

Chapter 1

Kinematics of a Particle

1.1 Introduction

1.1.1 Position, Velocity, and Acceleration

The position of a particle P relative to a given reference frame with origin O is given by the position vector \mathbf{r} from point O to point P , as shown in Fig. 1.1. If the particle

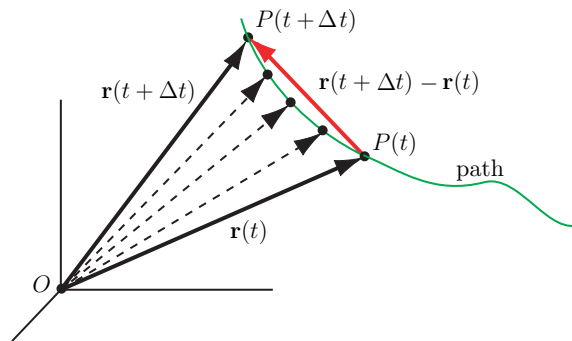


Fig. 1.1 Position of a particle P

P is in motion relative to the reference frame, the position vector \mathbf{r} is a function of time t , Fig. 1.1, and can be expressed as

$$\mathbf{r} = \mathbf{r}(t).$$

The velocity of the particle P relative to the reference frame at time t is defined by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}, \quad (1.1)$$

where the vector $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ is the change in position, or displacement of P , during the interval of time Δt , Fig. 1.1. The velocity is the rate of change of the position of the particle P . The magnitude of the velocity \mathbf{v} is the speed $v = |\mathbf{v}|$. The dimensions of \mathbf{v} are (distance)/(time). The position and velocity of a particle can be specified only relative to a reference frame.

The acceleration of the particle P relative to the given reference frame at time t is defined by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}, \quad (1.2)$$

where $\mathbf{v}(t + \Delta t) - \mathbf{v}(t)$ is the change in the velocity of P during the interval of time Δt , Fig. 1.1. The acceleration is the rate of change of the velocity of P at time t (the second time derivative of the displacement), and its dimensions are (distance)/(time)².

1.1.2 Angular Motion of a Line

The angular motion of the line L , in a plane, relative to a reference line L_0 , in the plane, is given by an angle θ , Fig. 1.2. The angular velocity of L relative to L_0 is defined by

$$\omega = \frac{d\theta}{dt} = \dot{\theta}, \quad (1.3)$$

and the angular acceleration of L relative to L_0 is defined by

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \dot{\omega} = \ddot{\theta}. \quad (1.4)$$

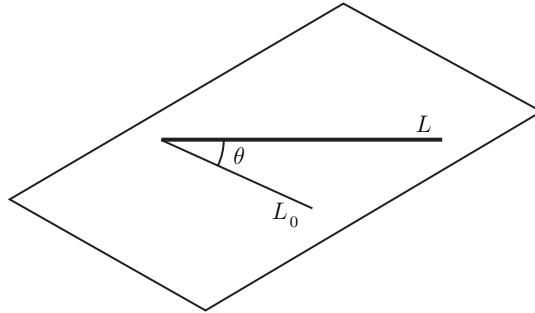


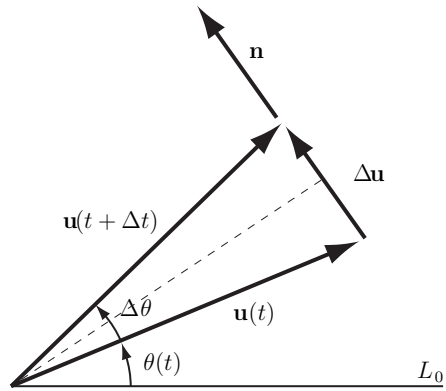
Fig. 1.2 Angular motion of line L relative to a reference line L_0

The dimensions of the angular position, angular velocity, and angular acceleration are [rad], [rad/s], and [rad/s²], respectively. The scalar coordinate θ can be positive or negative. The counterclockwise (ccw) direction is considered positive.

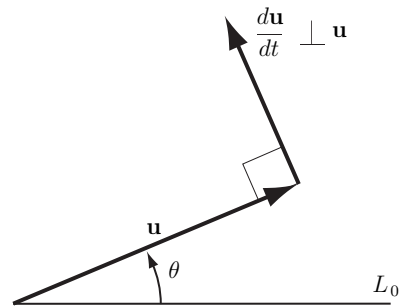
1.1.3 Rotating Unit Vector

The angular motion of a unit vector \mathbf{u} in a plane can be described as the angular motion of a line. The direction of \mathbf{u} relative to a reference line L_0 , is specified by the angle θ in Fig. 1.3(a), and the rate of rotation of \mathbf{u} relative to L_0 is defined by the angular velocity

$$\omega = \frac{d\theta}{dt} = \dot{\theta}.$$



(a)



(b)

Fig. 1.3 Angular motion of a unit vector \mathbf{u} in plane

The time derivative of \mathbf{u} is specified by

$$\frac{d\mathbf{u}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{u}(t + \Delta t) - \mathbf{u}(t)}{\Delta t}.$$

Figure 1.3(a) shows the vector \mathbf{u} at time t and at time $t + \Delta t$. The change in \mathbf{u} during this interval is $\Delta\mathbf{u} = \mathbf{u}(t + \Delta t) - \mathbf{u}(t)$, and the angle through which \mathbf{u} rotates is $\Delta\theta = \theta(t + \Delta t) - \theta(t)$. The triangle in Fig. 1.3(a) is isosceles, so the magnitude of $\Delta\mathbf{u}$ is

$$|\Delta\mathbf{u}| = 2|\mathbf{u}|\sin(\Delta\theta/2) = 2\sin(\Delta\theta/2).$$

The vector $\Delta\mathbf{u}$ is

$$\Delta\mathbf{u} = |\Delta\mathbf{u}|\mathbf{n} = 2\sin(\Delta\theta/2)\mathbf{n},$$

where \mathbf{n} is a unit vector that points in the direction of $\Delta\mathbf{u}$, Fig. 1.3(a). The time derivative of \mathbf{u} is

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta\mathbf{u}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{2\sin(\Delta\theta/2)\mathbf{n}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\sin(\Delta\theta/2)}{\Delta\theta/2} \frac{\Delta\theta}{\Delta t} \mathbf{n} = \\ &= \lim_{\Delta t \rightarrow 0} \frac{\sin(\Delta\theta/2)}{\Delta\theta/2} \frac{\Delta\theta}{\Delta t} \mathbf{n} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} \mathbf{n} = \frac{d\theta}{dt} \mathbf{n}, \end{aligned}$$

where $\lim_{\Delta t \rightarrow 0} \frac{\sin(\Delta\theta/2)}{\Delta\theta/2} = 1$ and $\lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$.

So the time derivative of the unit vector \mathbf{u} is

$$\frac{d\mathbf{u}}{dt} = \frac{d\theta}{dt} \mathbf{n} = \dot{\theta} \mathbf{n} = \boldsymbol{\omega} \mathbf{n},$$

where \mathbf{n} is a unit vector that is perpendicular to \mathbf{u} , $\mathbf{n} \perp \mathbf{u}$, and points in the positive θ direction, Fig. 1.3(b).

1.2 Rectilinear Motion

The position of a particle P on a straight line relative to a reference point O can be indicated by the coordinate s measured along the line from O to P , as shown in Fig. 1.4. In this case the reference frame is the straight line and the origin of the

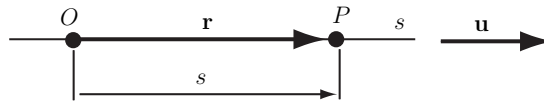


Fig. 1.4 Straight line motion of P

the reference frame is the point O . The reference frame and its origin are used to describe the position of particle P . The coordinate s is considered to be positive to the right of the origin O and is considered to be negative to the left of the origin.

Let \mathbf{u} be a unit vector parallel to the straight line and pointing in the positive s , Fig. 1.4. The position vector of the point P relative to the origin O is

$$\mathbf{r} = s\mathbf{u}.$$

The velocity of the particle P relative to the origin O is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt}\mathbf{u} = \dot{s}\mathbf{u}.$$

The magnitude v of the velocity vector $\mathbf{v} = v\mathbf{u}$ is the speed (velocity scalar)

$$v = \frac{ds}{dt} = \dot{s}.$$

The speed v of the particle P is equal to the slope at time t of the line tangent to the graph of s as a function of time.

