MECH 2110 - Statics & Dynamics

Chapter D2 Problem 3 Solution
Page 27, Engineering Mechanics - Dynamics, 4th Edition, Meriam and Kraige

Given: Particle moving along a straight line (s-axis) with velocity, v, given in terms of time, t, by:
$v = A - B\,t + C\,t^{3/2}$ $A = 2$ m/s, $B = 4$ m/s^2 $C = 5$ m/s$^{5/2}$
The position of the particle at time t = 0 is given by s_0 equal to 3 m.

Find: The position, s, velocity, v, and acceleration, a, when the time, t, is equal to 3 s.

0. Observations:
1. Interested in motion only without regard to the forces causing the motion, free body diagram is not of interest.

2. The motion is along a single straight line. The motion diagram is simple enough that it can be omitted.

1. Mechanical System - Particle during the interval from t = 0 to t = 3 s.

3. Equations
$v = A - B\,t + C\,t^{3/2}$
Relationship between velocity, acceleration and time:
$a = dv/dt = -B + 3/2\,C\,t^{1/2}$
Relationship between velocity, position, and time:
$ds/dt = v = A - B\,t + C\,t^{3/2}$
Separating variables and integrating:
$\int_{s0}^{s} dx = \int_{0}^{t} v\,dt$
$s|_{s0}^{s} = \int_{0}^{t} \{ A - B\,t + C\,t^{3/2} \}\,dt$
$s - s_0 = \{ A\,t - B\,t^2/2 + 2/5\,C\,t^{5/2} \}|_0^t$
$s = s_0 + A\,t - B\,t^2/2 + 2/5\,C\,t^{5/2}$

4. Solve
Evaluating each of the three expressions at t equal to 3 s:
$v = A - B\,t + C\,t^{3/2}$
$v(t{=}3s) = 2$ m/s $- 4$ m/s^2 3 s + 5 m/s$^{5/2}$ (3 s)$^{3/2}$
= 15.98 m/s

$a = -B + 3/2\,C\,t^{1/2}$
$a(t{=}3s) = -4$ m/s^2 + 3/2 5 m/s$^{5/2}$ (3 s)$^{1/2}$
= 8.99 m/s^2

$s = s_0 + A\,t - B\,t^2/2 + 2/5\,C\,t^{5/2}$
$s(t = 3s) = 3$ m + 2 m/s 3 s - 4 m/s^2 1/2 (3 s)^2 + 2/5 5 m/s$^{5/2}$ (3 s)$^{5/2}$

= 22.2 m

Results
Position = s(t=3s) = 22.2 m
Velocity = v(t=3s) = 15.98 m/s
Acceleration = a(t=3s) = 8.99 m/s^2

Sample Problem 2/2

A particle moves along the x-axis with an initial velocity $v_x = 50$ ft/sec at the origin when $t = 0$. For the first 4 seconds it has no acceleration, and thereafter it is acted on by a retarding force which gives it a constant acceleration $a_x = -10$ ft/sec^2. Calculate the velocity and the x-coordinate of the particle for the conditions of $t = 8$ sec and $t = 12$ sec and find the maximum positive x-coordinate reached by the particle. ①

Helpful Hints

① Learn to be flexible with symbols. The position coordinate x is just as valid as s.

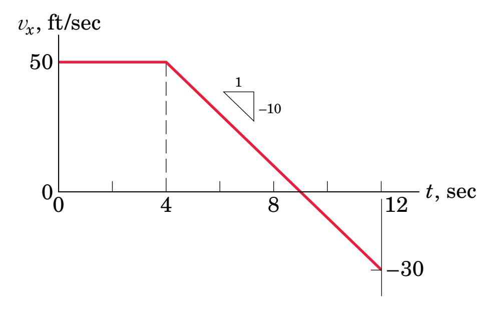
Solution. The velocity of the particle after $t = 4$ sec is computed from

② $$\left[\int dv = \int a\,dt\right] \qquad \int_{50}^{v_x} dv_x = -10\int_4^t dt \qquad v_x = 90 - 10t \text{ ft/sec}$$

and is plotted as shown. At the specified times, the velocities are

$$t = 8 \text{ sec}, \qquad v_x = 90 - 10(8) = 10 \text{ ft/sec}$$

$$t = 12 \text{ sec}, \qquad v_x = 90 - 10(12) = -30 \text{ ft/sec} \qquad \textit{Ans.}$$



