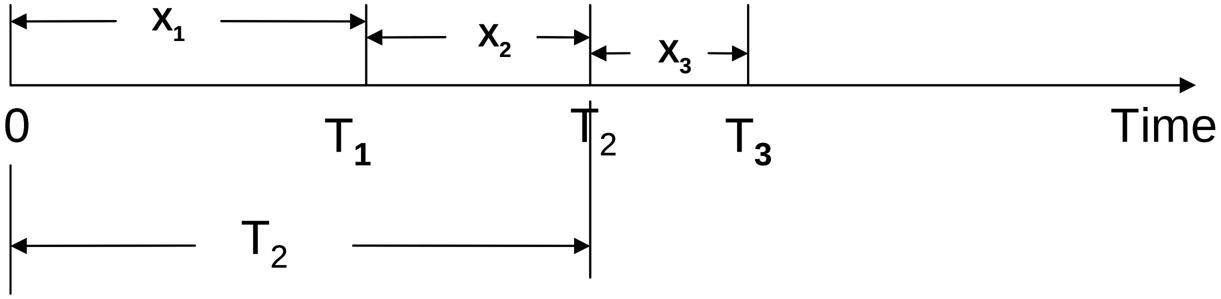
Renewal Processes Reference: Chapter 9 of Ebeling (2^{nd}) Maghsoodloo

As an example, suppose a machine component is replaced ASA it fails with a new one. Let N(t) be the number of replacements during the interval [0, t] of length t. Then, N(t) is called a renewal counting process. The study of renewal processes focus on the following topics:

(1) The pmf (Pr mass function) of N(t), (2) The expected number of renewals during [0, t] or [t_0, t_0+t], denoted by E[N(t)], where E[N(t)] = m(t), m for mean, is called the renewal function. Note that this case also covers the negligible repair-time., (3) The occurrence Pr (mass) or density function of a renewal at specific epochs of time, and (4) The time needed for the occurrence of k events (such as failures that are followed by a replacement) to occur. [For more details See U. N. Bhat (1984), *Elements of Applied Stochastic Processes*, 2^{nd} Ed. , Chapter 8.]

Suppose that failures occur at times T_k (k = 1, 2, 3, 4, …) measured from zero and assuming for the time being that replacement (or restoration time) is negligible relative to operational time, then T_k represents the operating time (measured from zero) until the k^{th} failure, where $T_0 = 0$. Because the pdf of T_1 may be different from the intervening times $X_2 = (T_2 – T_1)$, $X_3 = (T_3 – T_2)$, $X_4 = (T_4 – T_3)$, ..., we let $f_1(t)$ represent the pdf of the time to the 1^{st} failure (TTFF) and f(t) represent the pdf of intervening times X_2, X_3, X_4, … as depicted below.



Note that X_1, X_2, X_3, … represent intervening times between failures, while T_i represents time to the i^{th} renewal measured from zero. The above figure clearly shows that T_k (Time to the k^{th} renewal) = $\sum_{i=1}^{k} X_i$ = sum of the times to the 1^{st} failure plus the intervening times of 2^{nd} failure until the k^{th} failure. If k > 30, then the central theorem states that the distribution of T_k approaches

normality with mean μ_1 + (k – 1)μ, where μ = E(X_k)= mean time between successive renewals, k = 2, 3, 4, … and with variance σ_1^2 + (k – 1)σ^2, where σ^2 = V(X_k), k = 2, 3, 4, … (see pp. 224-226 of Ebeling). However, if the pdf of X_K is highly skewed and/or k is not sufficiently large, then the pdf of T_k = $\sum_{i=1}^{k} X_i$ is given by the kth-fold convolution given as $f_{T_k}(t) = f_1(t)*f_{(k-1)}(t)$, where $f_{(k-1)}(t)$ is the pdf of the sum x_2 +x_3 +…+x_k, or the (k – 1) convolution of f(t) with itself.