The acceleration of the particle P relative to O is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{u}) = \frac{dv}{dt}\mathbf{u} = \dot{v}\mathbf{u} = \ddot{s}\mathbf{u}.$$

The magnitude a of the acceleration vector $\mathbf{a} = a\mathbf{u}$ is the acceleration scalar

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The acceleration a is equal to the slope at time t of the line tangent to the graph of v as a function of time.

1.3 Curvilinear Motion

The motion of the particle P along a curvilinear path, relative to a reference frame, can be specified in terms of its position, velocity, and acceleration vectors. The directions and magnitudes of the position, velocity, and acceleration vectors do not depend on the particular coordinate system used to express them. The representations of the position, velocity, and acceleration vectors are different in different coordinate systems.

1.3.1 Cartesian Coordinates

Let \mathbf{r} be the position vector of a particle P relative to the origin O of a cartesian reference frame, Fig. 1.5. The components of \mathbf{r} are the $x, y,$ and z coordinates of the particle P

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \quad (1.5)$$

The velocity of the particle P relative to the reference frame is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}. \quad (1.6)$$

The velocity in terms of scalar components is

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}, \quad (1.7)$$

Three scalar equations can be obtained

$$v_x = \frac{dx}{dt} = \dot{x}, \quad v_y = \frac{dy}{dt} = \dot{y}, \quad v_z = \frac{dz}{dt} = \dot{z}. \quad (1.8)$$

The acceleration of the particle P relative to the reference frame is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \dot{\mathbf{r}} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} = \dot{v}_x\mathbf{i} + \dot{v}_y\mathbf{j} + \dot{v}_z\mathbf{k} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}.$$

Expressing the acceleration in terms of scalar components

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}, \quad (1.9)$$

three scalar equations can be obtained

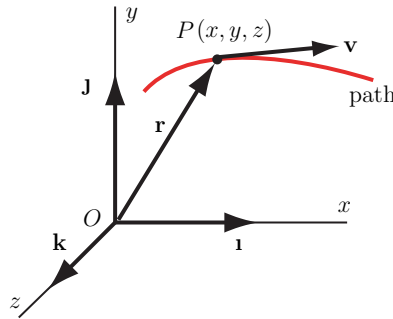


Fig. 1.5 Position vector of a particle P in a cartesian reference frame

$$a_x = \frac{dv_x}{dt} = \dot{v}_x = \ddot{x}, \quad a_y = \frac{dv_y}{dt} = \dot{v}_y = \ddot{y}, \quad a_z = \frac{dv_z}{dt} = \dot{v}_z = \ddot{z}. \quad (1.10)$$

Equations (1.8) and (1.10) describe the motion of a particle relative to a cartesian coordinate system.

1.3.2 Normal and Tangential Coordinates

The position, velocity, and acceleration of a particle will be specified in terms of their components tangential and normal (perpendicular) to the path. The particle P is moving along a plane, curvilinear path relative to a reference frame, Fig. 1.6. The

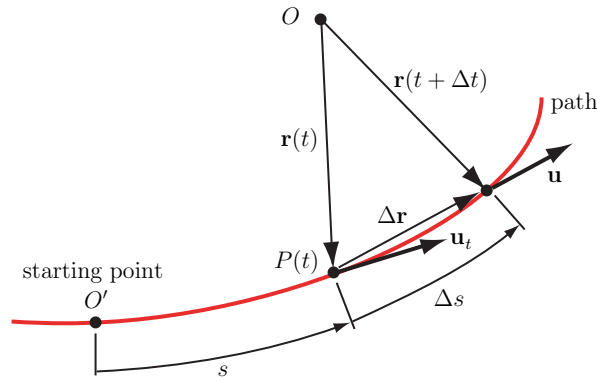


Fig. 1.6 Particle P moving along a plane, curvilinear path

position vector \mathbf{r} specifies the position of the particle P relative to the reference point O . The coordinate s measures the position of the particle P along the path relative to a point O' on the path. The velocity of P relative to O is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t}, \quad (1.11)$$

where $\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$, as shown in Fig. 1.6. The distance traveled along the path from t to $t + \Delta t$ is Δs . One can write Eq. (1.11) as

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} \mathbf{u},$$

where \mathbf{u} is a unit vector in the direction of $\Delta \mathbf{r}$. In the limit, Δt approaches zero, the magnitude of $\Delta \mathbf{r}$ equals ds because a chord progressively approaches the curve. For the same reason, the direction of $\Delta \mathbf{r}$ approaches tangency to the curve, \mathbf{u} becomes a unit vector, \mathbf{u}_t , tangent to the path at the position of P , as shown Fig. 1.6

$$\mathbf{v} = v \mathbf{u}_t = \frac{ds}{dt} \mathbf{u}_t. \quad (1.12)$$

The *tangent direction* is defined by the unit tangent vector \mathbf{u}_t (or $\boldsymbol{\tau}$), which is a path variable parameter

$$\frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \mathbf{u}_t,$$

or

$$\mathbf{u}_t = \frac{d\mathbf{r}}{ds}. \quad (1.13)$$

The velocity of a particle in curvilinear motion is a vector whose magnitude equals the rate of change of distance traveled along the path and whose direction is tangent to the path.

To determine the acceleration of P , the time derivative of Eq. (1.12) is taken

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt} \mathbf{u}_t + v \frac{d\mathbf{u}_t}{dt}. \quad (1.14)$$

If the path is not a straight line, the unit vector \mathbf{u}_t rotates as P moves on the path, and the time derivative of \mathbf{u}_t is not zero. The path angle θ defines the direction of \mathbf{u}_t relative to a reference line shown in Fig. 1.7. The time derivative of the rotating tangent unit vector \mathbf{u}_t is

$$\frac{d\mathbf{u}_t}{dt} = \frac{d\theta}{dt} \mathbf{u}_n,$$

where \mathbf{u}_n is a unit vector that is normal to \mathbf{u}_t , and points in the positive θ direction if $d\theta/dt$ is positive. The normal unit vector \mathbf{u}_n (or $\boldsymbol{\nu}$) defines the normal direction to the path. Substituting this expression into Eq. (1.14), the acceleration of P is obtained

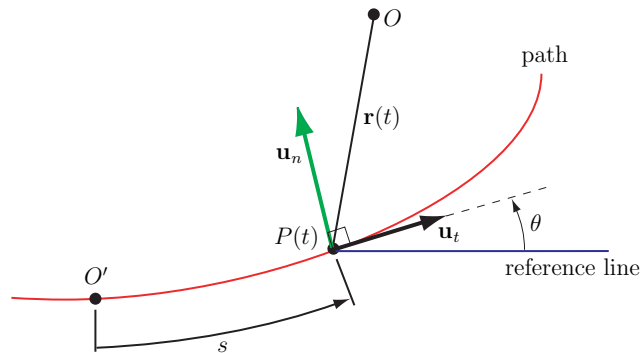


Fig. 1.7 Path angle θ and normal and tangent unit vectors to the path

$$\mathbf{a} = \frac{dv}{dt} \mathbf{u}_t + v \frac{d\theta}{dt} \mathbf{u}_n. \quad (1.15)$$

If the path is a straight line at time t , the normal component of the acceleration equals zero, because in that case $d\theta/dt$ is zero.

The tangential component of the acceleration arises from the rate of change of the magnitude of the velocity. The normal component of the acceleration arises from the rate of change in the direction of the velocity vector.

Figure 1.8 shows the positions on the path reached by P at time t , $P(t)$, and at time $t + dt$, $P(t + dt)$. If the path is curved, straight lines extended from these points

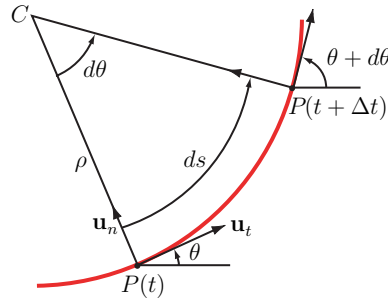


Fig. 1.8 Instantaneous radius of curvature ρ

$P(t)$ and $P(t + dt)$ perpendicular to the path will intersect at C as shown in Fig. 1.8. The distance ρ from the path to the particle where these two lines intersect is called the *instantaneous radius of curvature* of the path.