The x-coordinate of the particle at any time greater than 4 seconds is the distance traveled during the first 4 seconds plus the distance traveled after the discontinuity in acceleration occurred. Thus,

$$\left[\int ds = \int v\,dt\right] \qquad x = 50(4) + \int_4^t (90 - 10t)\,dt = -5t^2 + 90t - 80 \text{ ft}$$

For the two specified times,

$$t = 8 \text{ sec}, \qquad x = -5(8^2) + 90(8) - 80 = 320 \text{ ft}$$

$$t = 12 \text{ sec}, \qquad x = -5(12^2) + 90(12) - 80 = 280 \text{ ft} \qquad \textit{Ans.}$$

The x-coordinate for $t = 12$ sec is less than that for $t = 8$ sec since the motion is in the negative x-direction after $t = 9$ sec. The maximum positive x-coordinate is, then, the value of x for $t = 9$ sec which is

$$x_{max} = -5(9^2) + 90(9) - 80 = 325 \text{ ft} \qquad \textit{Ans.}$$

③ These displacements are seen to be the net positive areas under the v-t graph up to the values of t in question.

② Note that we integrate to a general time t and then substitute specific values.

③ Show that the total distance traveled by the particle in the 12 sec is 370 ft.

MECH 2110 - Statics & Dynamics

Chapter D2 Problem 29 Solution
Page 31, Engineering Mechanics - Dynamics, 4th Edition, Meriam and Kraige

Given: The acceleration of an arrow decreases linearly with distance, s, from a maximum of a_0 equal to 16,000 ft/s^2 upon release of the arrow to zero after a distance of travel L equal to 2 ft.

Find: The maximum velocity of the arrow.

0. Observations:
1. Interested exclusively in the motion of the arrow independent of the forces producing that motion, thus no free body diagram is of interest.

2. The motion is along a single straight line. The motion diagram is simple enough that it can be omitted.

3. The arrow will travel nearly in a straight line during that brief interval between release of the aroow and the launch point.

4. As the arrow continues accelerating until it reaches the distance L, the maximum velocity will occur at that point.

1. Mechanical System - Arrow from release until it has traveled a distance L.

3. Equations
Acceleration, a, is linear with distance, s:
a = m s + b
The acceleration is known at two points:
a(s=0) = $-a_0$/L
a(s=L) = 0
The "intercept", b, is the value of the acceleration at s = 0, that is a_0. The "slope", m, is the change in acceleration, a, divided by the change in distance, s, between two points where both of those quantities are known:
m = (0 - a_0) / (L - 0) = $-a_0$/L
The dependence of the acceleration on position can be expressed as:
a = $-a_0$/L s + a_0 = a_0 { 1 - s/L }
The relationship between acceleration, velocity, and position is:
a = v dv/ds
v dv/ds = a_0 { 1 - s/L }
Separating variables and integrating:
$\int_0^{vmax}$ v dv = $\int_0^L$ a_0 { 1 - s/L }ds
1/2 $v^2|_0^{vmax}$ = a_0 { s - 1/2 s^2/L }$|_0^L$
1/2 v_{max}^2 = a_0 { L - 1/2 L^2/L }
v_{max}^2 = a_0 L

4. Solve

$v_{max}^2 = a_0 \ L$

$v_{max} = (a_0 \ L)^{1/2}$

$= (16{,}000 \text{ ft/s}^2 \ 2 \text{ ft})^{1/2}$

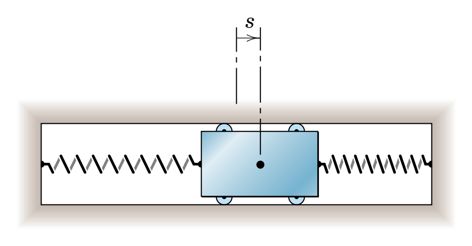
$= 178.9 \text{ ft/s}$

Results

Maximum velocity = v_{max} = 178.9 ft/s

Sample Problem 2/3

The spring-mounted slider moves in the horizontal guide with negligible friction and has a velocity v_0 in the s-direction as it crosses the mid-position where $s = 0$ and $t = 0$. The two springs together exert a retarding force to the motion of the slider, which gives it an acceleration proportional to the displacement but oppositely directed and equal to $a = -k^2 s$, where k is constant. (The constant is arbitrarily squared for later convenience in the form of the expressions.) Determine the expressions for the displacement s and velocity v as functions of the time t.