The most common counting process is the homogeneous Poisson process (HPP) , where the intervening times are exponentially distributed at the constant arrival (or failure) rate λ. Because λ is a constant and intervening times are iid, such a Poisson process is also referred to as a homogeneous renewal process. Then during the interval [0, t] the renewal function, m(t), is E[N(t)] = λt and the pmf of N(t) is given by pmf[(N(t)] = Pr[N(t) = k] = $\frac{(\lambda t)^k}{k!}e^{-\lambda t}$, k = 0, 1, 2, 3, …. For a homogeneous renewal process the pdf of X_1 is given by $f_1(t)$ = $\lambda e^{-\lambda t}$, and because f(t) = $f_1(t)$ also for i = 2, 3, 4, …, k then $f_{T_k}(t) = f_{(k)}(t)$, where $f_{(k)}(t)$ is the k-fold convolution of the exponential with itself, and hence the density of time to the kth occurrence is given by $f_{T_k}(t) = f_{(k)}(t) = \frac{\lambda}{\Gamma(k)}(\lambda t)^{k-1}e^{-\lambda t}$ (the Gamma density with parameters λ and k).

The Renewal function m(t) = E[N(t)]

Because the two events {N(t) ≥ k} and {$T_k \le t$} are equivalent, i.e., the P[N(t) ≥ k] = P($T_k \le t$) = $F_{(k)}(t)$, where $F_{(k)}(t) = F_{(1)}(t)*F_{(k-1)}(t)$ is the kth-fold convolution representing the cdf of T_k = $\sum_{i=1}^{k} X_i$. Thus, P[N(t) = k] = P[N(t) ≥ k] – P[N(t) ≥ k + 1] = $F_{(k)}(t) - F_{(k+1)}(t)$. Then the renewal function is obtained as follows:

m(t) = E[N(t)] = $\sum_{n=1}^{\infty} n \times P[N(t) = n]$ = $\sum_{n=1}^{\infty} n \times P_n(t)$, where $P_n(t)$ = Pr[N(t) = n]. Thus,

$$m(t) = \sum_{n=1}^{\infty} P_n(t) + \sum_{n=2}^{\infty} P_n(t) + \sum_{n=3}^{\infty} P_n(t) + \sum_{n=4}^{\infty} P_n(t) + \ldots = \sum_{n=1}^{\infty} P[N(t) \geq n] = \sum_{n=1}^{\infty} F_{(n)}(t) \qquad (91)$$

You may wish to replace m(t) in Eq. (91) with $\overline{N}(t)$ if the notation $\overline{N}(t)$ would be less confusing. Note that m(t) = $\overline{N}(t)$ may represent either the expected number of brand-new replacements, or number of repairs where the failed unit (or system) has been restored to its original as-good-as-new condition with minimal (or negligible) repair-time. For different time-to-repair (TTR) distributions, see pp. 219-223 of Ebeling (2^{nd}).

As an example, if intervening times $X_1, X_2, \ldots, X_k$ are all ii and exponentially distributed like f(x) = $\lambda e^{-\lambda x}$, then the density of the time to the n^{th} renewal is $f_{(n)}(t) = \frac{\lambda}{\Gamma(n)}(\lambda t)^{n-1} e^{-\lambda t}$ which is the n-fold convolution of f(t) with itself, and from Eq. (91) we obtain

$$E[N(t)] = m(t) = \sum_{n=1}^{\infty} F_{(n)}(t) = \sum_{n=1}^{\infty} \int_{x=0}^{t} \frac{\lambda}{\Gamma(n)}(\lambda x)^{n-1} e^{-\lambda x} dx = \int_{x=0}^{t} \sum_{n=1}^{\infty} \frac{\lambda}{\Gamma(n)}(\lambda x)^{n-1} e^{-\lambda x} dx$$

$$= \int_{x=0}^{t} \lambda e^{-\lambda x} \sum_{n=1}^{\infty} \frac{(\lambda x)^{n-1}}{(n-1)!} dx = \int_{x=0}^{t} \lambda e^{-\lambda x} e^{\lambda x} dx = \int_{x=0}^{t} \lambda dx = \lambda t,$$

which is the mean number of (homogenous) Poisson events occurring during an interval of length t, as expected.

