

or, alternatively,

$$x = C \sin (\omega_n t + \psi) \qquad (8/5)$$

Direct substitution of these expressions into Eq. 8/2 verifies that each expression is a valid solution to the equation of motion. We determine the constants *A* and *B*, or *C* and ψ, from knowledge of the initial displacement x_0 and initial velocity $\dot{x}_0$ of the mass. For example, if we work with the solution form of Eq.

8/4 and evaluate x and $\dot{x}$ at time $t = 0$, we obtain

$$x_0 = A \qquad \text{and} \qquad \dot{x}_0 = B\omega_n$$

Substitution of these values of A and B into Eq. 8/4 yields

$$x = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \qquad (8/6)$$

The constants C and ψ of Eq. 8/5 can be determined in terms of given initial conditions in a similar manner. Evaluation of Eq. 8/5 and its first time derivative at $t = 0$ gives

$$x_0 = C \sin \psi \qquad \text{and} \qquad \dot{x}_0 = C\omega_n \cos \psi$$

Solving for C and ψ yields

$$C = \sqrt{x_0^{\,2} + (\dot{x}_0 / \omega_n)^2} \qquad \psi = \tan^{-1}(x_0\omega_n / \dot{x}_0)$$

Substitution of these values into Eq. 8/5 gives

$$x = \sqrt{x_0^{\,2} + (\dot{x}_0 / \omega_n)^2} \sin\left[\omega_n t + \tan^{-1}(x_0\omega_n / \dot{x}_0)\right] \qquad (8/7)$$

Equations 8/6 and 8/7 represent two different mathematical expressions for the same time-dependent motion. We observe that $C = \sqrt{A^2 + B^2}$ and $\psi = \tan^{-1}(A / B)$.

Graphical Representation of Motion

The motion may be represented graphically, Fig. 8/2, where x is seen to be the projection onto a vertical axis of the rotating vector of length C. The vector rotates at the constant angular velocity $\omega_n = \sqrt{k / m}$, which is called the *natural circular frequency* and has the units radians per second. The number of complete cycles per unit time is the *natural frequency* $f_n = \omega_n / 2\pi$ and is expressed in hertz (1 hertz (Hz) = 1 cycle per second). The time required for one complete motion cycle (one rotation of the reference vector) is the *period* of the motion and is given by $\tau = 1 / f_n = 2\pi / \omega_n$.

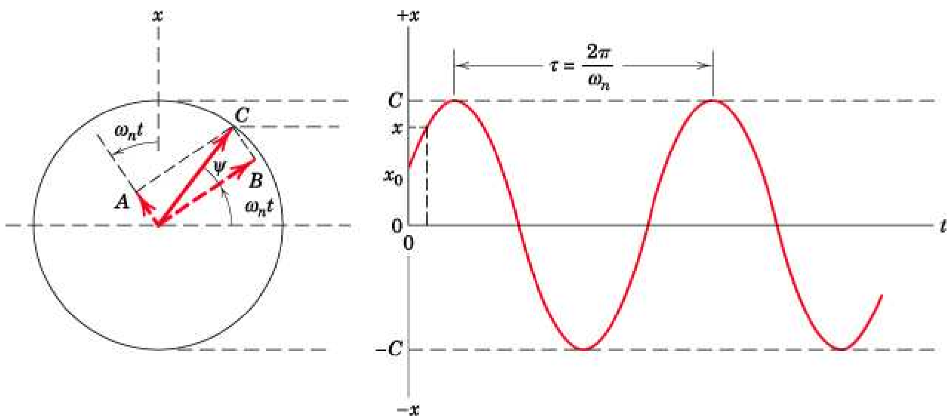


Figure 8/2

We also see from the figure that x is the sum of the projections onto the vertical axis of two perpendicular vectors whose magnitudes are A and B and whose vector sum C is the *amplitude*. Vectors A, B, and C rotate together with the constant angular velocity ω_n. Thus, as we have already seen, $C = \sqrt{A^2 + B^2}$ and $\psi = \tan^{-1}(A / B)$.

Equilibrium Position as Reference

As a further note on the free undamped vibration of particles, we see that, if the system of Fig. 8/1*a* is rotated 90° clockwise to obtain the system of Fig. 8/3 where the motion is vertical rather than horizontal, the equation of motion (and therefore all system properties) is unchanged if we continue to define x as the displacement from the equilibrium position. The equilibrium position now involves a nonzero spring deflection δ_{st}. From the free-body diagram of Fig. 8/3, Newton's second law gives

$$-k(\delta_{st} + x) + mg = m\ddot{x}$$

At the equilibrium position $x = 0$, the force sum must be zero, so that

$$-k\delta_{st} + mg = 0$$

Thus, we see that the pair of forces $-k\delta_{st}$ and mg on the left side of the motion equation cancel, giving

$$m\ddot{x} + kx = 0$$

which is identical to Eq. 8/1.

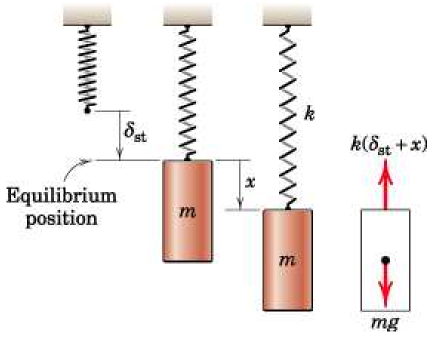


Figure 8/3

The lesson here is that by defining the displacement variable to be zero at equilibrium rather than at the position of zero spring deflection, we may ignore the equal and opposite forces associated with equilibrium.*

Equation of Motion for Damped Free Vibration

Every mechanical system possesses some inherent degree of friction, which dissipates mechanical energy. Precise mathematical models of the dissipative friction forces are usually complex. The dashpot or viscous damper is a device intentionally added to systems for the purpose of limiting or retarding vibration. It consists of a cylinder filled with a viscous fluid and a piston with holes or other passages by which the fluid can flow from one side of the piston to the other. Simple dashpots arranged as shown schematically in Fig. 8/4*a* exert a force F_d whose magnitude is proportional to the velocity of the mass, as depicted in Fig. 8/4*b*. The constant of proportionality *c* is called the *viscous damping coefficient* and has units of $\text{N} \cdot \text{s} / \text{m}$ or lb-sec/ft. The direction of the damping force as applied to the mass is opposite that of the velocity $\dot{x}$. Thus, the force on the mass is $-c\dot{x}$.

