

### Sample Problem 2/5

The curvilinear motion of a particle is defined by  $v_x = 50 - 16t$  and  $y = 100 - 4t^2$ , where  $v_x$  is in meters per second,  $y$  is in meters, and  $t$  is in seconds. It is also known that  $x = 0$  when  $t = 0$ . Plot the path of the particle and determine its velocity and acceleration when the position  $y = 0$  is reached.

**Solution.** The  $x$ -coordinate is obtained by integrating the expression for  $v_x$ , and the  $x$ -component of the acceleration is obtained by differentiating  $v_x$ . Thus,

$$\left[ \int dx = \int v_x dt \right] \quad \int_0^x dx = \int_0^t (50 - 16t) dt \quad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \quad a_x = \frac{d}{dt}(50 - 16t) \quad a_x = -16 \text{ m/s}^2$$

The  $y$ -components of velocity and acceleration are

$$[v_y = \dot{y}] \quad v_y = \frac{d}{dt}(100 - 4t^2) \quad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \quad a_y = \frac{d}{dt}(-8t) \quad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of  $x$  and  $y$  for various values of  $t$  and plot  $x$  against  $y$  to obtain the path as shown.

When  $y = 0$ ,  $0 = 100 - 4t^2$ , so  $t = 5$  s. For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

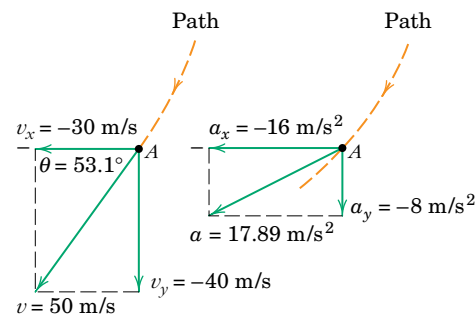
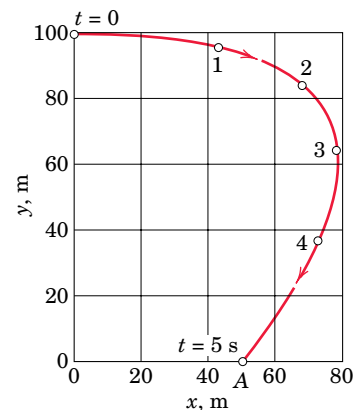
The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where  $y = 0$ . Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s}$$

Ans.

$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2$$

Ans.

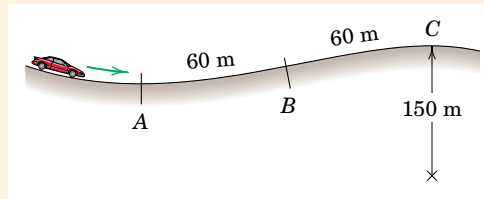


#### Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

### Sample Problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom  $A$  of the dip and 50 km/h at the top  $C$  of the hump, which is 120 m along the road from  $A$ . If the passengers experience a total acceleration of  $3 \text{ m/s}^2$  at  $A$  and if the radius of curvature of the hump at  $C$  is 150 m, calculate (a) the radius of curvature  $\rho$  at  $A$ , (b) the acceleration at the inflection point  $B$ , and (c) the total acceleration at  $C$ .



#### Helpful Hint

- ① The dimensions of the car are small compared with those of the path, so we will treat the car as a particle. The velocities are

$$v_A = \left(100 \frac{\text{km}}{\text{h}}\right) \left(\frac{1 \text{ h}}{3600 \text{ s}}\right) \left(1000 \frac{\text{m}}{\text{km}}\right) = 27.8 \text{ m/s}$$

$$v_C = 50 \frac{1000}{3600} = 13.89 \text{ m/s}$$

We find the constant deceleration along the path from

$$\left[ \int v \, dv = \int a_t \, ds \right] \quad \int_{v_A}^{v_C} v \, dv = a_t \int_0^s ds$$

$$a_t = \frac{1}{2s} (v_C^2 - v_A^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41 \text{ m/s}^2$$