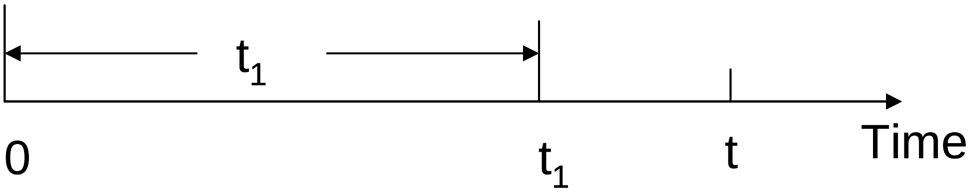
Unfortunately, obtaining a closed-form expression for the renewal function is not as simple as the case of exponential interarrival times. For the sake of illustration, consider, the Example 9.5 on pp. 225-6 of Ebeling, where a cutting tool has a N(5 hours, 1) TTF distribution and one production run is 12 hours. Then the P_0 = Pr[N_f(12 hours)= 0] = Pr(TTF_1 > 12) = Pr(Z > 7) = Φ(-7) =$1.279812543886 \times 10^{-12}$; P_1(12)= Pr[N_f(12 hours)=1] = Pr[N_f(12 hours) $\geq$ 1] − Pr[N_f(12 hours)$\geq$ 2] = Pr($TTF_1 \leq 12$) − Pr($TTF_2 \leq 12$) =Φ(7) − Pr($Z \leq \frac{12-10}{\sqrt{2}}$) = 0.99999999999872

$-0.92135039647486 = 0.07864960352386$. P_2 = Pr[N_f(12 hours)=2] = Pr[N_f(12 hours) $\geq$ 2] – Pr[N_f(12 hours) $\geq$ 3] = Pr(T_2 = $TTF_2 \leq 12$) – Pr(T_3 = $TTF_3 \leq 12$) = Pr($Z \leq \frac{12-10}{\sqrt{2}}$) – Pr($Z \leq \frac{12-15}{\sqrt{3}}$) = 0.92135039647486 – 0.04163225833178 = 0.87971813814308. P_3 = Pr[N_f(12 hours)=3] = Pr[N_f(12 hours) $\geq$ 3] – Pr[N_f(12 hours) $\geq$ 4] = Pr($TTF_3 \leq 12$) – Pr($TTF_4 \leq 12$) = Pr($Z \leq \frac{12-15}{\sqrt{3}}$) – Pr($Z \leq \frac{12-20}{\sqrt{4}}$) = 0.04163225833178 – 0.00003167124183312 = 0.0416005871. P_4 = Pr[N_f(12 hours)= 4] = Pr[N_f(12 hours) $\geq$ 4] – Pr[N_f(12 hours) $\geq$ 5] = Pr($TTF_4 \leq 12$) – Pr($TTF_5 \leq 12$) = Pr($Z \leq \frac{12-20}{\sqrt{4}}$) – Pr($Z \leq \frac{12-25}{\sqrt{5}}$) = 0.00003167124183312 – $3.053943179854770 \times 10^{-9}$ = 0.00003166818789; P_5 = Pr[N_f(12 hours)= 5] = Pr[N_f(12 hours) $\geq$ 5] – Pr[N_f(12 hours) $\geq$ 6] = Pr($TTF_5 \leq 12$) – Pr($TTF_6 \leq 12$) = Pr($Z \leq \frac{12-25}{\sqrt{5}}$) – Pr($Z \leq \frac{12-30}{\sqrt{6}}$) =$3.053943179854770 \times 10^{-9}$ – $1.002448040140151 \times 10^{-13}$ = $3.053842935050756 \times 10^{-9}$ → E[N_f(12)] = m(12) = mean no. of failures in 12 hrs = $\sum_{n=0}^{\infty} n \times P_n(12)$ = 1.96301433, which agrees with Ebeling's answer atop his page 226 to 2 decimals → The approximate value of renewal intensity $\rho \approx$ m(12)/12 =1.96301433/12 = 0.16358453 per hour (see Eqs. (9.12) and 9.15 on pages 227-229 of Ebeling). In other words, the MTBF $\approx$ 12 hours/m(12) = 12/1.96301433 = 6.1130476 hours, implying tool replacement of roughly 6 hours. Note that Ebeling uses the notation $\rho(t) = dm(t)/dt$ for the renewal intensity. As yet another good example, see the Example 9.6 on page 226 of Ebeling.

