

## Binomial and Sequential Testing Reference: pp. 342-348 of Ebeling(2<sup>nd</sup>)

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Ebeling provides an example of a single sampling plan, pp. 342-343, in RE testing where  $n$  units are tested for their TsTF and the number of failed units,  $N_f$ , by end of test time,  $T$ , is recorded as  $r$ . The objective is to determine the two decision variables  $n$  and  $N_f = r$  such that the producer's risk,  $\alpha = \text{Pr}$  of type I error at  $R = R_0$ , is roughly 0.05, and the consumer's risk,  $\beta = \text{Pr}$  of type II error, say is roughly 8% when  $R = R_1 < R_0$ . Note that in his Table 13.2, p. 343, his  $R_1 = \text{my } R_0$  and his  $R_2 = \text{my } R_1$ . I prefer 0 and 1 because of  $H_0$  and  $H_1$ . Since  $N_f$  has a binomial distribution, under  $H_0: R = R_0$ , the Pr of accepting a submitted lot of size  $n$  for RE testing is given by  $P_a(\text{Good lot}) = B(r; n, R_0) = \sum_{i=0}^r \binom{n}{i} R_0^{n-i} Q_0^i \geq 1 - \alpha = \text{Pr}$  of accepting a good lot,  $P_a(\text{Bad lot}) = B(r; n, R_1) =$

$$\sum_{i=0}^r \binom{n}{i} R_1^{n-i} Q_1^i \leq \beta = \text{The Pr of accepting a bad lot.}$$

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As an example, suppose an  $R_0 = 0.98$  is desired with  $\alpha = 0.05$  but when lots are as bad as  $R_1 = 0.92$ , we wish to reject them with a Pr of roughly 0.91 (or Consumer's risk = 0.09). Then, we

need to solve the two equations  $\sum_{i=0}^r \binom{n}{i} 0.02^i 0.98^{n-i} \geq 0.95$  and  $\sum_{i=0}^r \binom{n}{i} (0.08)^i (0.92)^{n-i} \leq 0.09$ . I

have provided an Excel file on my website that solves for possible values of  $n$  and  $r$ . My Excel file shows that  $n = 130$  and  $r = 6$  come close to meeting the above requirements. If an Excel file is

not available, then  $\sum_{i=0}^r \binom{n}{i} 0.02^i 0.98^{n-i} \cong \sum_{i=0}^r \frac{(0.02n)^i}{i!} e^{-0.02n} \geq 0.95$ , and  $\sum_{i=0}^r \binom{n}{i} R_1^{n-i} Q_1^i \cong$

$\sum_{i=0}^r \frac{(0.08n)^i}{i!} e^{-0.08n} \leq \beta$  can be solved to obtain approximate solutions for  $n$  and  $r$ . For the given

plan ( $130, r = 6$ ), this Poisson approximation to the Binomial gives  $1 - \alpha = 0.9828$ , while  $\beta = 0.1069$ .

We will discuss, as does Ebeling on his p. 345, the one-sided procedure for a sequential sampling plan, which is the most common in RE testing of the type  $H_0: \lambda = \lambda_0$  versus the alternative  $H_1: \lambda > \lambda_0$ . In the one-sided procedure, the alternative value of process failure rate is denoted by  $\lambda_1 > \lambda_0$  (a right-tailed test). In carrying out sequential testing, generally samples of

size  $n$  ( $= 1$ , but not always) is taken at each stage of the sampling plan, the sequential test statistic (SQTS) is updated, and a 3-way decision is made as follows: (1) accept  $H_0$ , (2) reject  $H_0$ , or (3) continue the experiment by taking additional observation(s). I would venture to say, that it is rare to take differing sample sizes at different stages in RE Engineering, although this is sometimes done in acceptance sampling for FNC =fraction nonconforming (specially in a 2-stage sampling plan where  $n_2 = 2 \times n_1$  but then in this case the decision about a submitted lot is definitely made after stage 2 because the difference between the acceptance and rejection numbers at the 2<sup>nd</sup> stage is always equal to 1).