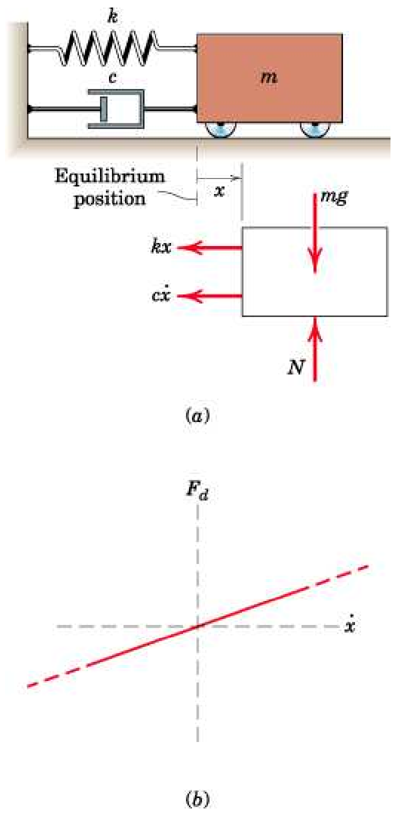


Figure 8/4

Complex dashpots with internal flow-rate-dependent one-way valves can produce different damping coefficients in extension and in compression; nonlinear characteristics are also possible. We will restrict our attention to the simple linear dashpot.

The equation of motion for the body with damping is determined from the free-body diagram as shown in Fig. 8/4*a*. Newton's second law gives

$$-kx - c\dot{x} = m\ddot{x} \qquad \text{or} \qquad m\ddot{x} + c\dot{x} + kx = 0 \tag{8/}$$

In addition to the substitution $\omega_n = \sqrt{k/m}$, it is convenient, for reasons which will shortly become evident, to introduce the combination of constants

$$\zeta = c / (2m\omega_n)$$

The quantity ζ (zeta) is called the *viscous damping factor* or *damping ratio* and is a measure of the severity of the damping. You should verify that ζ is nondimensional. Equation 8/8 may now be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad (8/9$$

Solution for Damped Free Vibration

In order to solve the equation of motion, Eq. 8/9, we assume solutions of the form

$$x = Ae^{\lambda t}$$

Substitution into Eq. 8/9 yields

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

which is called the *characteristic equation*. Its roots are

$$\lambda_1 = \omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right) \qquad \lambda_2 = \omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$$

Linear systems have the property of *superposition*, which means that the general solution is the sum of the individual solutions each of which corresponds to one root of the characteristic equation. Thus, the general solution is

$$\begin{aligned} x &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \\ &= A_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + A_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \end{aligned} \qquad (8/10$$

Categories of Damped Motion

Because $0 \le \zeta \le \infty$, the radicand $(\zeta^2 - 1)$ may be positive, negative, or even zero, giving rise to the following three categories of damped motion:

I. $\zeta > 1$ (*overdamped*). The roots λ_1 and λ_2 are distinct, real, and negative numbers. The motion as given by Eq. 8/10 decays so that *x* approaches zero fc large values of time *t*. There is no oscillation and therefore no period associated with the motion.

II. $\zeta = 1$ (*critically damped*). The roots λ_1 and λ_2 are equal, real, and negative numbers $(\lambda_1 = \lambda_2 = -\omega_n)$. The solution to the differential equation for the special case of equal roots is given by

$$x = (A_1 + A_2 t)e^{-\omega_n t}$$

Again, the motion decays with *x* approaching zero for large time, and the motion is nonperiodic. A critically damped system, when excited with an initial velocity or displacement (or both), will approach equilibrium faster than will an overdamped system. Figure 8/5 depicts actual responses for both an overdamped and a critically damped system to an initial displacement x_0 and no initial velocity $(\dot{x}_0 = 0)$.

III. $\zeta < 1$ (*underdamped*). Noting that the radicand $(\zeta^2 - 1)$ is negative and recalling that $e^{(a+b)} = e^a e^b$, we may rewrite Eq. 8/10 as

$$x = \left\{ A_1 e^{i\sqrt{1-\zeta^2}\,\omega_n t} + A_2 e^{-i\sqrt{1-\zeta^2}\,\omega_n t} \right\} e^{-\zeta\omega_n t}$$

where $i = \sqrt{-1}$. It is convenient to let a new variable ω_d represent the combination $\omega_n\sqrt{1-\zeta^2}$. Thus,

$$x = \left\{ A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t} \right\} e^{-\zeta\omega_n t}$$

Use of the Euler formula $e^{\pm ix} = \cos x \pm i \sin x$ allows the previous equation to be written as

$$\begin{aligned} x &= \{A_1(\cos\omega_d t + i\sin\omega_d t) + A_2(\cos\omega_d t - i\sin\omega_d t)\}\, e^{-\zeta\omega_n t} \\ &= \{(A_1 + A_2)\cos\omega_d t + i(A_1 - A_2)\sin\omega_d t\}\, e^{-\zeta\omega_n t} \\ &= \{A_3\cos\omega_d t + A_4\sin\omega_d t\}\, e^{-\zeta\omega_n t} \end{aligned} \qquad (8/1$$

where $A_3 = (A_1 + A_2)$ and $A_4 = i(A_1 - A_2)$. We have shown with Eqs. 8/4 and 8/5 that the sum of two equal-frequency harmonics, such as those i the braces of Eq. 8/11, can be replaced by a single trigonometric function which involves a phase angle. Thus, Eq. 8/11 can be written as

$$x = \{C\sin(\omega_d t + \psi)\}\, e^{-\zeta\omega_n t}$$

or

$$x = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \psi) \qquad (8/1$$

Equation 8/12 represents an exponentially decreasing harmonic function, as shown in Fig. 8/6 for specific numerical values. The frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

is called the *damped natural frequency*. The *damped period* is given by $\tau_d = 2\pi / \omega_d = 2\pi / \left(\omega_n\sqrt{1-\zeta^2}\right)$.