Solution I. Since the acceleration is specified in terms of the displacement, the differential relation $v\,dv = a\,ds$ may be integrated. Thus,

① $$\int v\,dv = \int -k^2 s\,ds + C_1 \text{ a constant,} \quad \text{or} \quad \frac{v^2}{2} = -\frac{k^2 s^2}{2} + C_1$$

When $s = 0$, $v = v_0$, so that $C_1 = v_0{}^2/2$, and the velocity becomes

$$v = +\sqrt{v_0{}^2 - k^2 s^2}$$

The plus sign of the radical is taken when v is positive (in the plus s-direction). This last expression may be integrated by substituting $v = ds/dt$. Thus,

② $$\int \frac{ds}{\sqrt{v_0{}^2 - k^2 s^2}} = \int dt + C_2 \text{ a constant,} \quad \text{or} \quad \frac{1}{k}\sin^{-1}\frac{ks}{v_0} = t + C_2$$

With the requirement of $t = 0$ when $s = 0$, the constant of integration becomes $C_2 = 0$, and we may solve the equation for s so that

$$s = \frac{v_0}{k}\sin kt \qquad \textit{Ans.}$$

The velocity is $v = \dot{s}$, which gives

$$v = v_0 \cos kt \qquad \textit{Ans.}$$

Solution II. Since $a = \ddot{s}$, the given relation may be written at once as

$$\ddot{s} + k^2 s = 0$$

This is an ordinary linear differential equation of second order for which the solution is well known and is

$$s = A \sin Kt + B \cos Kt$$

where A, B, and K are constants. Substitution of this expression into the differential equation shows that it satisfies the equation, provided that $K = k$. The velocity is $v = \dot{s}$, which becomes

$$v = Ak \cos kt - Bk \sin kt$$

The initial condition $v = v_0$ when $t = 0$ requires that $A = v_0/k$, and the condition $s = 0$ when $t = 0$ gives $B = 0$. Thus, the solution is

③ $$s = \frac{v_0}{k}\sin kt \quad \text{and} \quad v = v_0 \cos kt \qquad \textit{Ans.}$$

Helpful Hints

① We have used an indefinite integral here and evaluated the constant of integration. For practice, obtain the same results by using the definite integral with the appropriate limits.

② Again try the definite integral here as above.

③ This motion is called *simple harmonic motion* and is characteristic of all oscillations where the restoring force, and hence the acceleration, is proportional to the displacement but opposite in sign.

Sample Problem 2/4

A freighter is moving at a speed of 8 knots when its engines are suddenly
① stopped. If it takes 10 minutes for the freighter to reduce its speed to 4 knots, determine and plot the distance s in nautical miles moved by the ship and its speed v in knots as functions of the time t during this interval. The deceleration of the ship is proportional to the square of its speed, so that $a = -kv^2$.

Solution. The speeds and the time are given, so we may substitute the expression for acceleration directly into the basic definition $a = dv/dt$ and integrate. Thus,

$$-kv^2 = \frac{dv}{dt} \qquad \frac{dv}{v^2} = -k\,dt \qquad \int_8^v \frac{dv}{v^2} = -k\int_0^t dt$$

② $$-\frac{1}{v} + \frac{1}{8} = -kt \qquad v = \frac{8}{1 + 8kt}$$

Now we substitute the end limits of $v = 4$ knots and $t = \frac{10}{60} = \frac{1}{6}$ hour and get

$$4 = \frac{8}{1 + 8k(1/6)} \qquad k = \frac{3}{4}\text{ mi}^{-1} \qquad v = \frac{8}{1 + 6t} \qquad \textit{Ans.}$$

The speed is plotted against the time as shown.