For example, the double sampling plan  $n_1 = 50$ ,  $c_1 = 1$ ,  $n_2 = 100$ , and  $c_2 = 4$  implies that a random sample of size 50 is taken from a large lot (usually  $N > 1000$  units) and the lot is immediately accepted if the number of defectives,  $D_1$ , at stage 1 is  $\leq 1$  and the lot is immediately rejected at stage 1 if  $D_1 > 4$ . If the lot quality is mediocre at stage 1, where  $D_1 = 2, 3$ , or 4, then the lot is given a second chance by taking an additional sample of size  $n_2 = 100$ . At the 2<sup>nd</sup> stage, the lot is accepted only if the total number of defectives in the 150 sampled units is 4 or less, and it is rejected at stage 2 only if  $D_1 + D_2 > 4$ .

When sequential sampling is applied in SPC or RE Engineering, first the reader should be cognizant of the fact that the inspection is not always by attributes (like on p. 347 of Ebeling) and that the decision about the MTTF or failure of a Weibull (or Exponential) process is carried out with a statistic (like TTF) whose range space is on a continuous (or dense) scale. The very basic statistical idea behind testing the hypothesis  $H_0: \lambda = \lambda_0$  versus the alternative  $H_1: \lambda > \lambda_0$  is to make use of the Neyman-Pearson Lemma by using the likelihood ratio statistic

$$y_r = \frac{\prod_{i=1}^r f(t_i; \lambda_1) dx_i}{\prod_{i=1}^r f(t_i; \lambda_0) dx_i} = \frac{\prod_{i=1}^r f(t_i; \lambda_1)}{\prod_{i=1}^r f(t_i; \lambda_0)} = \frac{P_a(\lambda = \lambda_1)}{P_a(\lambda = \lambda_0)} \quad (121)$$

In equation (121) the statistic in the numerator on the RHS is related to the occurrence Pr of the sample  $(t_1, t_2, \dots, t_r)$  given that  $\lambda = \lambda_1 > \lambda_0$  because the likelihood (or Pr) of obtaining such a

sample is given by  $[f(t_1; \lambda)dt_1] \times [f(t_2; \lambda_1)dt_2] \times \dots \times [f(t_r; \lambda_1)dt_r] = \prod_{i=1}^r [f(t_i; \lambda_1)dt_i]$ , where  $f(t; \lambda)$

represents the failure density of the sampled population. The quantity  $\prod_{i=1}^r f(t_i; \lambda_1)$  in this last Pr

statement is called the likelihood function under  $H_1$ , and after the sample is drawn and the sample values  $t_1, t_2, \dots, t_r$  are known numbers (no longer rvs), then the likelihood function

$\prod_{i=1}^r f(t_i; \lambda_1)$  is only a function of the parameter  $\lambda$ , which is denoted by  $L(\lambda) = \prod_{i=1}^r f(t_i; \lambda)$ . In

other words, the numerator of (121) gives the likelihood that  $\lambda$  lies in an extremely small interval around  $\lambda_1$  and its denominator provides the Pr that  $\lambda$  lies in an infinitesimally small interval around  $\lambda_0$ . Therefore, when  $y_r$  is very large, we will reject  $H_0$  in favor of  $H_1$  at the stage  $r$  (or at the time  $t_r$ ); when  $y_r$  is too small, we will favor  $H_0$  over  $H_1$ , and when it is neither, then the sample point is still in the indecision region of sample space so that we have to continue on sampling to the stage  $r+1$ . Further, for notational convenience, let  $\mathbf{L}(\lambda) = \ln[L(\lambda)]$ , i.e.,  $\mathbf{L}(\lambda)$  is the natural logarithm of the likelihood function, as before. Working with the natural logarithm is easier than using likelihood function  $L(\lambda)$  itself. Before providing such an example, we must provide some partial answer as to how large  $y_r$  should be before we adopt  $\lambda = \lambda_1$  and how small  $y_r$  should be before we adopt  $\lambda = \lambda_0$ . In other words, we will reject  $H_0$  if  $y_r \geq B$ , we will accept  $H_0$  if  $y_r \leq A$ , and will continue sampling if the value of  $y_r$  lies in the indecision interval  $A < y_r < B$ , which depends on the specified values of the LOS  $\alpha$  and the type II error Pr  $\beta$ . Note that in general  $B \gg A$ .