If the path is circular with radius a , then the radius of curvature equals the radius of the path, $\rho = a$. The angle $d\theta$ is the change in the path angle, and ds is the distance traveled, from t to $t + \Delta t$. The radius of curvature ρ is related to ds by, Fig. 1.8

$$ds = \rho d\theta.$$

Dividing by dt , one can obtain

$$\frac{ds}{dt} = v = \rho \frac{d\theta}{dt}.$$

Using this relation, one can write Eq. (1.15) as

$$\mathbf{a} = \frac{dv}{dt} \mathbf{u}_t + \frac{v^2}{\rho} \mathbf{u}_n.$$

For a given value of v , the normal component of the acceleration depends on the instantaneous radius of curvature. The greater the curvature of the path, the greater the normal component of acceleration. When the acceleration is expressed in this

way, the normal unit vector \mathbf{u}_n must be defined to point toward the concave side of the path, Fig. 1.9. The velocity and acceleration in terms of normal and tangential

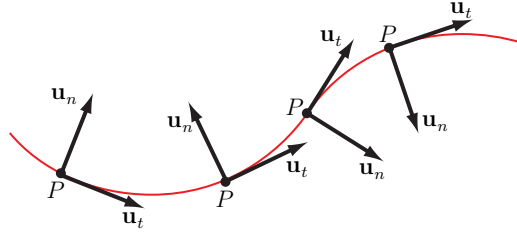


Fig. 1.9 Direction of the normal unit vector \mathbf{u}_n

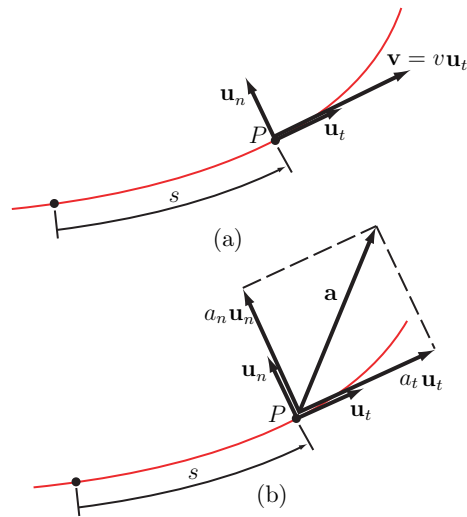


Fig. 1.10 (a) Velocity and (b) acceleration in terms of normal and tangential components

components are, Fig.1.10

$$\mathbf{v} = v \mathbf{u}_t = \frac{ds}{dt} \mathbf{u}_t, \quad (1.16)$$

$$\mathbf{a} = a_t \mathbf{u}_t + a_n \mathbf{u}_n, \quad (1.17)$$

where

$$a_t = \frac{dv}{dt}, \quad a_n = v \frac{d\theta}{dt} = \frac{v^2}{\rho}. \quad (1.18)$$

If the motion occurs in the $x - y$ plane of a cartesian reference frame, Fig. 1.11, and θ is the angle between the x axis and the unit vector $\boldsymbol{\tau}$, the unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are related to the cartesian unit vectors by

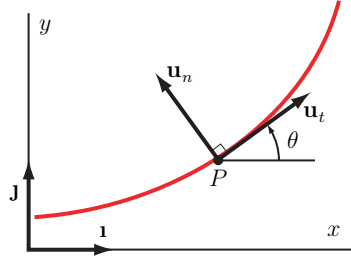


Fig. 1.11 $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ in a cartesian reference frame

$$\begin{aligned} \mathbf{u}_t &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \\ \mathbf{u}_n &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}. \end{aligned} \quad (1.19)$$

If the path in the $x - y$ plane is described by a function $y = y(x)$, it can be shown that the instantaneous radius of curvature is given by

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\left| \frac{d^2y}{dx^2} \right|}. \quad (1.20)$$

1.3.3 Circular Motion

The particle P moves in a plane circular path of radius R as shown in Fig. 1.12. The distance s is

$$s = R\theta, \quad (1.21)$$

where the angle θ specifies the position of the particle P along the circular path. The velocity is obtained taking the time derivative of Eq. (1.21)

$$v = \dot{s} = R\dot{\theta} = R\omega, \quad (1.22)$$

where $\omega = \dot{\theta}$ is the angular velocity of the line from the center of the path O to the particle P . The tangential component of the acceleration is $a_t = dv/dt$ and the

$$a_t = \dot{v} = R\dot{\omega} = R\alpha, \quad (1.23)$$

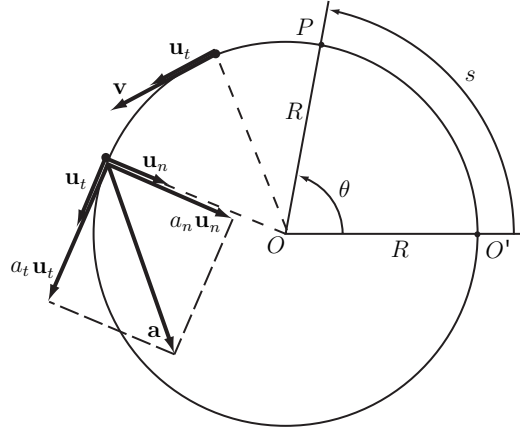


Fig. 1.12 Circular path of radius R

where $\alpha = \dot{\omega}$ is the angular acceleration. The normal component of the acceleration is

$$a_n = \frac{v^2}{R} = R\omega^2. \quad (1.24)$$

For the circular path the instantaneous radius of curvature is $\rho = R$.

1.3.4 Polar Coordinates

A particle P is considered in the $x - y$ plane of a cartesian coordinate system. The position of the point P relative to the origin O may be specified either by its cartesian coordinates x, y or by its polar coordinates r, θ as shown in Fig. 1.13. The polar coordinates are defined by:

- the unit vector \mathbf{u}_r , that points in the direction of the radial line from the origin O to the particle P ;
- the unit vector \mathbf{u}_θ that is perpendicular to \mathbf{u}_r , and points in the direction of increasing the angle θ .

The unit vectors \mathbf{u}_r and \mathbf{u}_θ are related to the cartesian unit vectors \mathbf{i} and \mathbf{j} by

$$\begin{aligned} \mathbf{u}_r &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \\ \mathbf{u}_\theta &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}. \end{aligned} \quad (1.25)$$

The position vector \mathbf{r} from O to P is

$$\mathbf{r} = r\mathbf{u}_r, \quad (1.26)$$

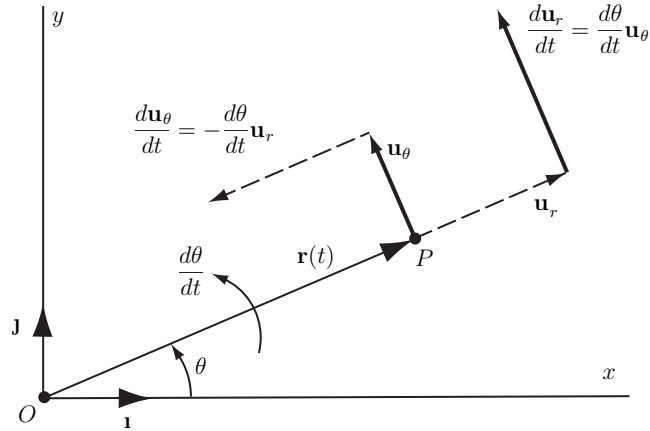


Fig. 1.13 Polar coordinates r and θ

where r is the magnitude of the vector \mathbf{r} , $r = |\mathbf{r}|$.

The velocity of the particle P in terms of polar coordinates is obtained by taking the time derivative of Eq. (1.26)

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt}. \quad (1.27)$$

The time derivative of the rotating unit vector \mathbf{u}_r is

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\theta}{dt}\mathbf{u}_\theta = \omega\mathbf{u}_\theta, \quad (1.28)$$

where $\omega = d\theta/dt$ is the angular velocity.