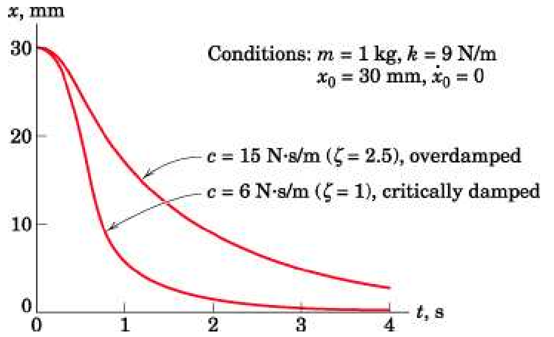


Figure 8/5

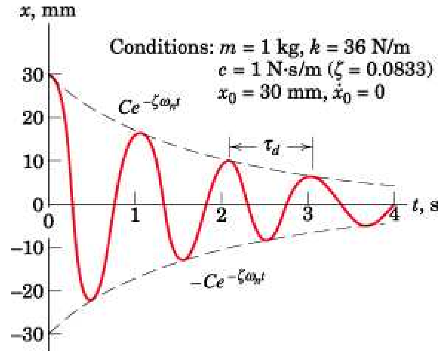


Figure 8/6

It is important to note that the expressions developed for the constants C and ψ in terms of initial conditions for the case of no damping are not valid for the case of damping. To find C and ψ if damping is present, you must begin anew, setting the general displacement expression of Eq. 8/12 and its first time derivative, both evaluated at time $t = 0$, equal to the initial displacement x_0 and initial velocity $\dot{x}_0$, respectively.

Determination of Damping by Experiment

We often need to experimentally determine the value of the damping ratio ζ for an underdamped system. The usual reason is that the value of the viscous damping coefficient c is not otherwise well known. To determine the damping, we may excite the system by initial conditions and obtain a plot of the displacement x versus time t, such as that shown schematically in Fig. 8/7. We then measure two successive amplitudes x_1 and x_2 a full cycle apart and compute their ratio

$$\frac{x_1}{x_2} = \frac{Ce^{-\zeta\omega_n t_1}}{Ce^{-\zeta\omega_n(t_1+\tau_d)}} = e^{\zeta\omega_n\tau_d}$$

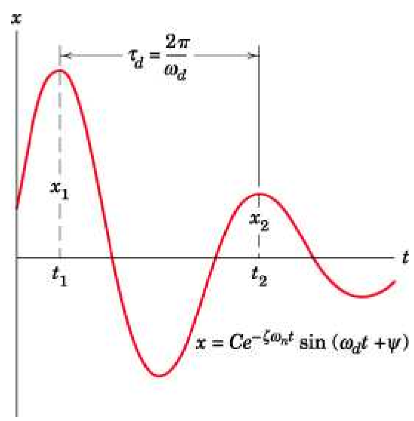


Figure 8/7

The *logarithmic decrement* δ is defined as

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

From this equation, we may solve for ζ and obtain

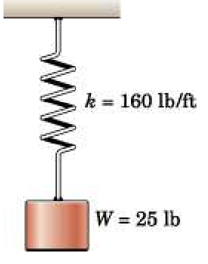
$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

For a small damping ratio, $x_1 \cong x_2$ and $\delta << 1$, so that $\zeta \cong \delta / 2\pi$. If x_1 and x_2 are so close in value that experimental distinction between them is impractical, the above analysis may be modified by using two observed amplitudes which are *n* cycles apart.

Sample Problem 8/1

A body weighing 25 lb is suspended from a spring of constant $k = 160$ lb / ft. At time $t = 0$, it has a downward velocity of 2 ft/sec as it passes through the position of static equilibrium. Determine

(a) the static spring deflection δ_{st}
(b) the natural frequency of the system in both rad/sec (ω_n) and cycles/sec (f_n)
(c) the system period τ
(d) the displacement *x* as a function of time, where *x* is measured from the position of static equilibrium
(e) the maximum velocity v_{max} attained by the mass
(f) the maximum acceleration a_{max} attained by the mass.



Solution.

a. From the spring relationship $F_s = kx$, we see that at equilibrium
①

$$mg = k\delta_{st} \qquad \delta_{st} = \frac{mg}{k} = \frac{25}{160} = 0.1562 \text{ ft or } 1.875 \text{ in}. \qquad \textit{Ans}.$$

b.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{160}{25/32.2}} = 14.36 \text{ rad/sec} \qquad \textit{Ans}.$$

$$f_n = (14.36)\left(\frac{1}{2\pi}\right) = 2.28 \text{ cycles/sec} \qquad \textit{Ans}.$$

c.

$$\tau = \frac{1}{f_n} = \frac{1}{2.28} = 0.438 \text{ sec} \qquad \textit{Ans}.$$

d. ② From Eq. 8/6:

$$\begin{aligned} x &= x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \\ &= (0) \cos 14.36t + \frac{2}{14.36} \sin 14.36t \\ &= 0.1393 \sin 14.36t \qquad \textit{Ans}. \end{aligned}$$

As an exercise, let us determine *x* from the alternative Eq. 8/7:

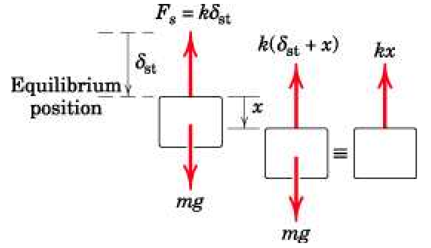
$$\begin{aligned} x &= \sqrt{{x_0}^2 + (\dot{x}_0/\omega_n)^2} \sin\left[\omega_n t + \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right)\right] \\ &= \sqrt{0^2 + \left(\frac{2}{14.36}\right)^2} \sin\left[14.36t + \tan^{-1}\left(\frac{(0)(14.36)}{2}\right)\right] \\ &= 0.1393 \sin 14.36t \end{aligned}$$

e. The velocity is $\dot{x} = 14.36(0.1393)\cos 14.36t = 2\cos 14.36t$. Because the cosine function cannot be greater than 1 or less than -1, the maximum velocity v_{max} is 2 ft/sec, which, in this case, is the initial velocity. *Ans.*

f. The acceleration is

$$\ddot{x} = -14.36(2)\sin 14.36t = -28.7\sin 14.36t$$

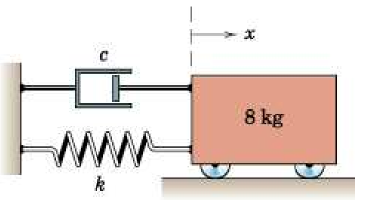
The maximum acceleration a_{max} is $28.7\ \text{ft/sec}^2$. *Ans.*



Sample Problem 8/2

Click here to see a computer extension of this problem.