Suppose now that the 1st failure occurs at $T_1 = X_1 = t_1$; then, the renewal function must satisfy the following relationship:

$$m(t) = E[N_f(t)] = \int_0^{\infty} E[N_f(t) | X_1 = t_1)] f_1(t_1) dt_1 , \qquad t_1 < t$$

Clearly, if $t_1 > t$, then no failures have occurred by t and thus E[N(t)] = 0, as depicted atop the next page. Therefore, $E[N(t)|X_1 = t_1)]$ = 1 + $E[N(t-t_1)]$ and hence m(t) = E[N_f(t)] =



$$\int_0^t \{1 + E[N_f(t-t_1)]\}f_1(t_1)dt_1 = F_1(t) + \int_0^t E[N_f(t-t_1)]f_1(t_1)dt_1 \rightarrow$$

$$m(t) = F_1(t) + \int_0^t m(t-t_1)f_1(t_1)dt_1 \qquad (92)$$

Eq. (92) is called the Renewal Equation and in simple cases (i.e., in cases when the integrand can be made free of t) can be used to obtain the renewal function m(t), which in turn can be used to obtain ρ(t) =dm(t)/dt. For example, suppose the time to the 1st failure of a component is uniformly distributed over [500, 1000 hours]; then $F_1(t) = (t-500)/500$ and $f_1(t_1)$ = 1/500, $500 \le t_1 \le 1000$ hours. Upon substitution into (92), we obtain m(t) = E[N_f(t)] = $\frac{t-500}{500}$ + $\int_0^t m(t-t_1)dt_1/500$; letting $t - t_1 = x$ yields m(t) = $\frac{t-500}{500} + \frac{1}{500}\int_t^0 m(x)(-dx)$

$$m(t) = \frac{t-500}{500} + \frac{1}{500}\int_0^t m(x)dx \qquad (93)$$

The above Eq. (93) shows that the Renewal intensity Function, ρ(t), is given by

$$\rho(t) = \frac{\partial m(t)}{\partial t} = \frac{1}{500} + \frac{m(t)}{500} \rightarrow \frac{dm(t)}{dt} - \frac{m(t)}{500} = \frac{1}{500} \rightarrow \frac{dm(t)}{dt}\,e^{-t/500} - \frac{m(t)}{500}\,e^{-t/500} =$$

$$\frac{1}{500}\,e^{-t/500} \rightarrow \frac{d}{dt}[m(t)e^{-t/500}] = \frac{1}{500}\,e^{-t/500} \rightarrow m(t)e^{-t/500} = \int \frac{1}{500}e^{-t/500}dt + C$$

$m(t) = (-e^{-t/500}) \times e^{t/500} + C\,e^{t/500} = C\,e^{t/500} - 1$; applying the boundary condition m(t = 0) = 0 →

0 = C – 1 → C = 1 → $m(t) = E[N_f(t)] = e^{t/500} - 1$; for example, the expected number of renewals during [600, 900 hours] is given by $m(300) = e^{300/500} - 1 = 1.8221 - 1 = 0.8221$, while the mean number of failures during the mission time [500, 1000 hours] is given by $m(500) = e^1 - 1 = 1.7183$. Note that the renewal intensity $\rho(t) = \frac{1}{500}e^{t/500}$ is not a pdf; in fact its integral over [0, ∞) diverges in this case. The renewal intensity, $\rho(t)$, gives the instantaneous renewal rate at time t, i.e., $\rho(t) = \lim_{\Delta t \to 0} \frac{m(t+\Delta t) - m(t)}{\Delta t}$ so that $\rho(t) \times \Delta t$ gives the Pr element of a renewal during the interval $(t, t+\Delta t)$, and in the case of negligible repair-time, $\rho(t)$ also represents the instantaneous failure intensity function. When $m(t) = E[N_f(t)] = F_1(t) + \int_0^t m(t - t_1) f_1(t_1) dt_1$ depends on both t and t_1 (t_1 = the TTFF), then the above procedure applied in the case of the uniform distribution for the time to the 1st failure will not generally work because the renewal intensity, $\rho(t)$, will still have an integral on the RHS of the equation (93), and hence we will have to resort to Laplace Transforms described below.