Substituting Eq. (1.28) into Eq. (1.27), the velocity of P is

$$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta = \dot{r}\mathbf{u}_r + r\omega\mathbf{u}_\theta = \dot{r}\mathbf{u}_r + r\omega\mathbf{u}_\theta, \quad (1.29)$$

or

$$\mathbf{v} = v_r\mathbf{u}_r + v_\theta\mathbf{u}_\theta, \quad (1.30)$$

where

$$v_r = \frac{dr}{dt} = \dot{r} \text{ and } v_\theta = r\omega. \quad (1.31)$$

The acceleration of the particle P is obtained by taking the time derivative of Eq. (1.29)

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\mathbf{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta +$$

$$r \frac{d^2\theta}{dt^2} \mathbf{u}_\theta + r \frac{d\theta}{dt} \frac{d\mathbf{u}_\theta}{dt} \quad (1.32)$$

As P moves, \mathbf{u}_θ also rotates with angular velocity $d\theta/dt$. The time derivative of the unit vector \mathbf{u}_θ is in the $-\mathbf{u}_r$ direction if $d\theta/dt$ is positive

$$\frac{d\mathbf{u}_\theta}{dt} = -\frac{d\theta}{dt} \mathbf{u}_r. \quad (1.33)$$

Substituting Eq. (1.33) and Eq. (1.28) into Eq. (1.32), the acceleration of the particle P is

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \mathbf{u}_r + \left[r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] \mathbf{u}_\theta.$$

Thus the acceleration of P is

$$\mathbf{a} = a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta, \quad (1.34)$$

where

$$\begin{aligned} a_r &= \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = \frac{d^2r}{dt^2} - r\omega^2 = \ddot{r} - r\omega^2, \\ a_\theta &= r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} = r\alpha + 2\omega \frac{dr}{dt} = r\alpha + 2\dot{r}\omega. \end{aligned} \quad (1.35)$$

The term

$$\alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

is called the angular acceleration.

The radial component of the acceleration $-r\omega^2$ is called *the centripetal* acceleration. The transverse component of the acceleration $2\omega(dr/dt)$ is called the *Coriolis* acceleration.

1.3.5 Cylindrical Coordinates

The cylindrical coordinates r, θ , and z describe the motion of a particle P in the xyz space as shown in Fig. 1.14. The cylindrical coordinates r and θ are the polar coordinates of P measured in the plane parallel to the $x-y$ plane, and the unit vectors \mathbf{u}_r , and \mathbf{u}_θ are the same. The coordinate z measure the position of the particle P perpendicular to the $x-y$ plane. The unit vector \mathbf{k} attached to the coordinate z points in the positive z axis direction. The position vector \mathbf{r} of the particle P in terms of cylindrical coordinates is

$$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}. \quad (1.36)$$

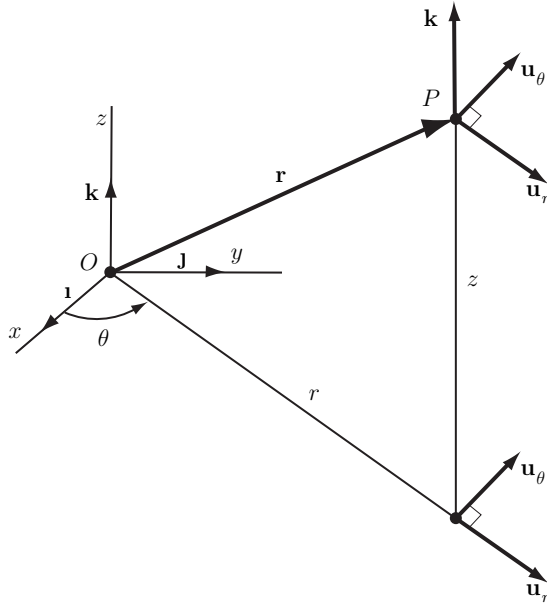


Fig. 1.14 Cylindrical coordinates r , θ , and z

The coordinate r in Eq. (1.36) is not equal to the magnitude of \mathbf{r} except when the particle P moves along a path in the $x - y$ plane.

The velocity of the particle P is

$$\begin{aligned}
 \mathbf{v} &= \frac{d\mathbf{r}}{dt} = v_r \mathbf{u}_r + v_\theta \mathbf{u}_\theta + v_z \mathbf{k} \\
 &= \frac{dr}{dt} \mathbf{u}_r + r\omega \mathbf{u}_\theta + \frac{dz}{dt} \mathbf{k} \\
 &= \dot{r} \mathbf{u}_r + r\omega \mathbf{u}_\theta + \dot{z} \mathbf{k},
 \end{aligned} \tag{1.37}$$

and the acceleration of the particle P is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta + a_z \mathbf{k}, \tag{1.38}$$

where

$$\begin{aligned}
 a_r &= \frac{d^2 r}{dt^2} - r\omega^2 = \ddot{r} - r\omega^2, \\
 a_\theta &= r\alpha + 2\frac{dr}{dt}\omega = r\alpha + 2\dot{r}\omega, \\
 a_z &= \frac{d^2 z}{dt^2} = \ddot{z}.
 \end{aligned} \tag{1.39}$$

1.4 Relative Motion

Suppose that A and B are two particles that move relative to a reference frame with origin at point O , Fig. 1.15. Let \mathbf{r}_A and \mathbf{r}_B be the position vectors of points A and

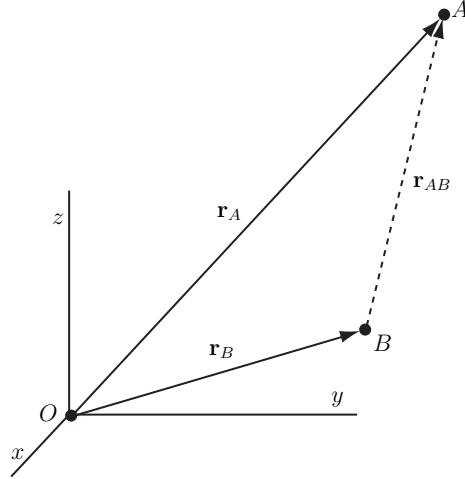


Fig. 1.15 Relative motion of two particles A and B

B relative to O . The vector \mathbf{r}_{BA} is the position vector of point A relative to point B . These vectors are related by

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{BA}. \quad (1.40)$$

The time derivative of Eq. (1.40) is

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}, \quad (1.41)$$

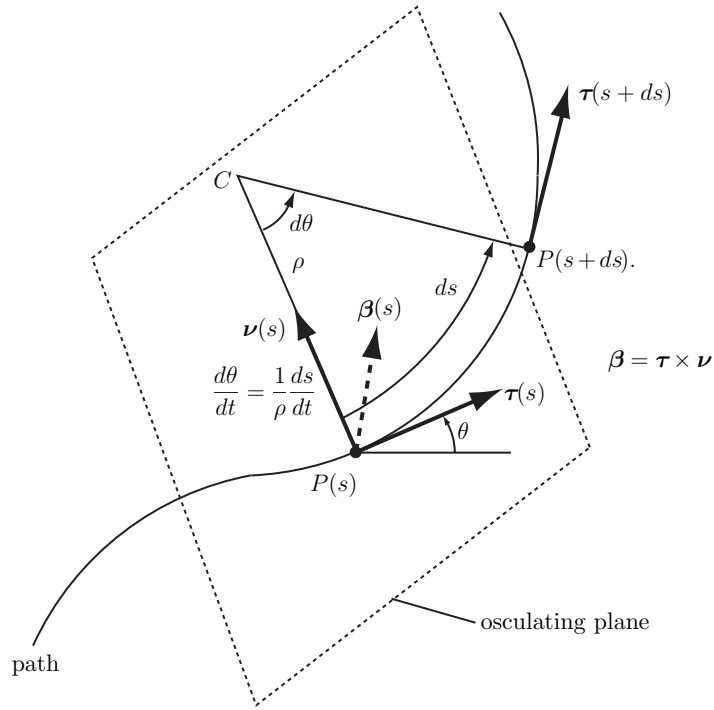
where \mathbf{v}_A is the velocity of A relative to O , \mathbf{v}_B is the velocity of B relative to O , and $\mathbf{v}_{AB} = d\mathbf{r}_{AB}/dt = \dot{\mathbf{r}}_{AB}$ is the velocity of A relative to B . The time derivative of Eq. (1.41) is

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{AB}, \quad (1.42)$$

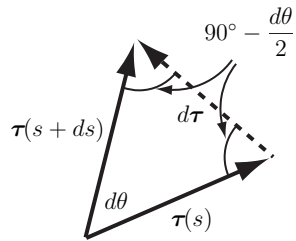
where \mathbf{a}_A and \mathbf{a}_B are the accelerations of A and B relative to O and $\mathbf{a}_{AB} = d\mathbf{v}_{AB}/dt = \dot{\mathbf{v}}_{AB}$ is the acceleration of A relative to B .