The 8-kg body is moved 0.2 m to the right of the equilibrium position and released from rest at time $t = 0$. Determine its displacement at time $t = 2$ s. The viscous damping coefficient c is $20\ \text{N}\cdot\text{s/m}$, and the spring stiffness k is 32 N/m.



Solution. We must first determine whether the system is underdamped, critically damped, or overdamped. For that purpose, we compute the damping ratio ζ.

$$\omega_n = \sqrt{k/m} = \sqrt{32/8} = 2\ \text{rad/s} \qquad \zeta = \frac{c}{2m\omega_n} = \frac{20}{2(8)(2)} = 0.625$$

Since $\zeta < 1$, the system is underdamped. The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 2\sqrt{1-(0.625)^2} = 1.561\ \text{rad/s}$. The motion is given by Eq. 8/12 and is

$$x = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \psi) = Ce^{-1.25t}\sin(1.561t + \psi)$$

The velocity is then

$$\dot{x} = -1.25Ce^{-1.25t}\sin(1.561t + \psi) + 1.561Ce^{-1.25t}\cos(1.561t + \psi)$$

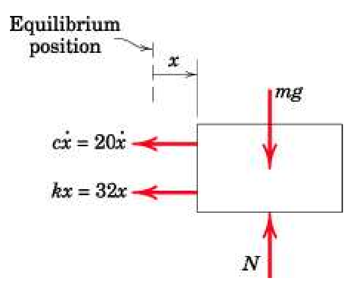
Evaluating the displacement and velocity at time $t = 0$ gives

$$x_0 = C\sin\psi = 0.2 \qquad \dot{x}_0 = -1.25C\sin\psi + 1.561C\cos\psi = 0$$

Solving the two equations for C and ψ yields $C = 0.256\ \text{m}$ and $\psi = 0.896\ \text{rad}$. Therefore, the displacement in meters is

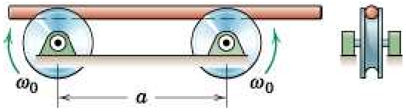
$$x = 0.256e^{-1.25t}\sin(1.561t + 0.896)$$

① Evaluation for time $t = 2$ s gives $x_2 = -0.01616\ \text{m}$. *Ans.*



Sample Problem 8/3

The two fixed counterrotating pulleys are driven at the same angular speed ω_0. A round bar is placed off center on the pulleys as shown. Determine the natural frequency of the resulting bar motion. The coefficient of kinetic friction between the bar and pulleys is μ_k.



Solution. The free-body diagram of the bar is constructed for an arbitrary displacement *x* from the central position as shown. The governing equations are

$$[\Sigma F_x = m\ddot{x}] \qquad \mu_k N_A - \mu_k N_B = m\ddot{x}$$

$$[\Sigma F_y = 0] \qquad N_A + N_B - mg = 0$$

$$[\Sigma M_A = 0] \qquad aN_B - \left(\frac{a}{2} + x\right)mg = 0$$

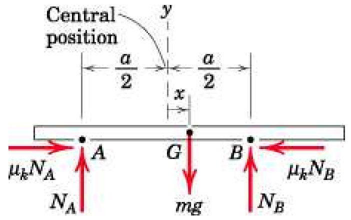
① Eliminating N_A and N_B from the first equation yields

②

$$\ddot{x} + \frac{2\mu_k g}{a}x = 0$$

We recognize the form of this equation as that of Eq. 8/2, so that the natural frequency in radians per second is $\omega_n = \sqrt{2\mu_k g/a}$ and the natural frequency in cycles per second is

$$f_n = \frac{1}{2\pi}\sqrt{2\mu_k g/a}$$



Student Sample Solution - Free Vibration of Particles (Undamped) Example #1

Student Sample Solution - Free Vibration of Particles (Undamped) Example #2

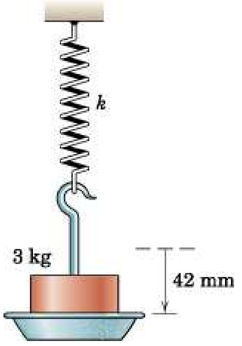
PROBLEMS

(Unless otherwise indicated, all motion variables are referred to the equilibrium position.)

Introductory Problems—Undamped, Free Vibrations

8/1. When a 3-kg collar is placed upon the pan which is attached to the spring of unknown constant, the additional static deflection of the pan is observed to be 42 mm. Determine the spring constant *k* in N/m, lb/in., and lb/ft.

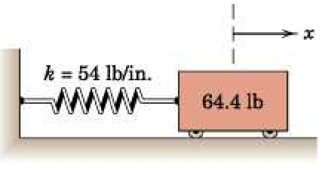
Answer



Problem 8/1

8/2. Show that the natural frequency of a vertically oriented spring-mass system, such as that of Prob. 8/1, may be expressed as $\omega_n = \sqrt{g/\delta_{st}}$ where δ_{st} is the static deflection.

8/3. Determine the natural frequency of the spring-mass system in both radians per second and cycles per second (Hz). Answer

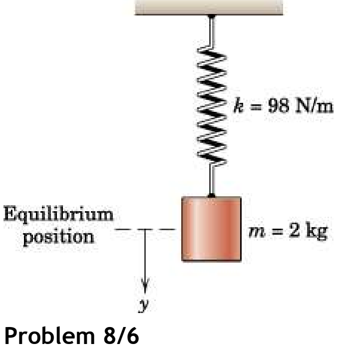


Problem 8/3

8/4. For the system of Prob. 8/3, determine the position x of the mass as a function of time if the mass is released from rest at time $t = 0$ from a position 2 inches to the left of the equilibrium position. Determine the maximum velocity and maximum acceleration of the mass over one cycle of motion.