Let f(t), not necessarily a pdf, be any function whose range space is [0, ∞); then the Laplace transform of f(t) is defined as

$$L\{f(t)\} = \bar{f}(s) = f^*(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad s > 0.$$

For example, the Laplace transform of f(t) = 1 is given by $L\{1\} =$

$\int_0^\infty e^{-st} dt = \left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$; note that some authors use θ in lieu of s, while Ebeling also uses s for the arbitrary real numbers. Further, $L\{t\} = \int_0^\infty e^{-st} t\,dt = \left[\frac{-te^{-st}}{s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s^2}$,

$$L\{t^n\} = \int_0^\infty e^{-st}t^n\,dt = \left[\frac{-t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt = \ldots = \frac{n!}{s^{n+1}} = \Gamma(n+1)/(s^{n+1})$$

$$L\{e^{-\lambda t}\} = \int_0^\infty e^{-st}e^{-\lambda t}dt = \int_0^\infty e^{-(s+\lambda)t}\,dt = \left[\frac{-e^{-t(s+\lambda)}}{s+\lambda}\right]_0^\infty = \frac{1}{s+\lambda},\ \lambda > 0.$$

$$L\{df(t)/dt\} = \int_0^\infty e^{-st}f'(t)dt = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt =$$

$$= -f(0) + s\bar{f}(s) = sf^*(s) - f(0) = sL\{f(t)\} - f(0)$$

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-(s-a)t}f(t)dt = \bar{f}(s-a) = f^*(s-a)$$

$$L\{\int_o^t f(x)dx\} = \int_0^\infty e^{-st}\int_0^t f(x)dx\,dt = \left[\frac{-e^{-st}}{s}\int_0^t f(x)dx\right]_0^\infty +$$

$$\frac{1}{s}\int_0^\infty e^{-st}f(t)dt = 0 + \bar{f}(s)/s = \bar{f}(s)/s = f^*(s)/s$$

$$L\{\frac{1}{a-b}(e^{-bt} - e^{-at})\} = \int_0^\infty e^{-st}\,\frac{1}{a-b}(e^{-bt} - e^{-at})dt = \int_0^\infty \frac{1}{a-b}[e^{-(b+s)t} - e^{-(a+s)t}]dt$$

$= \frac{1}{a-b}[\frac{1}{(b+s)} - \frac{1}{(a+s)}] = \frac{1}{(s+a)(s+b)}$. Table 5 gives a summary of basic Laplace transforms.

Properties of Laplace Transforms

L is a linear operator because if C_1 and C_2 are any constants, it can easily be verified that (1) $L\{C_1f_1(t) + C_2f_2(t)\} = C_1L\{f_1(t)\} + C_2\,L\{f_2(t)\}$. (2) Let $f{*}g = \int_0^t f(t-x)g(x)dx$ be the convolution of f(t) with g(t). Then it can be proven that $L\{f(t){*}g(t)\} = \bar{f}(s)\bar{g}(s)$, where $\bar{g}(s) = L\{g(t)\}$.

Table 5. A summary of $\mathcal{L}\{f(t)\}$ for different useful f(t).