1.5 Frenet's Formulas

The motion of a particle P along a three-dimensional path is considered, Fig. 1.16 (a). The tangent direction is defined by the unit tangent vector $\boldsymbol{\tau}$ ($|\boldsymbol{\tau}| = 1$)



(a)



(b)

Fig. 1.16 Frenet's reference frame

$$\boldsymbol{\tau} = \frac{d\mathbf{r}}{ds}. \quad (1.43)$$

The second unit vector is derived by considering the dependence of $\boldsymbol{\tau}$ on s , $\boldsymbol{\tau} = \boldsymbol{\tau}(s)$. The dot product $\boldsymbol{\tau} \cdot \boldsymbol{\tau}$ gives the magnitude of the unit vector $\boldsymbol{\tau}$, i.e.

$$\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 1. \quad (1.44)$$

Equation (1.44) can be differentiated with respect to the path variable s

$$\frac{d\boldsymbol{\tau}}{ds} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{ds} = 0 \implies \boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{ds} = 0. \quad (1.45)$$

Equation (1.45) means that the vector $d\boldsymbol{\tau}/ds$ is always perpendicular to the vector $\boldsymbol{\tau}$. The normal direction, with the unit vector is $\boldsymbol{\nu}$, is defined to be parallel to the derivative $d\boldsymbol{\tau}/ds$. Because parallelism of two vectors corresponds to their proportionality, the normal unit vector may be written as

$$\boldsymbol{\nu} = \rho \frac{d\boldsymbol{\tau}}{ds}, \quad (1.46)$$

or

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{1}{\rho} \boldsymbol{\nu}, \quad (1.47)$$

where ρ is the radius of curvature.

Figure 1.16(a) depicts the tangent and normal vectors associated with two points, $P(s)$ and $P(s+ds)$. The two points are separated by an infinitesimal distance ds measured along an arbitrary planar path. The point C is the intersection of the normal vectors at the two positions along the curve, and it is the center of curvature. Because ds is infinitesimal, the arc $P(s)P(s+ds)$ seems to be circular. The radius ρ of this arc is the radius of curvature. The formula for the arc of a circle is

$$d\theta = ds/\rho.$$

The angle $d\theta$ between the normal vectors in Fig. 1.16(a) is also the angle between the tangent vectors $\boldsymbol{\tau}(s+ds)$ and $\boldsymbol{\tau}(s)$. The vector triangle $\boldsymbol{\tau}(s+ds)$, $\boldsymbol{\tau}(s)$, $d\boldsymbol{\tau} = \boldsymbol{\tau}(s+ds) - \boldsymbol{\tau}(s)$ in Fig. 1.16(b) is isosceles because $|\boldsymbol{\tau}(s+ds)| = |\boldsymbol{\tau}(s)| = 1$. Hence, the angle between $d\boldsymbol{\tau}$ and either tangent vector is $90^\circ - d\theta/2$. Since $d\theta$ is infinitesimal, the vector $d\boldsymbol{\tau}$ is perpendicular to the vector $\boldsymbol{\tau}$ in the direction of $\boldsymbol{\nu}$. A unit vector has a length of one, so

$$|d\boldsymbol{\tau}| = d\theta |\boldsymbol{\tau}| = \frac{ds}{\rho}.$$

Any vector may be expressed as the product of its magnitude and a unit vector defining the sense of the vector

$$d\boldsymbol{\tau} = |d\boldsymbol{\tau}|\boldsymbol{\nu} = \frac{ds}{\rho}\boldsymbol{\nu}. \quad (1.48)$$

Note that the radius of curvature ρ is generally not a constant.

The tangent ($\boldsymbol{\tau}$) and normal ($\boldsymbol{\nu}$) unit vectors at a selected position form a plane, the *osculating plane*, that is tangent to the curve. Any plane containing $\boldsymbol{\tau}$, is tangent to the curve. When the path is not planar, the orientation of the osculating plane containing the $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ pair will depend on the position along the curve. The direction perpendicular to the osculating plane is called the *binormal*, and the corresponding unit vector is $\boldsymbol{\beta}$. The cross product of two unit vectors is a unit vector perpendicular to the original two, so the binormal direction may be defined such that

$$\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}. \quad (1.49)$$

Next the derivative of the $\boldsymbol{\nu}$ unit vector with respect to s in terms of its tangent, normal, and binormal components will be calculated. The component of any vector in a specific direction may be obtained from a dot product with a unit vector in that direction

$$\frac{d\boldsymbol{\nu}}{ds} = \left(\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\nu}}{ds}\right)\boldsymbol{\tau} + \left(\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\nu}}{ds}\right)\boldsymbol{\nu} + \left(\boldsymbol{\beta} \cdot \frac{d\boldsymbol{\nu}}{ds}\right)\boldsymbol{\beta}. \quad (1.50)$$

The orthogonality of the unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, $\boldsymbol{\tau} \perp \boldsymbol{\nu}$, requires that

$$\boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0. \quad (1.51)$$

Equation (1.51) can be differentiated with respect to the path variable s

$$\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\nu}}{ds} + \boldsymbol{\nu} \cdot \frac{d\boldsymbol{\tau}}{ds} = 0,$$

or

$$\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\nu}}{ds} = -\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\tau}}{ds} = -\boldsymbol{\nu} \cdot \left(\frac{1}{\rho}\boldsymbol{\nu}\right) = -\frac{1}{\rho}. \quad (1.52)$$

Because $\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1$ one may find that

$$\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\nu}}{ds} = 0. \quad (1.53)$$

The derivative of the binormal component is

$$\frac{1}{T} = \boldsymbol{\beta} \cdot \frac{d\boldsymbol{\nu}}{ds}, \quad (1.54)$$

or

$$\frac{d\boldsymbol{\nu}}{ds} = \frac{1}{T}\boldsymbol{\beta}, \quad (1.55)$$

where T is the *torsion*. The reciprocal is used for consistency with Eq. (1.47). The torsion T has the dimension of length.

Substitution of Eqs. (1.52)(1.53) and (1.54) into Eq. (1.50) results in

$$\frac{d\boldsymbol{\nu}}{ds} = -\frac{1}{\rho}\boldsymbol{\tau} + \frac{1}{T}\boldsymbol{\beta}. \quad (1.56)$$

The derivative of $\boldsymbol{\beta}$

$$\frac{d\boldsymbol{\beta}}{ds} = \left(\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\beta}}{ds}\right)\boldsymbol{\tau} + \left(\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\beta}}{ds}\right)\boldsymbol{\nu} + \left(\boldsymbol{\beta} \cdot \frac{d\boldsymbol{\beta}}{ds}\right)\boldsymbol{\beta}. \quad (1.57)$$

may be obtained by a similar approach.

Using the fact that $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ are mutually orthogonal, and Eqs. (1.47)(1.56), yields

$$\begin{aligned} \boldsymbol{\tau} \cdot \boldsymbol{\beta} = 0 &\implies \boldsymbol{\tau} \cdot \frac{d\boldsymbol{\beta}}{ds} = -\frac{d\boldsymbol{\tau}}{ds} \cdot \boldsymbol{\beta} = -\frac{1}{\rho}\boldsymbol{\nu} \cdot \boldsymbol{\beta} = 0, \\ \boldsymbol{\nu} \cdot \boldsymbol{\beta} = 0 &\implies \boldsymbol{\nu} \cdot \frac{d\boldsymbol{\beta}}{ds} = -\frac{d\boldsymbol{\nu}}{ds} \cdot \boldsymbol{\beta} = -\frac{1}{T}, \\ \boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1 &\implies \boldsymbol{\beta} \cdot \frac{d\boldsymbol{\beta}}{ds} = 0. \end{aligned} \quad (1.58)$$

The result is

$$\frac{d\boldsymbol{\beta}}{ds} = -\frac{1}{T}\boldsymbol{\nu}. \quad (1.59)$$

Because $\boldsymbol{\nu}$ is a unit vector, this relation provides an alternative to Eq. (1.55) for the torsion

$$\frac{1}{T} = -\left|\frac{d\boldsymbol{\beta}}{ds}\right|. \quad (1.60)$$

Equations (1.47), (1.56), and (1.59) are the Frenet's formulas for a spatial curve.