**(a) Condition at A.** With the total acceleration given and  $a_t$  determined, we can easily compute  $a_n$  and hence  $\rho$  from

$$[a^2 = a_n^2 + a_t^2] \quad a_n^2 = 3^2 - (2.41)^2 = 3.19 \quad a_n = 1.785 \text{ m/s}^2$$

$$[a_n = v^2/\rho] \quad \rho = v^2/a_n = (27.8)^2/1.785 = 432 \text{ m} \quad \text{Ans.}$$

**(b) Condition at B.** Since the radius of curvature is infinite at the inflection point,  $a_n = 0$  and

$$a = a_t = -2.41 \text{ m/s}^2 \quad \text{Ans.}$$

**(c) Condition at C.** The normal acceleration becomes

$$[a_n = v^2/\rho] \quad a_n = (13.89)^2/150 = 1.286 \text{ m/s}^2$$

With unit vectors  $\mathbf{e}_n$  and  $\mathbf{e}_t$  in the  $n$ - and  $t$ -directions, the acceleration may be written

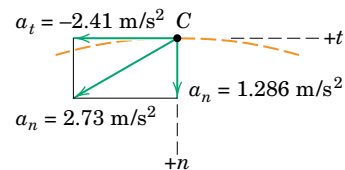
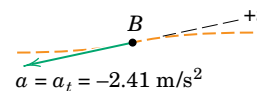
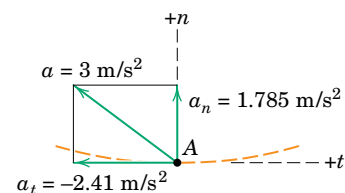
$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t \text{ m/s}^2$$

where the magnitude of  $\mathbf{a}$  is

$$[a = \sqrt{a_n^2 + a_t^2}] \quad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73 \text{ m/s}^2 \quad \text{Ans.}$$

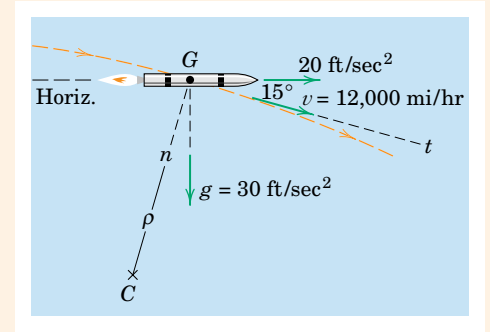
The acceleration vectors representing the conditions at each of the three points are shown for clarification.

- ① Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.



### Sample Problem 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of  $20 \text{ ft/sec}^2$ , and the downward acceleration component is the acceleration due to gravity at that altitude, which is  $g = 30 \text{ ft/sec}^2$ . At the instant represented, the velocity of the mass center  $G$  of the rocket along the  $15^\circ$  direction of its trajectory is  $12,000 \text{ mi/hr}$ . For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed  $v$  is increasing, (c) the angular rate  $\dot{\beta}$  of the radial line from  $G$  to the center of curvature  $C$ , and (d) the vector expression for the total acceleration  $\mathbf{a}$  of the rocket.



**Solution.** We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use  $n$ - and  $t$ -coordinates to describe the motion of  $G$ . The  $n$ - and  $t$ -components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their  $n$ - and  $t$ -components and then combining. From the figure we get

$$a_n = 30 \cos 15^\circ - 20 \sin 15^\circ = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15^\circ + 20 \cos 15^\circ = 27.1 \text{ ft/sec}^2$$

(a) We may now compute the radius of curvature from

$$\textcircled{2} [a_n = v^2/\rho] \quad \rho = \frac{v^2}{a_n} = \frac{[(12,000)(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft} \quad \text{Ans.}$$

(b) The rate at which  $v$  is increasing is simply the  $t$ -component of acceleration.

$$[\dot{v} = a_t] \quad \dot{v} = 27.1 \text{ ft/sec}^2 \quad \text{Ans.}$$

(c) The angular rate  $\dot{\beta}$  of line  $GC$  depends on  $v$  and  $\rho$  and is given by

$$[v = \rho\dot{\beta}] \quad \dot{\beta} = v/\rho = \frac{12,000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec} \quad \text{Ans.}$$