8/5. For the system of Prob. 8/3, determine the position x as a function of time if the mass is released at time $t = 0$ from a position 2 inches to the right of the equilibrium position with an initial velocity of 9 in./sec to the left. Determine the amplitude C and period τ of the motion. Answer

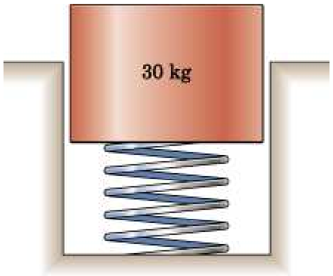
8/6. For the spring-mass system shown, determine the static deflection δ_{st}, the system period τ, and the maximum velocity v_{max} which result if the cylinder is displaced 100 mm downward from its equilibrium position and released.



Problem 8/6

8/7. The cylinder of the system of Prob. 8/6 is displaced 100 mm downward from its equilibrium position and released at time $t = 0$. Determine the position y, velocity v, and acceleration a when $t = 3$ s. What is the maximum acceleration? Answer

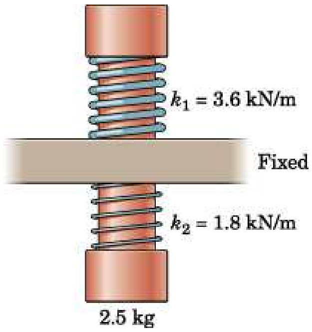
8/8. In the equilibrium position, the 30-kg cylinder causes a static deflection of 50 mm in the coiled spring. If the cylinder is depressed an additional 25 mm and released from rest, calculate the resulting natural frequency f_n of vertical vibration of the cylinder in cycles per second (Hz).



Problem 8/8

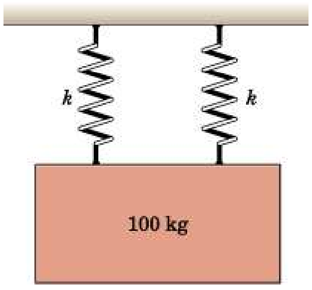
8/9. For the cylinder of Prob. 8/8, determine the vertical displacement x, measured positive down in millimeters from the equilibrium position, in terms of the time t in seconds measured from the instant of release from the position of 25 mm added deflection. Answer

8/10. The vertical plunger has a mass of 2.5 kg and is supported by the two springs, which are always in compression. Calculate the natural frequency f_n of vibration of the plunger if it is deflected from the equilibrium position and released from rest. Friction in the guide is negligible.



Problem 8/10

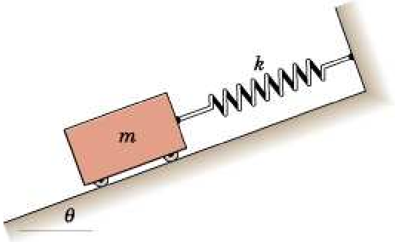
8/11. If the 100-kg mass has a downward velocity of 0.5 m/s as it passes through its equilibrium position, calculate the magnitude a_{max} of its maximum acceleration. Each of the two springs has a stiffness $k = 180\ \text{kN/m}$. Answer



Problem 8/11

Representative Problems—Undamped, Free Vibrations

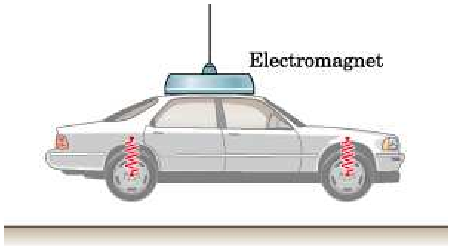
8/12. Prove that the natural frequency f_n of oscillation for the mass m is independent of θ.



Problem 8/12

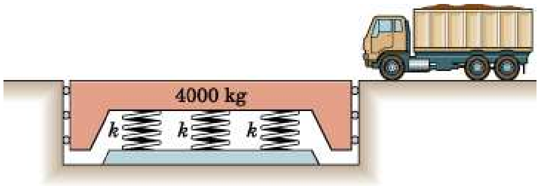
8/13. An old car being moved by a magnetic crane pickup is dropped from a short distance above the ground. Neglect any damping effects of its worn-out shock absorbers and calculate the natural frequency f_n in cycles per second (Hz) of the vertical vibration which occurs after impact with the ground. Each of the four springs on the 1000-kg car has a constant of 17.5 kN/m. Because the center of mass is located midway between the axles and the car is level when dropped, there is no rotational motion. State any assumptions.

Answer



Problem 8/13

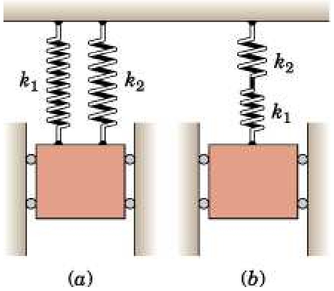
8/14. During the design of the spring-support system for the 4000-kg weighing platform, it is decided that the frequency of free vertical vibration in the unloaded condition shall not exceed 3 cycles per second. (*a*) Determine the maximum acceptable spring constant *k* for each of the three identical springs. (*b*) For this spring constant, what would be the natural frequency f_n of vertical vibration of the platform loaded by the 40-Mg truck?



Problem 8/14

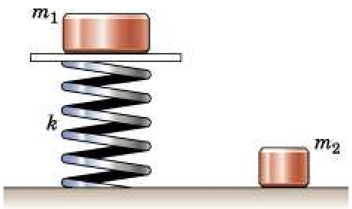
8/15. Replace the springs in each of the two cases shown by a single spring of stiffness *k* (equivalent spring stiffness) which will cause each mass to vibrate with its original frequency.