Next the path is given in parametric form, the x , y , and z coordinates are given in terms of a parameter α . The position vector may be written as

$$\mathbf{r} = x(\alpha)\mathbf{i} + y(\alpha)\mathbf{j} + z(\alpha)\mathbf{k}. \quad (1.61)$$

The unit tangent vector is

$$\boldsymbol{\tau} = \frac{d\mathbf{r}}{d\alpha} \frac{d\alpha}{ds} = \frac{\mathbf{r}'(\alpha)}{s'(\alpha)}, \quad (1.62)$$

where a prime denotes differentiation with respect to α and

$$\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}.$$

Using the fact that $|\boldsymbol{\tau}| = 1$ one may write

$$s' = (\mathbf{r}' \cdot \mathbf{r}')^{1/2} = [(x')^2 + (y')^2 + (z')^2]^{1/2}. \quad (1.63)$$

The arclength s may be computed with the relation

$$s = \int_{\alpha_0}^{\alpha} [(x')^2 + (y')^2 + (z')^2]^{1/2} d\alpha, \quad (1.64)$$

where α_0 is the value at the starting position. The value of s' found from Eq. (1.63) may be substituted into Eq. (1.62) to calculate the tangent vector

$$\boldsymbol{\tau} = \frac{x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}}{\sqrt{(x')^2 + (y')^2 + (z')^2}}. \quad (1.65)$$

From Eqs. (1.62),(1.47) the normal vector is

$$\begin{aligned} \boldsymbol{\nu} &= \rho \frac{d\boldsymbol{\tau}}{ds} = \rho \frac{d\boldsymbol{\tau}}{d\alpha} \frac{d\alpha}{ds} = \frac{\rho}{s'} \left(\frac{\mathbf{r}''}{s'} - \frac{\mathbf{r}'s''}{(s')^2} \right) \\ &= \frac{\rho}{(s')^3} (\mathbf{r}''s' - \mathbf{r}'s''). \end{aligned} \quad (1.66)$$

The value of s' is given by Eq. (1.63) and the value of s'' is obtained differentiating Eq. (1.63)

$$s'' = \frac{\mathbf{r}' \cdot \mathbf{r}''}{(\mathbf{r}' \cdot \mathbf{r}')^{1/2}} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{s'}. \quad (1.67)$$

The expression for the normal vector is obtained by substituting Eq. (1.67) into Eq. (1.66)

$$\boldsymbol{\nu} = \frac{\rho}{(s')^4} [\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')]. \quad (1.68)$$

Because $\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1$ the radius of curvature is

$$\begin{aligned} \frac{1}{\rho} &= \frac{\rho}{(s')^4} |[\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')]| \\ &= \frac{\rho}{(s')^4} [\mathbf{r}'' \cdot \mathbf{r}''(s')^4 - 2(\mathbf{r}' \cdot \mathbf{r}'')^2(s')^2 + \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')^2]^{1/2}. \end{aligned}$$

which simplifies to

$$\frac{1}{\rho} = \frac{1}{(s')^3} [\mathbf{r}'' \cdot \mathbf{r}''(s')^2 - (\mathbf{r}' \cdot \mathbf{r}'')^2]^{1/2}. \quad (1.69)$$

In the case of a planar curve $y = y(x)$ ($\alpha = x$) Eq. (1.69) reduces to Eq. (1.20).

The binormal vector may be calculated with the relation

$$\begin{aligned}\boldsymbol{\beta} &= \boldsymbol{\tau} \times \boldsymbol{\nu} = \frac{\mathbf{r}'}{s'} \times \frac{\rho}{(s')^4} [\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')] \\ &= \frac{\rho}{(s')^3} \mathbf{r}' \times \mathbf{r}''.\end{aligned}\quad (1.70)$$

The result of differentiating Eq. (1.71) may be written as

$$\frac{d\boldsymbol{\beta}}{ds} = \frac{1}{s'} \frac{d\boldsymbol{\beta}}{d\alpha} = \frac{1}{s'} \frac{d}{d\alpha} \left[\frac{\rho}{(s')^3} \right] (\mathbf{r}' \times \mathbf{r}'') + \frac{\rho}{(s')^4} (\mathbf{r}' \times \mathbf{r}'''). \quad (1.71)$$

The torsion T may be obtained by applying the formula

$$\begin{aligned}\frac{1}{T} &= -\boldsymbol{\nu} \cdot \frac{d\boldsymbol{\beta}}{ds} \\ &= -\frac{\rho}{(s')^4} [\mathbf{r}''(s')^2 - \mathbf{r}'(\mathbf{r}' \cdot \mathbf{r}'')] \cdot \left[\frac{1}{s'} \frac{d}{d\alpha} \left(\frac{\rho}{(s')^3} \right) (\mathbf{r}' \times \mathbf{r}'') + \frac{\rho}{(s')^4} (\mathbf{r}' \times \mathbf{r}''') \right].\end{aligned}$$

The above equation may be simplified and T can be calculated from

$$\frac{1}{T} = -\frac{\rho^2}{(s')^6} [\mathbf{r}'' \cdot (\mathbf{r}' \times \mathbf{r}''')]. \quad (1.72)$$

The expressions for the velocity and acceleration in normal and tangential directions for three-dimensional motion are identical in form to the expressions for planar motion. The velocity is a vector whose magnitude equals the rate of change of distance, and whose direction is tangent to the path. The acceleration has a component tangential to the path equal to the rate of change of the magnitude of the velocity, and a component perpendicular to the path that depends on the magnitude of the velocity and the instantaneous radius of curvature of the path. In three-dimensional motion $\boldsymbol{\nu}$ is parallel to the osculating plane, whose orientation depends on the nature of the path. The binormal vector $\boldsymbol{\beta}$ is a unit vector that is perpendicular to the osculating plane and therefore def

1.6 Examples

Example 1.1

A particle is moving along a straight line $OABC$ starting from the origin O when $t = 0$, as shown in Fig. E1.1(a), and has no initial velocity. For the first segment OA the motion has a constant acceleration. The velocity of the particle after $t_1=1$ s at the point A is $V=1$ m/s. For the segment AB , the velocity of the particle remains constant for the next $t_2=1$ s. For the last segment BC the particle is decelerating with a constant acceleration until it comes to a complete stop at C . It takes the particle $t_3=3$ s to go from point B to point C .

Determine and plot the acceleration, velocity and position versus time of the particle for the segment OC .

Solution.

1. *Segment OA.*

For the segment OA the acceleration is constant $a=\text{constant}$

$$\ddot{x}(t) = a_1.$$

The velocity equation is obtained taking the integral of the acceleration

$$\dot{x}(t) = \int a_1 dt = a_1 t + c_1.$$

The constant c_1 is obtained from the initial condition for the velocity at O

$$t = 0 \Rightarrow \dot{x}(0) = 0 \text{ or } c_1 = 0.$$

The constant acceleration a_1 is calculated from the fact that the velocity is V at A when $t = t_1$.