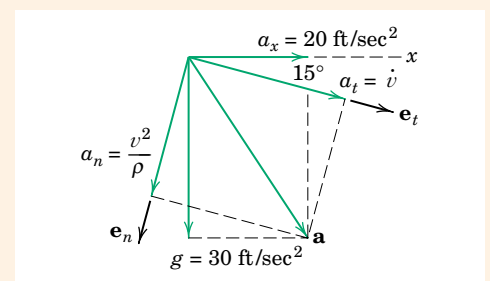
(d) With unit vectors  $\mathbf{e}_n$  and  $\mathbf{e}_t$  for the  $n$ - and  $t$ -directions, respectively, the total acceleration becomes

$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2 \quad \text{Ans.}$$

### Helpful Hints

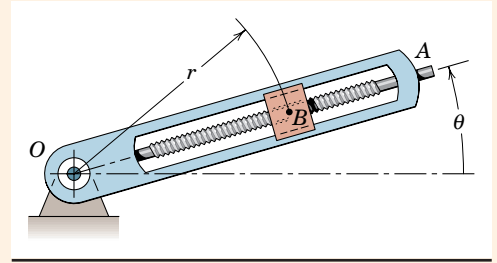
① Alternatively, we could find the resultant acceleration and then resolve it into  $n$ - and  $t$ -components.

② To convert from mi/hr to ft/sec, multiply by  $\frac{5280 \text{ ft/mi}}{3600 \text{ sec/hr}} = \frac{44 \text{ ft/sec}}{30 \text{ mi/hr}}$  which is easily remembered, as  $30 \text{ mi/hr}$  is the same as  $44 \text{ ft/sec}$ .



### Sample Problem 2/9

Rotation of the radially slotted arm is governed by  $\theta = 0.2t + 0.02t^3$ , where  $\theta$  is in radians and  $t$  is in seconds. Simultaneously, the power screw in the arm engages the slider  $B$  and controls its distance from  $O$  according to  $r = 0.2 + 0.04t^2$ , where  $r$  is in meters and  $t$  is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when  $t = 3$  s.



- Solution.** The coordinates and their time derivatives which appear in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for  $t = 3$  s.

$$\begin{aligned} r &= 0.2 + 0.04t^2 & r_3 &= 0.2 + 0.04(3^2) = 0.56 \text{ m} \\ \dot{r} &= 0.08t & \dot{r}_3 &= 0.08(3) = 0.24 \text{ m/s} \\ \ddot{r} &= 0.08 & \ddot{r}_3 &= 0.08 \text{ m/s}^2 \\ \theta &= 0.2t + 0.02t^3 & \theta_3 &= 0.2(3) + 0.02(3^3) = 1.14 \text{ rad} \\ & & & \text{or } \theta_3 = 1.14(180/\pi) = 65.3^\circ \\ \dot{\theta} &= 0.2 + 0.06t^2 & \dot{\theta}_3 &= 0.2 + 0.06(3^2) = 0.74 \text{ rad/s} \\ \ddot{\theta} &= 0.12t & \ddot{\theta}_3 &= 0.12(3) = 0.36 \text{ rad/s}^2 \end{aligned}$$

The velocity components are obtained from Eq. 2/13 and for  $t = 3$  s are

$$\begin{aligned} [v_r = \dot{r}] & & v_r &= 0.24 \text{ m/s} \\ [v_\theta = r\dot{\theta}] & & v_\theta &= 0.56(0.74) = 0.414 \text{ m/s} \\ [v = \sqrt{v_r^2 + v_\theta^2}] & & v &= \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s} \quad \text{Ans.} \end{aligned}$$

The velocity and its components are shown for the specified position of the arm.