One advantage that sequential testing has over the fixed-size sample tests is the fact that we can decide in advance on the sizes of type I and II errors (generally  $\alpha \leq 0.05$  and  $\beta \leq 0.10$ ) that we are willing to tolerate, which is unlike the fixed-size sampling where  $\alpha$  is specified a priori and  $\beta$  is computed for different parameter values. Suppose we fix  $\alpha$  at 0.05 and  $\beta$  at 0.10 for detecting an upward shift in a process failure rate; then clearly the value of  $B$  must be proportional to  $1 - \beta$  (= the Pr of rejecting a false  $H_0$ ). It was shown by A. Wald and G. A. Barnard

(1945 and 1946) that the acceptance and rejection limits must satisfy  $\frac{\beta}{1-\alpha} \leq A$  and  $B \leq \frac{1-\beta}{\alpha}$ ,

respectively. The statistic  $y_r$  is referred to as the sequential Pr ratio test (SPRT) with approximate boundaries  $A \cong \frac{\beta}{1-\alpha}$  and  $B \cong \frac{1-\beta}{\alpha}$ . Note that the boundaries for the indecision interval is designed in such a manner that the true type I and II Prs of the sequential test are at most  $\alpha$  and  $\beta$ , respectively. The reader should observe that the denominator of the SPRT statistic  $y_r =$

$$\frac{\prod_{i=1}^r f(t_i; \lambda_1) dt_i}{\prod_{i=1}^r f(t_i; \lambda_0) dt_i}$$

gives the likelihood Pr that  $H_0$  is true, and therefore, this Pr (i.e., the

denominator) must be directly proportional to  $1 - \alpha$  (= the Pr of accepting a true  $H_0$ ) and inversely proportional to  $\beta$  (= the Pr of accepting a false  $H_0$ ). This is how the expression for lower boundary A for the indecision interval  $A < y_r < B$  was constituted.

Now consider the Example 13.8 on pp. 345-346 of Ebeling, where the objective is to develop a SPRT for a CFR, where  $\lambda_0 = 0.00125$ ,  $\alpha = 0.05$ , and  $\lambda_1 = 0.0014286$ ,  $\beta = 0.10$ . Because

the underlying failure distribution is exponential, Eq. (121) becomes  $y_r = \frac{\prod_{i=1}^r f(t_i; \lambda_1) dx_i}{\prod_{i=1}^r f(t_i; \lambda_0) dx_i} =$

$$= \frac{\prod_{i=1}^r f(t_i; \lambda_1)}{\prod_{i=1}^r f(t_i; \lambda_0)} = \frac{\prod_{i=1}^r \lambda_1 e^{-\lambda_1 t_i}}{\prod_{i=1}^r \lambda_0 e^{-\lambda_0 t_i}}, \quad A = 0.10 / (0.95) = 0.10526, \text{ and } B = 0.90 / 0.05 = 18. \text{ Thus, the}$$

indecision interval is given by  $0.10526 \leq \frac{\prod_{i=1}^r \lambda_1 e^{-\lambda_1 t_i}}{\prod_{i=1}^r \lambda_0 e^{-\lambda_0 t_i}} = \frac{\lambda_1^r e^{-\sum_{i=1}^r \lambda_1 t_i}}{\lambda_0^r e^{-\sum_{i=1}^r \lambda_0 t_i}} \leq 18$ . Taking ln of all terms

we obtain,  $-2.2513 \leq r \ln(\lambda_1) - \sum_{i=1}^r \lambda_1 t_i - r \ln(\lambda_0) + \sum_{i=1}^r \lambda_0 t_i \leq 2.8904$ , or

$$-2.2513 \leq r \ln(\lambda_1 / \lambda_0) - (\lambda_1 - \lambda_0) \sum_{i=1}^r t_i \leq 2.8904 \rightarrow -2.2513 - r \ln(\lambda_1 / \lambda_0) \leq$$

$$-(\lambda_1 - \lambda_0) \sum_{i=1}^r t_i \leq 2.8904 - r \ln(\lambda_1 / \lambda_0) \rightarrow [-2.8904 + r \ln(\lambda_1 / \lambda_0)] / (\lambda_1 - \lambda_0) \leq \sum_{i=1}^r t_i = \tau_r \leq$$

$$[2.2513 + r \ln(\lambda_1 / \lambda_0)] / (\lambda_1 - \lambda_0) \rightarrow [-2.8904 + r \ln(\lambda_1 / \lambda_0)] / (\lambda_1 - \lambda_0) \leq \tau_r \leq [2.2513$$

$$+ r \ln(\lambda_1 / \lambda_0)] / (\lambda_1 - \lambda_0)]. \text{ Substituting } \ln(0.0014286/0.00125) = 0.1336 \text{ and } \lambda_1 - \lambda_0 =$$

$$0.00017857, \text{ we obtain } \tau_L = [-2.8904 + r \ln(\lambda_1 / \lambda_0)] / (\lambda_1 - \lambda_0) = -16186.24 + 747.7758r, \text{ and } \tau_U$$

$$= 12607.28 + 747.7758r. \text{ These are consistent with what Ebeling gives on his p. 346. Thus, the}$$

plan is to compute the value of  $\tau_r = \sum_{i=1}^r t_i$  for  $r = 1, 2, 3, 4, \dots$  and test this against the limits  $\tau_L$  and