Answer



Problem 8/15

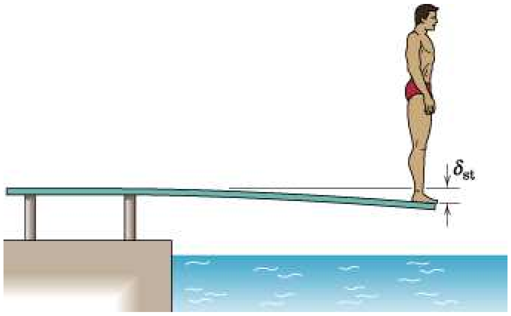
8/16. Explain how the values of the mass m_1 and the spring constant *k* may be experimentally determined if the mass m_2 is known. Develop expressions for m_1 and *k* in terms of specified experimental results. Note the existence of at least three ways to solve the problem.



Problem 8/16

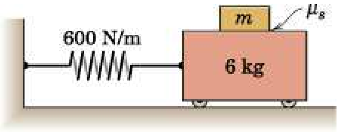
8/17. A 90-kg man stands at the end of a diving board and causes a vertical oscillation which is observed to have a period of 0.6 s. What is the static deflection δ_{st} at the end of the board? Neglect the mass of the board.

Answer



Problem 8/17

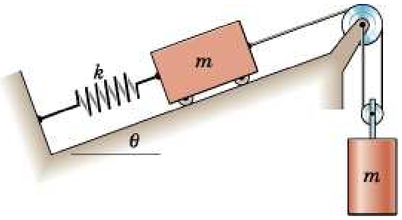
8/18. With the assumption of no slipping, determine the mass *m* of the block which must be placed on the top of the 6-kg cart in order that the system period be 0.75 s. What is the minimum coefficient of static friction μ_s for which the block will not slip relative to the cart if the cart is displaced 50 mm from the equilibrium position and released?



Problem 8/18

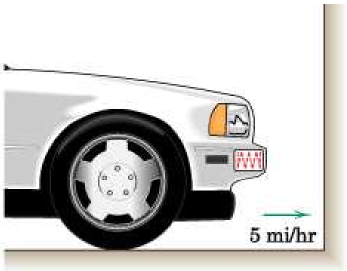
8/19. Calculate the natural frequency ω_n of the system shown in the figure. The mass and friction of the pulleys are negligible.

Answer



Problem 8/19

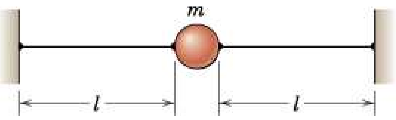
8/20. An energy-absorbing car bumper with its springs initially undeformed has an equivalent spring constant of 3000 lb/in. If the 2500-lb car approaches a massive wall with a speed of 5 mi/hr, determine (*a*) the velocity *v* of the car as a function of time during contact with the wall, where $t = 0$ is the beginning of the impact, and (*b*) the maximum deflection x_{max} of the bumper.



Problem 8/20

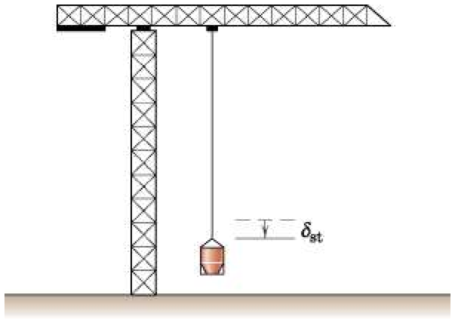
8/21. A small particle of mass *m* is attached to two highly tensioned wires as shown. Determine the system natural frequency ω_n for small vertical oscillations if the tension *T* in both wires is assumed to be constant. Is the calculation of the small static deflection of the particle necessary?

Answer



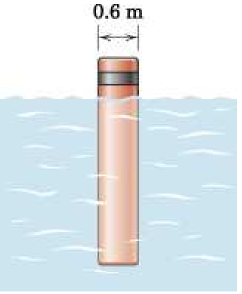
Problem 8/21

8/22. The large cement bucket suspended from the crane by an elastic cable has a mass of 4000 kg. When the bucket is disturbed, a vertical oscillation of period 0.5 s is observed. What is the static deflection δ_{st} of the bucket? Neglect the mass of the cable and assume that the crane is rigid for the inboard support position shown.



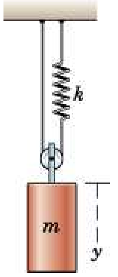
Problem 8/22

8/23. The cylindrical buoy floats in salt water (density $1030\ \text{kg}/m^3$) and has a mass of 800 kg with a low center of mass to keep it stable in the upright position. Determine the frequency f_n of vertical oscillation of the buoy. Assume that the water level remains undisturbed adjacent to the buoy. Answer



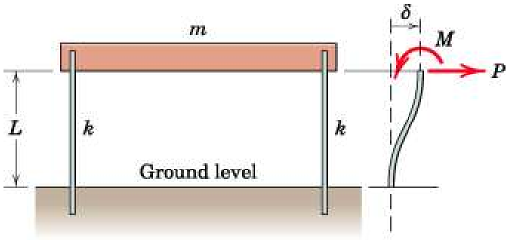
Problem 8/23

8/24. The cylinder of mass *m* is given a vertical displacement y_0 from its equilibrium position and released. Write the differential equation for the vertical vibration of the cylinder and find the period τ of its motion. Neglect the friction and mass of the pulley.



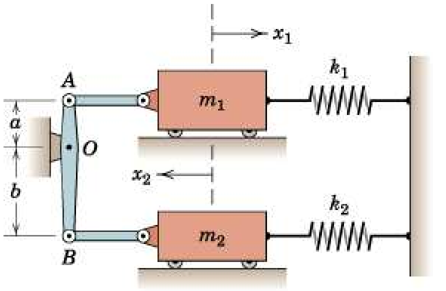
Problem 8/24

8/25. Shown in the figure is a model of a one-story building. The bar of mass *m* is supported by two light elastic upright columns whose upper and lower ends are fixed against rotation. For each column, if a force *P* and corresponding moment *M* were applied as shown in the right-hand part of the figure, the deflection δ would be given by $\delta = PL^3/12EI$, where *L* is the effective column length, *E* is Young's modulus, and *I* is the area moment of inertia of the column cross section with respect to its neutral axis. Determine the natural frequency of horizontal oscillation of the bar when the columns bend as shown in the figure. Answer



Problem 8/25

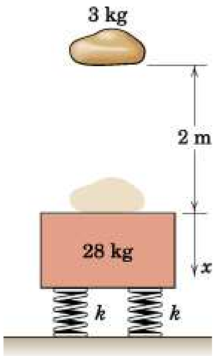
8/26. Derive the differential equation of motion for the system shown in terms of the variable x_1. The mass of the linkage is negligible. State the natural frequency ω_n in rad/s for the case $k_1 = k_2 = k$ and $m_1 = m_2 = m$. Assume small oscillations throughout.