The acceleration components are obtained from Eq. 2/14 and for  $t = 3$  s are

$$\begin{aligned} [a_r = \ddot{r} - r\dot{\theta}^2] & & a_r &= 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2 \\ [a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] & & a_\theta &= 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2 \\ [a = \sqrt{a_r^2 + a_\theta^2}] & & a &= \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2 \quad \text{Ans.} \end{aligned}$$

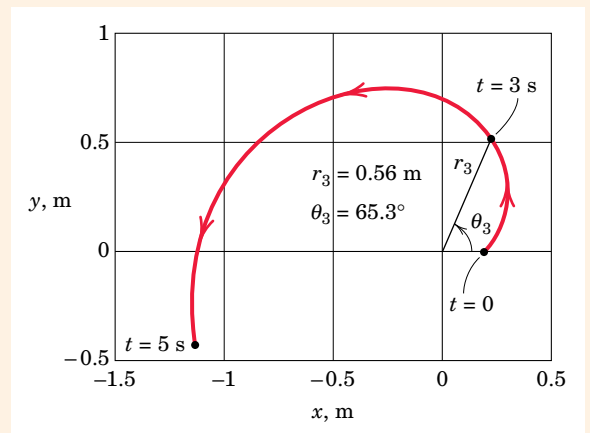
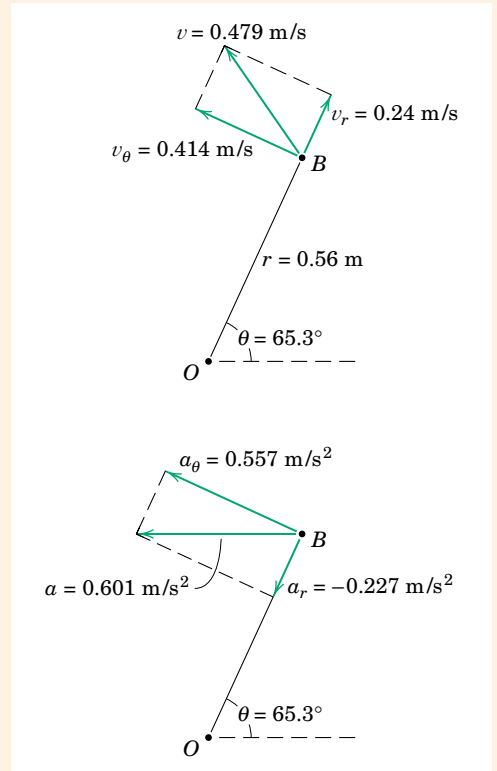
The acceleration and its components are also shown for the  $65.3^\circ$  position of the arm.

Plotted in the final figure is the path of the slider  $B$  over the time interval  $0 \leq t \leq 5$  s. This plot is generated by varying  $t$  in the given expressions for  $r$  and  $\theta$ . Conversion from polar to rectangular coordinates is given by

$$x = r \cos \theta \quad y = r \sin \theta$$

#### Helpful Hint

- ① We see that this problem is an example of constrained motion where the center  $B$  of the slider is mechanically constrained by the rotation of the slotted arm and by engagement with the turning screw.



### Sample Problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when  $\theta = 30^\circ$ , the tracking data give  $r = 25(10^4)$  ft,  $\dot{r} = 4000$  ft/sec, and  $\dot{\theta} = 0.80$  deg/sec. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is  $31.4$  ft/sec<sup>2</sup> vertically down. For these conditions determine the velocity  $v$  of the rocket and the values of  $\ddot{r}$  and  $\ddot{\theta}$ .

**Solution.** The components of velocity from Eq. 2/13 are

$$[v_r = \dot{r}] \quad v_r = 4000 \text{ ft/sec}$$

$$\textcircled{1} [v_\theta = r\dot{\theta}] \quad v_\theta = 25(10^4)(0.80)\left(\frac{\pi}{180}\right) = 3490 \text{ ft/sec}$$

$$[v = \sqrt{v_r^2 + v_\theta^2}] \quad v = \sqrt{(4000)^2 + (3490)^2} = 5310 \text{ ft/sec}$$

Since the total acceleration of the rocket is  $g = 31.4$  ft/sec<sup>2</sup> down, we can easily find its  $r$ - and  $\theta$ -components for the given position. As shown in the figure, they are