Function f(t), or $\mathcal{L}^{-1}\{f*(s)\}$	Laplace Transform $\mathcal{L}\{f(t)\} = \overline{f}(s) = f^*(s)$
λ	λ/s
t	$1/s^2$
t^n	$n!/s^{n+1} = \Gamma(n+1)/s^{n+1}$
$e^{-\lambda t}$	$1/(s+\lambda)$
$\frac{1}{a-b}(e^{-bt}-e^{-at})$	$\frac{1}{(s+a)(s+b)}$
$\frac{t^{n-1}e^{-\lambda t}}{(n-1)!}$	$\frac{1}{(s+\lambda)^n}$
$\frac{(n-1)-\lambda t}{(n-1)!}t^{n-2}e^{-\lambda t}$	$\frac{s}{(s+\lambda)^n}$
df/dt	$s\,f^*(s) - f(0)$
d^2f/dt^2	$s^2\,f^*(s) - sf(0) - df(0)/dt$
$f(at)$	$f^*(s/a)/a$
$\frac{1}{a^2}(e^{-at}+at-1)$	$1/[s^2(s+a)]$
$\frac{1}{a^2}[(1/a)-t+(at^2/2)-e^{-at}/a]$	$1/[s^3(s+a)]$
$\frac{1}{ab}\left[1+\frac{1}{a-b}(ae^{-bt}-be^{-at})\right]$	$\frac{1}{s(s+a)(s+b)}$
$s/[(s+a)(s+b)]$	$[b\exp(-bt)-a\exp(-at)]/(b-a)$
$\frac{1}{b-a}(e^{bt}-e^{at})$	$\frac{1}{(s-b)(s-a)}$
$\frac{1}{ab}\left[1+(ae^{bt}-be^{at})/(b-a)\right]$	$\frac{1}{s(s-b)(s-a)}$

Now let $f_1(t)$ be the pdf of the time to the 1st failure and $\phi_1(s)$ be its Laplace transform, i.e., $\phi_1(s) = L\{f_1(t)\} = \int_0^\infty e^{-st} f_1(t)dt = f_1*(s)$. Similarly, let f(t) be the lifetime density between the 1st & 2nd, 2nd & 3rd, … renewals (or failures). Then $f^*(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \bar{f}(s)$. In the special case $f_1(t) = f(t)$, then $\phi_1(s) = f^*(s)$. Similarly, let $L\{m(t)\} = \int_0^\infty e^{-st} m(t)dt = m^*(s)$ and $L\{\rho(t)\} = \int_0^\infty e^{-st}\rho(t)dt = \rho^*(s)$. Thus, $\rho^*(s) = \int_0^\infty e^{-st}\rho(t)dt = \int_0^\infty e^{-st}[dm(t)/dt]dt = \int_0^\infty e^{-st}\frac{d}{dt}\sum_{n=1}^\infty F_{(n)}(t)dt = \sum_{n=1}^\infty \int_0^\infty e^{-st} f_{(n)}(t)dt = \sum_{n=1}^\infty \bar{f}_{(n)}(s) = \sum_{n=1}^\infty E[e^{-(X_1+X_2+\ldots+X_n)s}] = \sum_{n=1}^\infty \phi_1(s)[f^*(s)]^{n-1} = \frac{\phi_1(s)}{1-f^*(s)}$. Because $\rho(t) = dm(t)/dt$, then its Laplace transform $\rho^*(\theta) = -m(0) + sm^*(s) = 0 + sm^*(s) = sm^*(s) \rightarrow m^*(s) = \frac{\rho^*(s)}{s} \rightarrow m^*(s) = \frac{\phi_1(s)}{s[1-f^*(s)]}$. As an example, suppose $f_1(t) = f(t) = \lambda e^{-\lambda t}$; then $\phi_1(s) = f^*(s) = \int_0^\infty e^{-st}\lambda e^{-\lambda t}dt = \frac{\lambda}{s+\lambda} \rightarrow m^*(s) = \frac{\lambda/(s+\lambda)}{s[1-\lambda/(s+\lambda)]} = \frac{\lambda}{s[(s+\lambda)-\lambda]} = \frac{\lambda}{s^2} \rightarrow m(t) = E[N(t)] = L^{-1}\{m^*(s)\} = L^{-1}\{\frac{\lambda}{s^2}\} = \lambda L^{-1}\{\frac{1}{s^2}\} = \lambda t$, as expected because the above is a HPP.