The distance is obtained by substituting the expression for v into the definition $v = ds/dt$ and integrating. Thus,

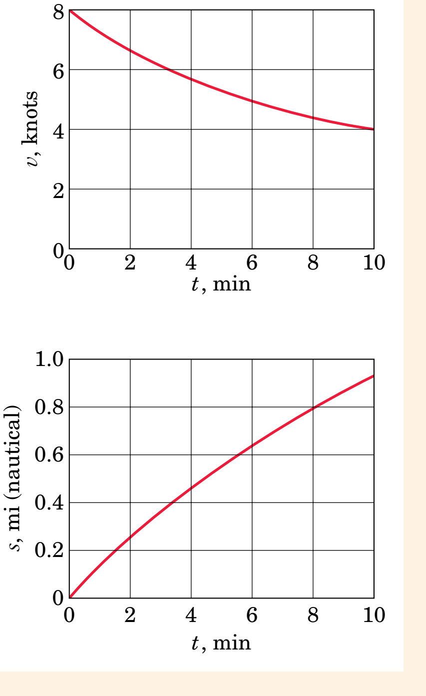
$$\frac{8}{1 + 6t} = \frac{ds}{dt} \qquad \int_0^t \frac{8\,dt}{1 + 6t} = \int_0^s ds \qquad s = \frac{4}{3}\ln(1 + 6t) \qquad \textit{Ans.}$$

The distance s is also plotted against the time as shown, and we see that the ship has moved through a distance $s = \frac{4}{3}\ln(1 + \frac{6}{6}) = \frac{4}{3}\ln 2 = 0.924$ mi (nautical) during the 10 minutes.

Helpful Hints

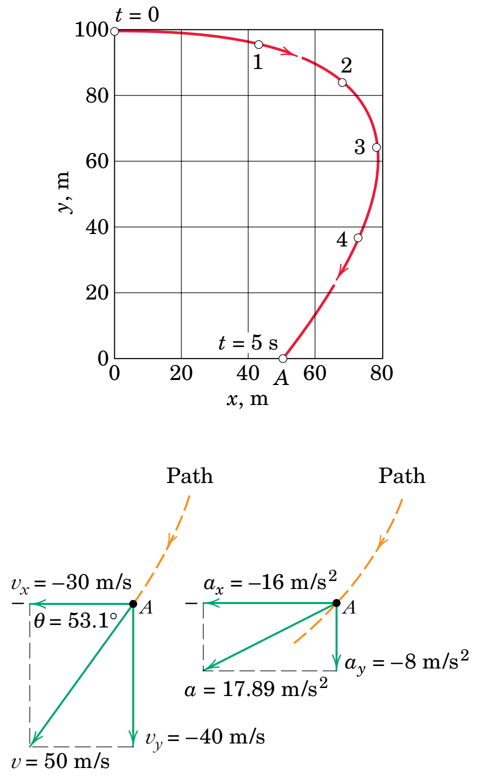
① Recall that one knot is the speed of one nautical mile (6076 ft) per hour. Work directly in the units of nautical miles and hours.

② We choose to integrate to a general value of v and its corresponding time t so that we may obtain the variation of v with t.



Sample Problem 2/5

The curvilinear motion of a particle is defined by $v_x = 50 - 16t$ and $y = 100 - 4t^2$, where v_x is in meters per second, y is in meters, and t is in seconds. It is also known that $x = 0$ when $t = 0$. Plot the path of the particle and determine its velocity and acceleration when the position $y = 0$ is reached.



Solution. The x-coordinate is obtained by integrating the expression for v_x, and the x-component of the acceleration is obtained by differentiating v_x. Thus,

$$\left[\int dx = \int v_x\,dt\right] \qquad \int_0^x dx = \int_0^t (50 - 16t)\,dt \qquad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \qquad a_x = \frac{d}{dt}(50 - 16t) \qquad a_x = -16 \text{ m/s}^2$$

The y-components of velocity and acceleration are

$$[v_y = \dot{y}] \qquad v_y = \frac{d}{dt}(100 - 4t^2) \qquad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \qquad a_y = \frac{d}{dt}(-8t) \qquad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of x and y for various values of t and plot x against y to obtain the path as shown.

When $y = 0$, $0 = 100 - 4t^2$, so $t = 5$ s. For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where $y = 0$. Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \qquad \textit{Ans.}$$

$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2 \qquad \textit{Ans.}$$

Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.

Problem 12-11

The acceleration of a particle as it moves along a straight line is given by $a = b\,t + c$. If $s = s_0$ and $v = v_0$ when $t = 0$, determine the particle's velocity and position when $t = t_1$. Also, determine the total distance the particle travels during this time period.

