

STAT3600 S05 Part of 3/24/2005 Lecture on LCs S. Maghsoodloo

Consider a final assembly with individual components X_1, X_2 & X_3 . Then the linear combination

$$Y = 2X_1 - 3X_2 + X_3 = \text{Final Assembly dimension}$$

NIID = Normally, independently, and identically distributed.

Given that X_1 & X_3 are NIID(50, 0.20) while X_2 is N(45, 0.15) and independent of X_1 & X_3 , compute the $P(Y \leq 14.5)$:

$$E(Y) = 2 \times 50 - 3 \times 45 + 50 = 15$$

$$V(Y) = \sum_{i=1}^3 c_i^2 \sigma_i^2 = 4 \times 0.20 + (-3)^2 \times 0.15 + 0.20 = 2.35 \rightarrow Y \sim N(15, 2.35) \rightarrow \sigma_y = 1.533$$

$$P(Y \leq 14.5) = P(Z \leq -0.3262) = \Phi(-0.3262) = 0.37215$$

Suppose now X_1, X_2 & X_3 are correlated such that $\text{COV}(X_1, X_2) = \sigma_{12} = 0.08, \sigma_{13} = 0.14, \sigma_{23} = \text{COV}(X_3, X_2) = 0.06$. Again compute $P(Y \leq 14.5)$. $E(Y) = 15$ does not change and Y is still normally distributed, but the variance of the LC, Y , does change as shown below:

$$\begin{aligned} V(Y) &= E[(2X_1 - 3X_2 + X_3) - (2\mu_1 - 3\mu_2 + \mu_3)]^2 \\ &= E[2(X_1 - \mu_1) - 3(X_2 - \mu_2) + (X_3 - \mu_3)]^2 \\ &= E[4(X_1 - \mu_1)^2 + 9(X_2 - \mu_2)^2 + (X_3 - \mu_3)^2 + 2(2)(-3)(X_1 - \mu_1)(X_2 - \mu_2) + \\ &\quad + (2)(2)(1)(X_1 - \mu_1)(X_3 - \mu_3) + (2)(-3)(1)(X_2 - \mu_2)(X_3 - \mu_3)] = \\ &= (4\sigma_{11} + 9\sigma_{22} + \sigma_{33}) + (-12\sigma_{12} + 4\sigma_{13} - 6\sigma_{23}) = 2.35 + (-12 \times 0.08 + 4 \times 0.14 - 6 \times 0.06) = 1.59. \\ &\rightarrow \sigma_y = 1.2609520 \rightarrow P(Y \leq 14.5) = P(Z \leq -0.3965258) = \Phi(-0.3965258) = 0.345858591989. \end{aligned}$$

The corresponding correlation coefficients are

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{0.08}{\sqrt{(0.2)(0.15)}} = 0.461880215352$$

$$\rho_{13} = \frac{\sigma_{13}}{\sigma_1 \sigma_3} = \frac{0.14}{\sqrt{(0.20)(0.20)}} = 0.700000000$$

$$\rho_{23} = \frac{\sigma_{23}}{\sigma_2 \sigma_3} = \frac{0.06}{\sqrt{(0.15)(0.20)}} = 0.34641016151378; \text{ note that } |\rho| \text{ can never exceed 1.}$$