$\tau_U$ . If  $\tau_r > \tau_U$ , immediately accept the submitted lot; If  $\tau_r$  is less than  $\tau_L$ , immediately reject the lot; otherwise, test one more unit to failure.

Ebeling also applies the SQTS to binomial testing, where  $H_0 : R = R_0$  versus the alternative  $H_1 : R < R_0$ . Let  $y_n =$  equal to number of survivors out of  $n$  units on test. Then, for  $y_{n,y} = y$  survivors

the LRTS is given by  $\frac{{}^n C_y R_1^y Q_1^{n-y}}{{}^n C_y R_0^y Q_0^{n-y}} = \frac{R_1^y Q_1^{n-y}}{R_0^y Q_0^{n-y}}$ , where  $R_1 < R_0$  is the alternative value of  $R$

under  $H_1$ , and as before  $A = \beta / (1 - \alpha)$  and  $B = (1 - \beta) / \alpha$ , and the most common values of  $\alpha$  and  $\beta$  are 5% and 10%, respectively. As Ebeling, I will also go through his Example 13.9 on pp. 347-348.

Here,  $R_0 = 0.90$  and  $R_1 = 0.85$ , and  $\alpha, \beta$  are the standard 5% and 10%, respectively. Then,  $A =$

$$0.10/0.95 = 0.105263 \text{ and } B = 18. \text{ Thus, we accept the lot iff } y_{n,r} = \frac{R_1^y Q_1^{n-y}}{R_0^y Q_0^{n-y}} < 0.105263 \text{ and}$$

reject the lot iff  $y_{n,r} > 18$ . We will continue testing iff

$$0.105263 < y_{n,r} = \frac{R_1^y Q_1^{n-y}}{R_0^y Q_0^{n-y}} < 18 \quad (122)$$

$$\text{Or: } -2.2512918 < \ln(R_1^y Q_1^{n-y}) - \ln(R_0^y Q_0^{n-y}) < 2.890371757896165$$

$$\text{Or: } -2.2512918 < y \ln(R_1) + (n-y) \ln(Q_1) - y \ln(R_0) - (n-y) \ln(Q_0) < 2.890371758$$

$$\text{Or: } -2.2512918 < -0.462624y + n \ln(Q_1 / Q_0) < 2.890371758 \rightarrow$$

$$\text{Or: } -2.2512918 < -0.462624y + 0.405465n < 2.890371758 \rightarrow$$

$$\text{Or: } -2.2512918 - 0.405465n < -0.462624y < 2.890371758 - 0.405465n \rightarrow$$

$$\text{Or: } (-2.2512918 - 0.405465n) / (-0.462624) > y > (2.890371758 - 0.405465n) / -0.462624 \rightarrow$$

$$\text{Or: } -6.247783826 + 0.87644724n < y < 4.8663583 + 0.87644724n$$

Thus, the above plan dictates that we test one unit to failure at-a-time and if  $N_f(n) < -6.247783826 + 0.87644724n$ , we accept the lot; if  $N_f(n) > 4.8663583 + 0.87644724n$ , we immediately reject the lot. Otherwise, test another unit to failure.

As Ebeling states on his page 348, Eq. (122) can also be used for testing acceptable and reject-able cumulative repair-hazard functions  $H(t^*) = P_0$  and  $H(t^*) = P_1$ , where  $R_1$  in (122) is

replaced with  $P_1$  and  $R_0$  is replaced with  $P_0$ . ASA, the LRTS  $\frac{P_1^y Q_1^{n-y}}{P_0^y P_0^{n-y}}$  exceeds 18 reject the lot,

and accept iff  $\frac{P_1^y Q_1^{n-y}}{P_0^y P_0^{n-y}} < 0.105263$  ( $\alpha = 0.05$  and  $\beta = 0.10$ ), and continue obtaining repair data as

long as  $A < \frac{P_1^y Q_1^{n-y}}{P_0^y P_0^{n-y}} < B$ .