$$t = t_1 \Rightarrow \dot{x}(t_1) = V \text{ or } a_1 t_1 = V.$$

The acceleration is

$$a_1 = \frac{V}{t_1} = \frac{1 \text{ m/s}}{1 \text{ s}} = 1 \text{ m/s}^2.$$

The velocity equation is

$$\dot{x}(t) = \frac{V}{t_1} t = t \quad \text{for } t \in [0; t_1].$$

The position is obtained taking the integral of the velocity

$$x(t) = \frac{V}{t_1} \int t dt = \frac{V}{t_1} \frac{t^2}{2} + c_2.$$

From the initial condition for displacement at O , the constant c_2 is obtained

$$t = 0 \Rightarrow x(0) = 0 \text{ or } c_2 = 0.$$

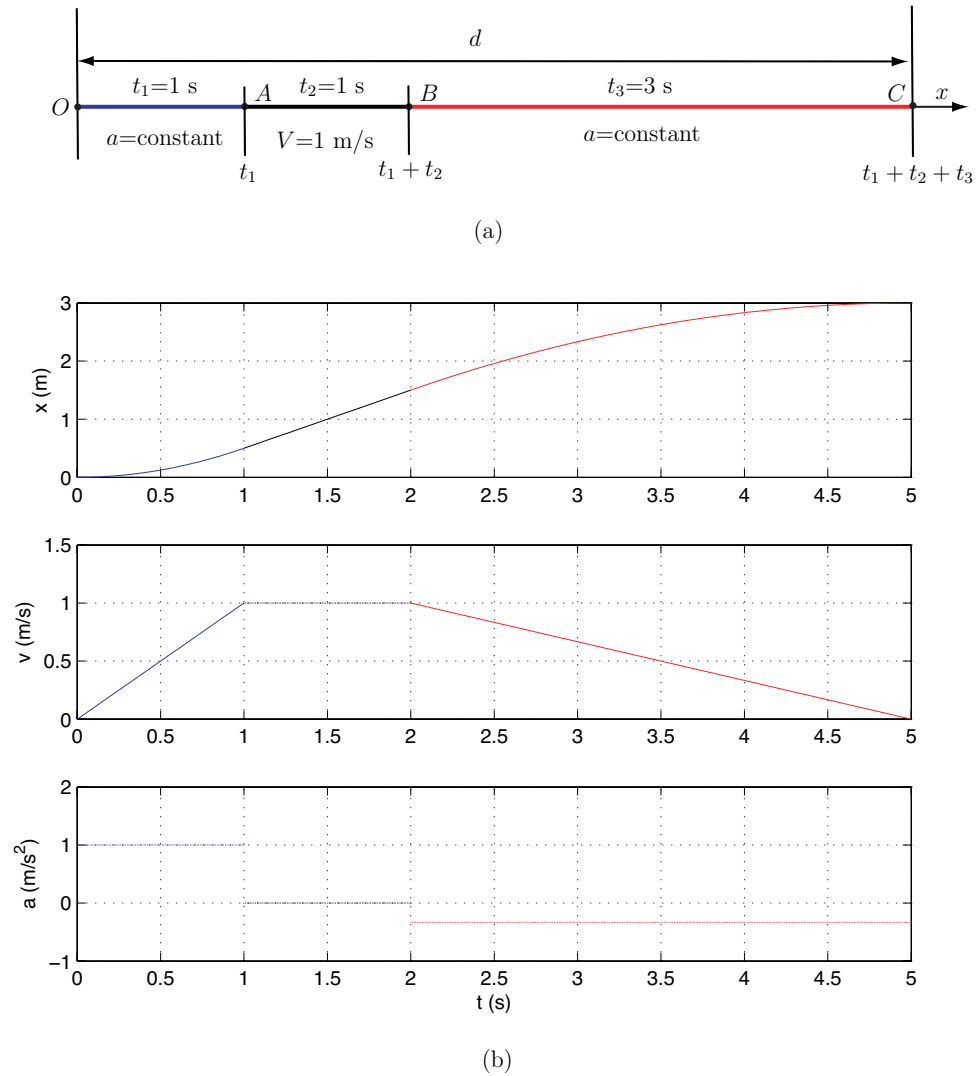


Fig. E1.1 Example 1.1

The distance equation for the first segment OA is

$$x(t) = \frac{V}{2t_1} t^2 = \frac{t^2}{2} \quad \text{for } t \in [0; t_1].$$

The distance $d_1 = OA$ for this segment is

$$d_1 = x(t_1) = \frac{Vt_1}{2} = \frac{1(1)}{2} = \frac{1}{2} \text{ m.}$$

In Fig. E1.1(b) are represented the position, velocity, and acceleration as a function of time for the first segment.

2. Segment AB.

For the segment *AB* the velocity of the particle is constant V

$$\dot{x}(t) = V \quad \text{for } t \in [t_1; t_1 + t_2],$$

where $t_2=1$ s is the time interval for the particle to travel the segment *AB*.

The acceleration is obtained differentiating the velocity

$$\ddot{x}(t) = \frac{d}{dt}V = \dot{V} = 0 \quad \text{or } \ddot{x}(t) = 0 \quad \text{for } t \in [t_1; t_1 + t_2].$$

The position equation for the segment *AB* is obtained integrating the velocity

$$x(t) = \int V dt = V t + c_3.$$

At the moment $t = t_1$ the displacement is $d_1 = \frac{V t_1}{2}$, and the constant c_3 is calculated from

$$t = t_1 \Rightarrow x(t_1) = V t_1 + c_3 = \frac{V t_1}{2},$$

or

$$c_3 = -\frac{V t_1}{2} = -\frac{1}{2}.$$

The position function of time is

$$x(t) = V t - \frac{V t_1}{2} = t - \frac{1}{2} \quad \text{for } t \in [t_1; t_1 + t_2].$$

For the second segment the distance traveled by the particle is $d_2 = AB$

$$t = t_1 + t_2 \Rightarrow x(t_1 + t_2) = d_1 + d_2 \quad \text{or} \quad V(t_1 + t_2) - \frac{V t_1}{2} = \frac{V t_1}{2} + d_2,$$

and $d_2 = V t_2 = 1(1) = 1$ m. The distance $s_2 = OB = d_1 + d_2 = V t_1/2 + V t_2$ is $s_2 = 0.5 + 1.5 = 1.5$ m.

3. Segment BC.

For the segment *BC* the acceleration is constant and negative because the particle stops at *C*

$$\ddot{x}(t) = -a_3 \quad \text{for } t \in [t_1 + t_2; t_1 + t_2 + t_3],$$

where a_3 is the constant acceleration of the particle for the last segment. The velocity equation is given by

$$\dot{x}(t) = - \int a_3 dt = -a_3 t + c_4.$$

At the moment $t = t_1 + t_2$ the velocity is V

$$t = t_1 + t_2 \Rightarrow \dot{x}(t_1 + t_2) = V \text{ or } -a_3(t_1 + t_2) + c_4 = V \text{ or } c_4 = V + a_3(t_1 + t_2).$$

The velocity equation is

$$\dot{x}(t) = -a_3t + V + a_3(t_1 + t_2),$$

or

$$\dot{x}(t) = V - a_3[t - (t_1 + t_2)].$$

For $t = t_1 + t_2 + t_3$ at C the velocity is zero

$$\dot{x}(t_1 + t_2 + t_3) = V - a_3[t_1 + t_2 + t_3 - (t_1 + t_2)] = V - a_3t_3 = 0.$$

The magnitude of the acceleration for the last segment is

$$a_3 = \frac{V}{t_3} = \frac{1}{3} \text{ m/s}^2.$$

The velocity equation for the last segment will be

$$\dot{x}(t) = V - \frac{V}{t_3}[t - (t_1 + t_2) = 1 - \frac{1}{3}(t - 2)] = -\frac{1}{3}t + \frac{5}{3}.$$

The position equation is

$$x(t) = \int \dot{x}(t) dt = -\frac{V}{2t_3}t^2 + V \frac{(t_3 + t_1 + t_2)}{t_3}t + c_5.$$

At the moment $t = t_1 + t_2$ at B the displacement is $d_1 + d_2$

$$t = t_1 + t_2 \Rightarrow x(t_1 + t_2) = d_1 + d_2 \text{ or}$$

$$-\frac{V}{2t_3}(t_1 + t_2)^2 + V \frac{(t_3 + t_1 + t_2)}{t_3}(t_1 + t_2) + c_5 = \frac{Vt_1}{2} + Vt_2.$$

The integration constant c_5 is

$$c_5 = -\frac{V}{2t_3} [(t_1 + t_2)^2 + t_1 t_3] = -\frac{1}{2(3)} [2^2 + 1(3)] = -\frac{7}{6}.$$

The position equation is

$$x(t) = -\frac{V}{2t_3}t^2 + V \frac{(t_3 + t_1 + t_2)}{t_3}t - \frac{V}{2t_3} [(t_1 + t_2)^2 + t_1 t_3] = -\frac{1}{6}t^2 + \frac{5}{3}t - \frac{7}{6}.$$