Given: $b := 2\frac{m}{s^3}$ $c := -1\frac{m}{s^2}$ $s_0 := 1m$ $v_0 := 2\frac{m}{s}$ $t_1 := 6s$

Solution:

$$\int_{v_0}^{v} 1\,dv = \int_0^t (b\,t + c)\,dt \qquad v = v_0 + \frac{b \cdot t^2}{2} + c \cdot t$$

$$\int_{s_0}^{s} 1\,ds = \int_0^t \left(v_0 + \frac{b \cdot t^2}{2} + c \cdot t\right) dt \qquad s = s_0 + v_0 \cdot t + \frac{b}{6} \cdot t^3 + \frac{c}{2} \cdot t^2$$

When $t = t_1$ $\quad v_1 := v_0 + \frac{b \cdot t_1^2}{2} + c \cdot t_1 \qquad v_1 = 32.00\ \frac{m}{s}$

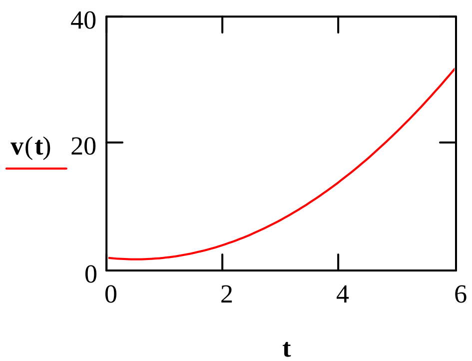
$$s_1 := s_0 + v_0 \cdot t_1 + \frac{b}{6} \cdot t_1^3 + \frac{c}{2} \cdot t_1^2 \qquad s_1 = 67.00\ m$$

The total distance traveled depends on whether the particle turned around or not. To tell we will plot the velocity and see if it is zero at any point in the interval

$t := 0, 0.01t_1 .. t_1 \qquad v(t) := v_0 + \frac{b \cdot t^2}{2} + c \cdot t$

If v never goes to zero then

$d := s_1 - s_0 \qquad d = 66.00\ m$



Problem 12-15

A particle travels to the right along a straight line with a velocity $v_p = a / (b + s_p)$. Determine its position when $t = t_1$ if $s_p = s_{p0}$ when $t = 0$.

Given: $a := 5\frac{m^2}{s}$ $\quad b := 4 \cdot m$ $\quad s_{p0} := 5m$ $\quad t_1 := 6s$

Solution:

$$\frac{ds_p}{dt} = \frac{a}{b + s_p} \qquad \int_{s_{p0}}^{s_p} (b + s_p)\, ds_p = \int_0^t a\, dt$$

$$b \cdot s_p + \frac{s_p^2}{2} - b \cdot s_{p0} - \frac{s_{p0}^2}{2} = a \cdot t$$

Guess $s_{p1} := 1m$

Given $b \cdot s_{p1} + \frac{s_{p1}^2}{2} - b \cdot s_{p0} - \frac{s_{p0}^2}{2} = a \cdot t_1$ $\quad s_{p1} := \text{Find}(s_{p1})$ $\quad s_{p1} = 7.87\, m$

Problem 12-39

A freight train starts from rest and travels with a constant acceleration a. After a time t_1 it maintains a constant speed so that when $t = t_2$ it has traveled a distance d. Determine the time t_1 and draw the v-t graph for the motion.

Given : $a := 0.5\frac{ft}{s^2}$ $\quad t_2 := 160s$ $\quad d := 2000ft$

Solution : **Guesses** $t_1 := 80s$ $\quad v_{max} := 30\frac{ft}{s}$

Given $v_{max} = a \cdot t_1$ $\quad d = \frac{1}{2} \cdot a \cdot t_1^2 + v_{max} \cdot (t_2 - t_1)$