Problem 8/26

8/27. A 3-kg piece of putty is dropped 2 m onto the initially stationary 28-kg block, which is supported by four springs, each of which has a constant $k = 800$ N/m. Determine the displacement *x* as a function of time during the resulting vibration, where *x* is measured from the initial position of the block as shown.

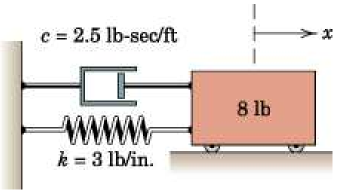
Answer



Problem 8/27

Introductory Problems—Damped, Free Vibrations

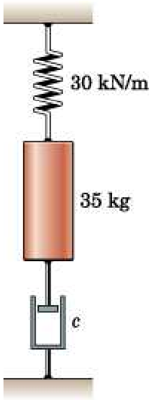
8/28. Determine the value of the damping ratio ζ for the simple spring-mass-dashpot system shown.



Problem 8/28

8/29. Determine the value of the viscous damping coefficient *c* for which the system shown is critically damped.

Answer

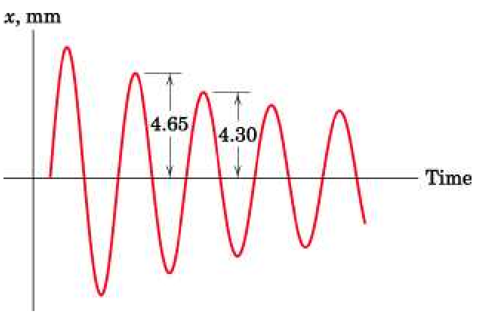


Problem 8/29

8/30. The 8-lb body of Prob. 8/28 is released from rest a distance x_0 to the right of the equilibrium position. Determine the displacement x as a function of time t, where $t = 0$ is the time of release.

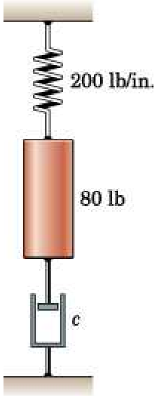
8/31. The addition of damping to an undamped spring-mass system causes its period to increase by 25 percent. Determine the damping ratio ζ. Answer

8/32. A linear harmonic oscillator having a mass of 1.10 kg is set into motion with viscous damping. If the frequency is 10 Hz and if two successive amplitudes a full cycle apart are measured to be 4.65 mm and 4.30 mm as shown, compute the viscous damping coefficient c.



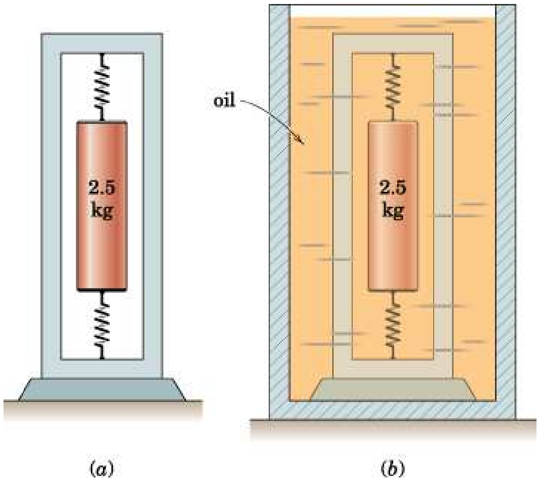
Problem 8/32

8/33. Determine the value of the viscous damping coefficient c for which the system shown is critically damped. Answer



Problem 8/33

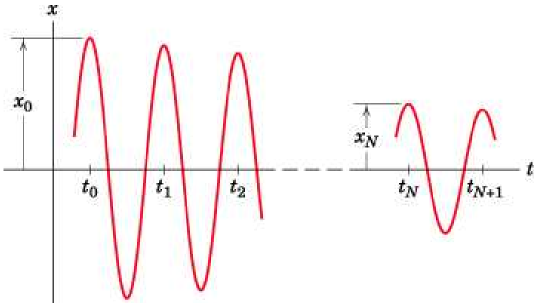
8/34. The 2.5-kg spring-supported cylinder is set into free vertical vibration and is observed to have a period of 0.75 s in part (a) of the figure. The system is then completely immersed in an oil bath in part (b) of the figure, and the cylinder is displaced from its equilibrium position and released. Viscous damping ensues, and the ratio of two successive positive-displacement amplitudes is 4. Calculate the viscous damping ratio ζ, the viscous damping constant c, and the equivalent spring constant k.



Problem 8/34

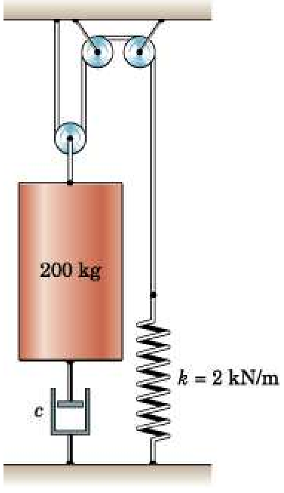
Representative Problems—Damped, Free Vibrations

8/35. The figure represents the measured displacement-time relationship for a vibration with small damping where it is impractical to achieve accurate results by measuring the nearly equal amplitudes of two successive cycles. Modify the expression for the viscous damping factor ζ based on the measured amplitudes x_0 and x_N which are N cycles apart. Answer



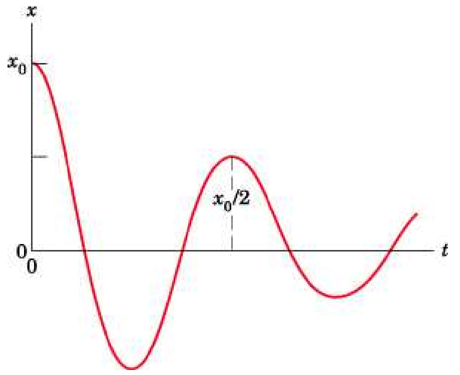
Problem 8/35

8/36. For the damped spring-mass system shown, determine the viscous damping coefficient for which critical damping will occur.