At the end of the motion $t = t_1 + t_2 + t_3$ the total displacement is d

$$t = t_1 + t_2 + t_3 \Rightarrow d = x(t_1 + t_2 + t_3)$$

or

$$d = -\frac{1}{6}5^2 + \frac{5}{3}5 - \frac{7}{6} = 3 \text{ m.}$$

Figure E1.1(b) shows the position, velocity, and acceleration as a function of time for the segment *OC*. Next the MATLAB program for the calculations is presented.

```
% E.1.1
clear all; clc; close all
syms t1 t2 t3 V a1 a3 x t real
syms c1 c2 c3 c4 c5 real
list={t1, t2, t3, V};
listn={1, 1, 3, 1};
fprintf('segment OA\n'); fprintf('\n')
ddx1 = a1;
fprintf('ddx1 = a1 \n')
fprintf('\n');
dx1 = int(ddx1, t) + c1;
fprintf('dx1 = %s \n', char(dx1))
dx10 = subs(dx1, t, 0);
dx11 = subs(dx1, t, t1)-V;
soll = solve(dx10, dx11, 'c1, a1');
c10 = soll.c1;
a10 = soll.a1;
fprintf('t=0 ; dx1=0 => c1 = %s \n', char(c10))
fprintf('t=t1; dx1=V => a1 = %s \n', char(a10))
fprintf('\n')
als = a10;
aln = subs(als, list, listn);
fprintf('a1 = %s = %s (m/s^2)\n', char(als), char(aln))
fprintf('\n')
v1 = int(a10, t) + c10;
v1n = subs(v1, list, listn);
fprintf('v1 = %s = %s (m/s)\n', char(v1), char(v1n))
fprintf('\n')
x1s = int(v1, t) + c2;
fprintf('x1 = %s \n', char(x1s))
x10 = subs(x1s, t, 0);
c20 = solve(x10, 'c2');
fprintf('t=0; x1=0 => c2 = %s \n', char(c20))
x1 = int(v1, t) + c20;
x1n = subs(x1, list, listn);
fprintf('x1 = %s = %s (m)\n', char(x1), char(x1n))
fprintf('\n')
d1 = subs(x1, t, t1);
d1n = subs(x1n, t, 1);
fprintf('d1 = %s = %g (m)\n', char(d1), d1n)
```

```

fprintf('\n')
fprintf('segment AB\n'); fprintf('\n')
dx2 = V;
fprintf('dx2 = v2 = V \n')
fprintf('\n')
ddx2 = diff(dx2, t);
fprintf('ddx2 = a2 = %s \n', char(ddx2))
fprintf('\n')
x2s = int(dx2, t) + c3;
fprintf('x2 = %s \n', char(x2s));
x21 = subs(x2s, t, t1)-d1;
c30 = solve(x21, 'c3');
fprintf('t=t1; x2=d1 => c3 = %s \n', char(c30))
x2 = int(dx2, t) + c30;
x2n = subs(x2, list, listn);
fprintf('x2 = %s = %s (m)\n', char(x2), char(x2n))
fprintf('\n')
s2 = subs(x2, t, t1+t2);
s2n = double(subs(s2, list, listn));
fprintf('s2 = d1 + d2 = %s = %g (m)\n', char(s2), s2n)
fprintf('\n')
fprintf('segment CD\n'); fprintf('\n')
ddx3 = -a3;
fprintf('ddx3 = -a3 \n');
fprintf('\n');
dx3 = int(ddx3, t) + c4;
fprintf('dx3 = %s \n', char(dx3));
dx32 = subs(dx3, t, t1+t2)-V;
dx33 = subs(dx3, t, t1+t2+t3);
sol3 = solve(dx32, dx33, 'c4, a3');
c40 = sol3.c4;
a30 = sol3.a3;
c4n = double(subs(c40, list, listn));
a3n = double(subs(a30, list, listn));
fprintf('t=t1+t2 ; dx3=V \n')
fprintf('t=t1+t2+t3; dx3=0 => \n')
fprintf('c4 = %s = %g \n', char(c40), c4n)
fprintf('a3 = %s = %g \n', char(a30), a3n)
fprintf('\n')
a3s = -a30;
v3 = int(-a30, t) + c40;
v3n = subs(v3, list, listn);
fprintf('v3 = %s = %s (m/s)\n', char(v3), char(v3n))
fprintf('\n')
x3s = int(v3, t) + c5;

```

```

fprintf('x3 = %s \n', char(x3s));
x12 = subs(x3s, t, t1+t2)-s2;
c50 = solve(x12, 'c5');
fprintf('t=t1+t2; x3=s2 => \n')
fprintf('c5 = %s \n', char(c50))
c50n = subs(c50, list, listn);
fprintf('c5 = %g \n', double(c50n))
fprintf('\n')
x3 = int(v3, t) + c50;
x3n = subs(x3, list, listn);
fprintf('x3 = %s \n', char(x3))
fprintf('x3 = %s (m)\n', char(x3n))
fprintf('\n')
s3 = subs(x3, t, t1+t2+t3);
s3n = subs(s3, list, listn);
fprintf('s3 = %s \n', char(s3))
fprintf('s3 = %g (m)\n', double(s3n))
fprintf('\n')
% Graphic
y1=0:.01:1;
y2=1:.01:2;
y3=2:.01:5;
y1= 0:.01:1;
y2= 1:.01:2;
y3= 2:.01:5;
px1=subs(x1n,t,y1);
px2=subs(x2n,t,y2);
px3=subs(x3n,t,y3);
pv1=subs(v1n,t,y1);
pv2=double(subs(V,list,listn));
pv3=subs(v3n,t,y3);
pa1=subs(a1n,t,y1);
pa2=0;
pa3=subs(a3n,t,y3);
subplot(3,1,1),...
plot(y1,px1,'b',y2,px2,'k',y3,px3,'r'),...
ylabel('x (m)'), grid,...
subplot(3,1,2),...
plot(y1,pv1,'b',y2,pv2,'k-',y3,pv3,'r'),...
ylabel('v (m/s)'), grid, axis([0 5 0 1.5])
subplot(3,1,3),...
plot(y1,pa1,'b',y2,pa2,'k',y3,pa3,'r')
xlabel('t (s)'), ylabel('a (m/s^2)'), grid,...
axis([0 5 -1 2])

```

segment OA

$$ddx1 = a1$$

$$dx1 = a1*t+c1$$

$$t=0 ; dx1=0 \Rightarrow c1 = 0$$

$$t=t1; dx1=V \Rightarrow a1 = V/t1$$

$$a1 = V/t1 = 1 \text{ (m/s}^2\text{)}$$

$$v1 = V/t1*t = t \text{ (m/s)}$$

$$x1 = 1/2*V/t1*t^2+c2$$

$$t=0; x1=0 \Rightarrow c2 = 0$$

$$x1 = 1/2*V/t1*t^2 = 1/2*t^2 \text{ (m)}$$

$$d1 = 1/2*V*t1 = 0.5 \text{ (m)}$$

segment AB

$$dx2 = v2 = V$$

$$ddx2 = a2 = 0$$

$$x2 = V*t+c3$$

$$t=t1; x2=d1 \Rightarrow c3 = -1/2*V*t1$$

$$x2 = V*t-1/2*V*t1 = t-1/2 \text{ (m)}$$

$$s2 = d1 + d2 = V*(t1+t2)-1/2*V*t1 = 1.5 \text{ (m)}$$

segment CD

$$ddx3 = -a3$$

$$dx3 = -a3*t+c4$$

$$t=t1+t2 ; dx3=V$$

$$t=t1+t2+t3; dx3=0 \Rightarrow$$

$$c4 = V*(t1+t2+t3)/t3 = 1.66667$$

$$a3 = 1/t3*V = 0.333333$$

$$v3 = -1/t3*V*t+V*(t1+t2+t3)/t3 = -1/3*t+5/3 \text{ (m/s)}$$

$$x3 = -1/2/t3*V*t^2+V*(t1+t2+t3)/t3*t+c5$$

$$t=t1+t2; x3=s2 \Rightarrow$$

$$c5 = -1/2*V*(t1^2+2*t1*t2+t2^2+t1*t3)/t3$$

$$c5 = -1.16667$$

$$x3 = -1/2/t3*V*t^2+V*(t1+t2+t3)/t3*t-1/2*V*(t1^2+2*t1*t2+t2^2+t1*t3)/t3$$

$$x3 = -1/6*t^2+5/3*t-7/6 \text{ (m)}$$

$$s3 = 1/2/t3*V*(t1+t2+t3)^2-1/2*V*(t1^2+2*t1*t2+t2^2+t1*t3)/t3$$

$$s3 = 3 \text{ (m)}$$