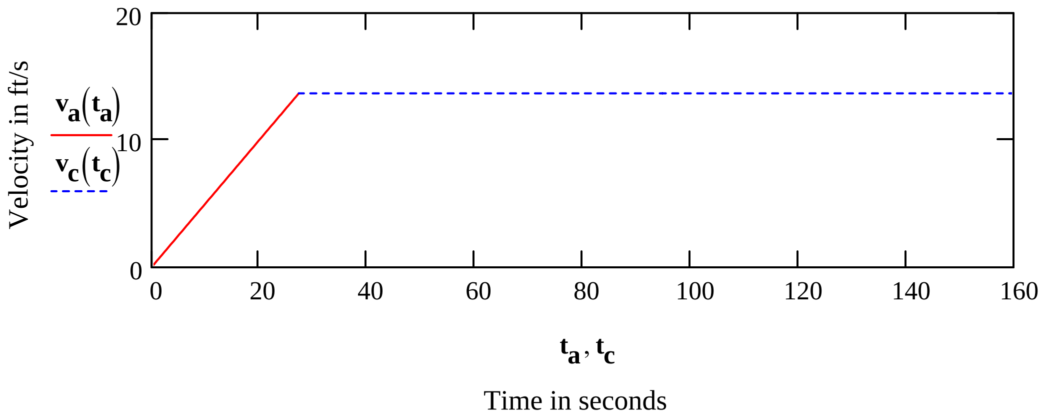
$$\begin{pmatrix} v_{max} \\ t_1 \end{pmatrix} := \text{Find}(v_{max}, t_1) \qquad v_{max} = 13.67\frac{ft}{s} \qquad t_1 = 27.34\ s$$

The equations of motion

$$t_a := 0, 0.01 \cdot t_1 .. t_1 \qquad t_c := t_1, 1.01 \cdot t_1 .. t_2$$

$$v_a(t_a) := a \cdot t_a \cdot \frac{s}{ft} \qquad v_c(t_c) := v_{max} \cdot \frac{s}{ft}$$

The plot



Problem 12-44

A motorcycle starts from rest at $s = 0$ and travels along a straight road with the speed shown by the v-t graph. Determine the motorcycle's acceleration and position when $t = t_4$ and $t = t_5$.

$s = 1.00\ \text{s}$

Given:

$v_0 := 5 \cdot \frac{\text{m}}{\text{s}}$

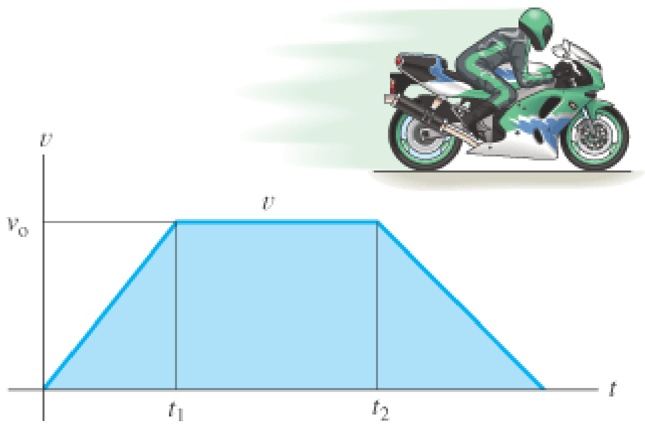
$t_1 := 4\text{s}$

$t_2 := 10\text{s}$

$t_3 := 15\text{s}$

$t_4 := 8\text{s}$

$t_5 := 12\text{s}$



Solution: At $t := t_4$ Because $t_1 < t_4 < t_2$ then $a_4 = \frac{dv}{dt} = 0$

$$s_4 := \frac{1}{2} \cdot v_0 \cdot t_1 + (t_4 - t_1) \cdot v_0$$

$s_4 = 30.00\ \text{m}$

At $t := t_5$ Because $t_2 < t_5 < t_3$ then

$$a_5 := \frac{-v_0}{t_3 - t_2}$$

$a_5 = -1.00\ \frac{\text{m}}{\text{s}^2}$

$$s_5 := \frac{1}{2} \cdot t_1 \cdot v_0 + v_0 \cdot (t_2 - t_1) + \frac{1}{2} \cdot v_0 \cdot (t_3 - t_2) - \frac{1}{2} \cdot \frac{t_3 - t_5}{t_3 - t_2} \cdot v_0 \cdot (t_3 - t_5)$$

$s_5 = 48.00\ \text{m}$

Problem 12-48

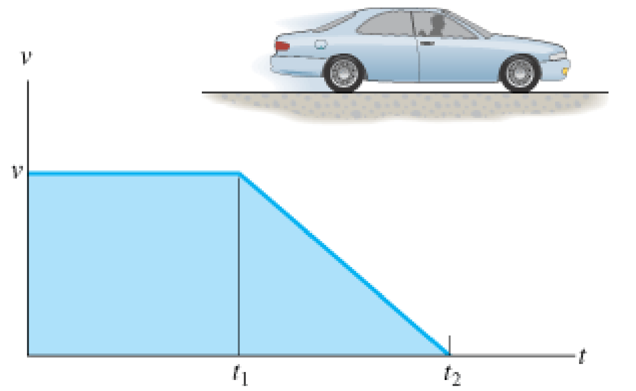
The velocity of a car is plotted as shown. Determine the total distance the car moves until it stops at time $t = t_2$. Construct the a-t graph.