$$\textcircled{2} \quad a_r = -31.4 \cos 30^\circ = -27.2 \text{ ft/sec}^2$$

$$a_\theta = 31.4 \sin 30^\circ = 15.70 \text{ ft/sec}^2$$

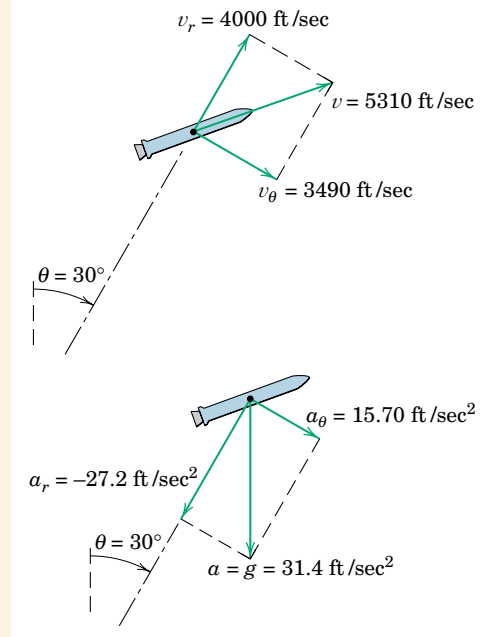
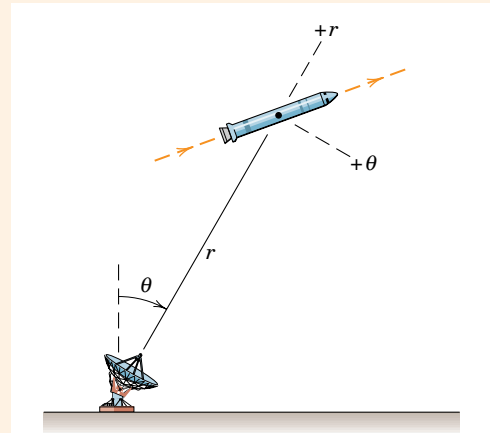
We now equate these values to the polar-coordinate expressions for  $a_r$  and  $a_\theta$  which contain the unknowns  $\ddot{r}$  and  $\ddot{\theta}$ . Thus, from Eq. 2/14

$$\textcircled{3} [a_r = \ddot{r} - r\dot{\theta}^2] \quad -27.2 = \ddot{r} - 25(10^4)\left(0.80 \frac{\pi}{180}\right)^2$$

$$\ddot{r} = 21.5 \text{ ft/sec}^2$$

$$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] \quad 15.70 = 25(10^4)\ddot{\theta} + 2(4000)\left(0.80 \frac{\pi}{180}\right)$$

$$\ddot{\theta} = -3.84(10^{-4}) \text{ rad/sec}^2$$



#### Helpful Hints

- ① We observe that the angle  $\theta$  in polar coordinates need not always be taken positive in a counterclockwise sense.
- ② Note that the  $r$ -component of acceleration is in the negative  $r$ -direction, so it carries a minus sign.
- ③ We must be careful to convert  $\dot{\theta}$  from deg/sec to rad/sec.

### Sample Problem 2/13

Passengers in the jet transport  $A$  flying east at a speed of 800 km/h observe a second jet plane  $B$  that passes under the transport in horizontal flight. Although the nose of  $B$  is pointed in the  $45^\circ$  northeast direction, plane  $B$  appears to the passengers in  $A$  to be moving away from the transport at the  $60^\circ$  angle as shown. Determine the true velocity of  $B$ .

**Solution.** The moving reference axes  $x$ - $y$  are attached to  $A$ , from which the relative observations are made. We write, therefore,

$$\textcircled{1} \quad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

Next we identify the knowns and unknowns. The velocity  $\mathbf{v}_A$  is given in both magnitude and direction. The  $60^\circ$  direction of  $\mathbf{v}_{B/A}$ , the velocity which  $B$  appears to the moving observers in  $A$ , is known, and the true velocity of  $B$  is in the  $45^\circ$  direction in which it is heading. The two remaining unknowns are the magnitudes of  $\mathbf{v}_B$  and  $\mathbf{v}_{B/A}$ . We may solve the vector equation in any one of three ways.