Problem 8/36

8/37. A damped spring-mass system is released from rest from a positive initial displacement x_0. If the succeeding maximum positive Answer displacement is $x_0 / 2$, determine the damping ratio ζ of the system.

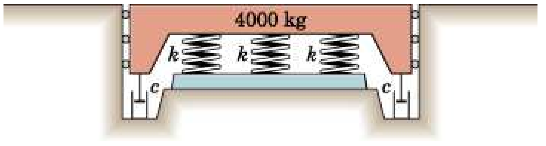


Problem 8/37

8/38. If the amplitude of the eighth cycle of a linear oscillator with viscous damping is sixteen times the amplitude of the twentieth cycle, calculate the damping ratio ζ.

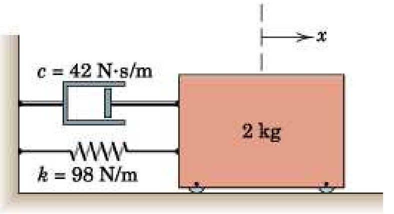
8/39. Further design refinement for the weighing platform of Prob. 8/14 is shown here where two viscous dampers are to be added to limit the ratio of successive positive amplitudes of vertical vibration in the unloaded condition to 4. Determine the necessary viscous damping coefficient c for each of the dampers.

Answer



Problem 8/39

8/40. The 2-kg mass is released from rest at a distance x_0 to the right of the equilibrium position. Determine the displacement x as a function of time.

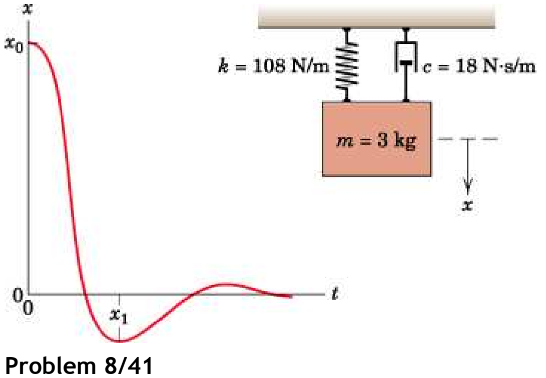


Problem 8/40

8/41. Click here to see an animation of a similar problem.

Answer

The system shown is released from rest from an initial position x_0. Determine the overshoot displacement x_1. Assume translational motion in the x-direction.

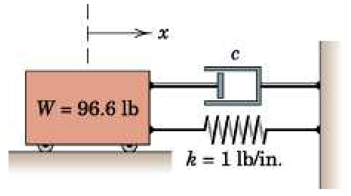


Problem 8/41

8/42. The mass of a given critically damped system is released at time $t = 0$ from the position $x_0 > 0$ with a negative initial velocity. Determine the critical value $(\dot{x}_0)_c$ of the initial velocity below which the mass will pass through the equilibrium position.

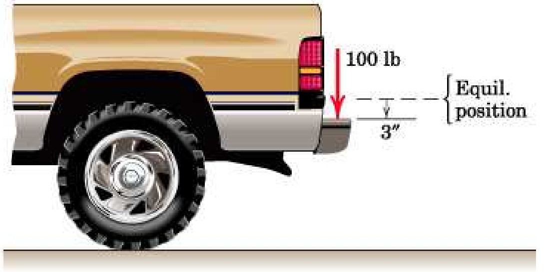
8/43. Click here to see an animation of a similar problem. Answer

The mass of the system shown is released from rest at $x_0 = 6$ in. when $t = 0$. Determine the displacement x at $t = 0.5$ sec if (*a*) $c = 12$ lb-sec / ft and (*b*) $c = 18$ lb-sec / ft.



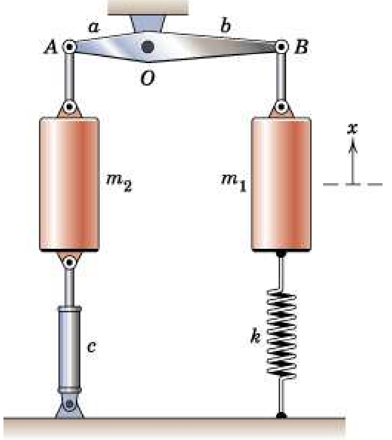
Problem 8/43

8/44. The owner of a 3400-lb pickup truck tests the action of his rear-wheel shock absorbers by applying a steady 100-lb force to the rear bumper and measuring a static deflection of 3 in. Upon sudden release of the force, the bumper rises and then falls to a maximum of $1/2$ in. below the unloaded equilibrium position of the bumper on the first rebound. Treat the action as a one-dimensional problem with an equivalent mass of half the truck mass. Find the viscous damping factor ζ for the rear end and the viscous damping coefficient c for each shock absorber assuming its action to be vertical.



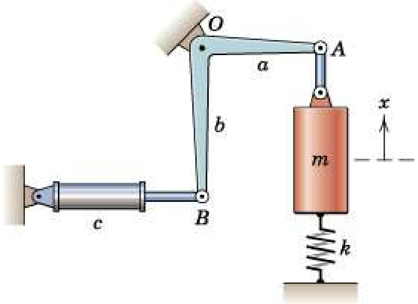
Problem 8/44

8/45. Derive the differential equation of motion for the system shown in its equilibrium position. Neglect the mass of link *AB* and assume small oscillations. Answer



Problem 8/45

8/46. Develop the equation of motion in terms of the variable x for the system shown. Determine an expression for the damping ratio ζ in terms of the given system properties. Neglect the mass of the crank *AB* and assume small oscillations about the equilibrium position shown.



Problem 8/46