Given :

$\mathbf{v} := 10\,\frac{\mathbf{m}}{\mathbf{s}}$

$\mathbf{t_1} := 40\mathrm{s}$

$\mathbf{t_2} := 80\mathrm{s}$



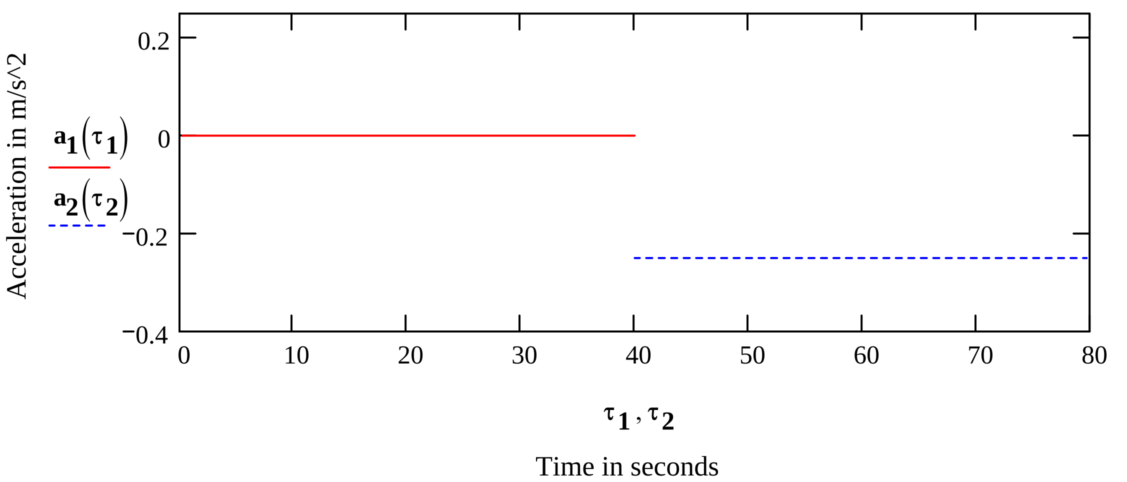
Solution :

$$\mathbf{d} := \mathbf{v}\cdot\mathbf{t_1} + \frac{1}{2}\cdot\mathbf{v}\cdot(\mathbf{t_2} - \mathbf{t_1}) \qquad \mathbf{d} = 600.00\ \mathbf{m}$$

The graph

$$\tau_1 := 0, 0.01\cdot\mathbf{t_1} .. \mathbf{t_1} \qquad \mathbf{a_1}(\tau_1) := 0\cdot\frac{\mathbf{s}^2}{\mathbf{m}}$$

$$\tau_2 := \mathbf{t_1}, 1.01\cdot\mathbf{t_1} .. \mathbf{t_2} \qquad \mathbf{a_2}(\tau_2) := \frac{-\mathbf{v}}{\mathbf{t_2} - \mathbf{t_1}}\cdot\frac{\mathbf{s}^2}{\mathbf{m}}$$



Problem 12-75

The path of a particle is defined by $y^2 = 4kx$, and the component of velocity along the y axis is $v_y = ct$, where both k and c are constants. Determine the x and y components of acceleration.

Solution :

$$y^2 = 4 \cdot k \cdot x$$

$$2 \cdot y \cdot v_y = 4 \cdot k \cdot v_x$$

$$2 \cdot v_y^{\,2} + 2 \cdot y \cdot a_y = 4 \cdot k \cdot a_x$$

$$v_y = c \cdot t$$

$$a_y = c$$

$$2 \cdot (c \cdot t)^2 + 2 \cdot y \cdot c = 4 \cdot k \cdot a_x$$

$$a_x = \frac{c}{2 \cdot k} \cdot \left(y + c \cdot t^2\right)$$