**(I) Graphical.** We start the vector sum at some point  $P$  by drawing  $\mathbf{v}_A$  to a convenient scale and then construct a line through the tip of  $\mathbf{v}_A$  with the known direction of  $\mathbf{v}_{B/A}$ . The known direction of  $\mathbf{v}_B$  is then drawn through  $P$ , and the intersection  $C$  yields the unique solution enabling us to complete the vector triangle and scale off the unknown magnitudes, which are found to be

$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$

**(II) Trigonometric.** A sketch of the vector triangle is made to reveal the trigonometry, which gives

$$\textcircled{4} \quad \frac{v_B}{\sin 60^\circ} = \frac{v_A}{\sin 75^\circ} \quad v_B = 800 \frac{\sin 60^\circ}{\sin 75^\circ} = 717 \text{ km/h} \quad \text{Ans.}$$

**(III) Vector Algebra.** Using unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ , we express the velocities in vector form as

$$\begin{aligned} \mathbf{v}_A &= 800\mathbf{i} \text{ km/h} & \mathbf{v}_B &= (v_B \cos 45^\circ)\mathbf{i} + (v_B \sin 45^\circ)\mathbf{j} \\ \mathbf{v}_{B/A} &= (v_{B/A} \cos 60^\circ)(-\mathbf{i}) + (v_{B/A} \sin 60^\circ)\mathbf{j} \end{aligned}$$

Substituting these relations into the relative-velocity equation and solving separately for the  $\mathbf{i}$  and  $\mathbf{j}$  terms give

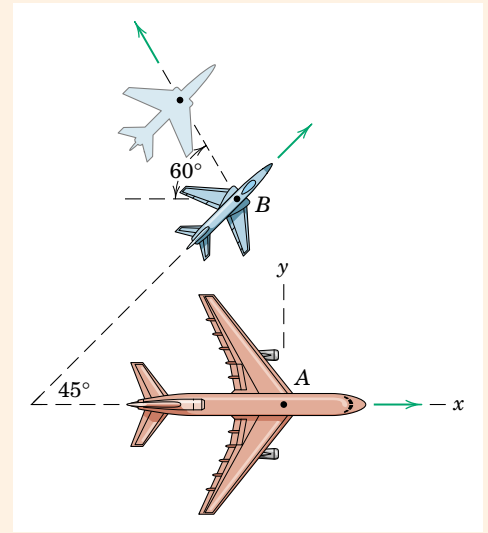
$$\text{(i-terms)} \quad v_B \cos 45^\circ = 800 - v_{B/A} \cos 60^\circ$$

$$\text{(j-terms)} \quad v_B \sin 45^\circ = v_{B/A} \sin 60^\circ$$

$\textcircled{5}$  Solving simultaneously yields the unknown velocity magnitudes

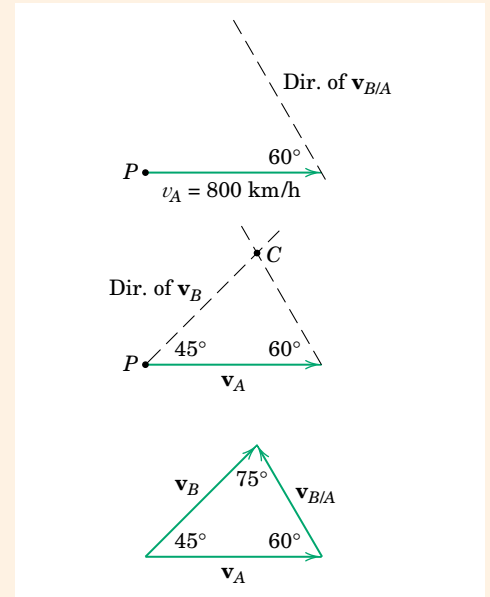
$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$

It is worth noting the solution of this problem from the viewpoint of an observer in  $B$ . With reference axes attached to  $B$ , we would write  $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ . The apparent velocity of  $A$  as observed by  $B$  is then  $\mathbf{v}_{A/B}$ , which is the negative of  $\mathbf{v}_{B/A}$ .



#### Helpful Hints

- $\textcircled{1}$  We treat each airplane as a particle.
- $\textcircled{2}$  We assume no side slip due to cross wind.
- $\textcircled{3}$  Students should become familiar with all three solutions.



- $\textcircled{4}$  We must be prepared to recognize the appropriate trigonometric relation, which here is the law of sines.
- $\textcircled{5}$  We can see that the graphical or trigonometric solution is shorter than the vector algebra solution in this particular problem.