As yet another example, suppose the Times TF of a machine component has the gamma pdf at the rate of $\lambda = 0.003$ and n = 2 and also $f_1(t) = f(t)$; we wish to obtain the expression for the renewal function m(t) and to compute the expected number of failures during t =1000 hours of operations. Note that X_i's are not exponentially distributed, and hence a NHHP.

$$f^*(s) = \int_0^\infty e^{-st}\lambda(\lambda t)e^{-\lambda t}dt = \lambda^2\int_0^\infty te^{-(\lambda+s)t}dt \;=\; \lambda^2\frac{\partial}{\partial s}\int_0^\infty -e^{-(\lambda+s)t}dt \;=\; \lambda^2\frac{\partial}{\partial s}\left[\frac{e^{-(\lambda+s)t}}{\lambda+s}\right]_0^\infty$$

$$= \lambda^2\frac{\partial}{\partial s}[\frac{-1}{\lambda+s}] \;=\; \frac{\lambda^2}{(\lambda+s)^2} \;\to\; m^*(s) = \frac{\phi_1(s)}{s[1-f^*(s)]} = \frac{\lambda^2/(s+\lambda)^2}{s[1-\lambda^2/(s+\lambda)^2]} = \frac{\lambda^2}{s[(s+\lambda)^2-\lambda^2]} =$$

$\frac{\lambda^2}{s[s^2+2\lambda s]} = \frac{\lambda^2}{s^2[s+2\lambda]}$. I will now use the partial fraction techniques to write $m^*(s)$ in such a manner that we can recognize its inverse Laplace transform.

$$m^*(s) = \frac{\lambda^2}{s^2[s+2\lambda]} = \lambda^2(\frac{a}{s}+\frac{b}{s^2}+\frac{c}{s+2\lambda}) \;=\; \lambda^2[\frac{-1/(4\lambda^2)}{s}+\frac{1/(2\lambda)}{s^2}+\frac{1/(4\lambda^2)}{s+2\lambda}]$$

$m^*(s) = \frac{-1}{4s}+\frac{\lambda}{2s^2}+\frac{1}{4(s+2\lambda)} \to$ m(t) = E[N(t)] = $L^{-1}\{m^*(s)\}$ = $L^{-1}\{\frac{-1}{4s}+\frac{\lambda}{2s^2}+\frac{1}{4(s+2\lambda)}\}$ = $-\frac{1}{4}$ + $\frac{\lambda}{2}t$ + $\frac{1}{4}e^{-2\lambda t}$; thus, during 1000 hours we expect m(1000) = $-\frac{1}{4}$ + $\frac{0.003}{2}1000$ + $\frac{1}{4}e^{-6}$ = 1.2506 failures; note that if the above process lifetime were exponential at the rate λ = 0.003, instead of the gamma with n =2, then m(1000) = λt = 3 with reliability R(1000) = exp(-3) = 0.0498 and renewal intensity λ, while for our gamma density TTF, the renewal intensity is ρ(t) = dm/dt = $\frac{\lambda}{2}$ – $\frac{\lambda}{2}e^{-2\lambda t}$, m(1000) = 1.2506, and R(1000) = (1+λt)$e^{-\lambda t}$ = 4 exp(-3) =0.1991.

Ebeling is his Example 9.11 on his page 232 computes the R(from 730 days to 1095 = 1 year duration) by integrating the intensity function ρ(t). If we apply his method to the Gamma pdf with λ = 0.003 and n = 2, we obtain R(1000 hrs) = exp(–1.2506) = 0.2863. As of right now, I am not sure that we can compute the value of the RE function in his manner for all cases because m(t) in general will not equal to the cumulative hazard function H(t). In fact for the Gamma density with n = 2, it can easily be verified that f(t) = $\frac{\lambda}{\Gamma(2)}(\lambda t)^{2-1}e^{-\lambda t} = \lambda(\lambda t)e^{-\lambda t}$, R(t) = (1+λt)× $e^{-\lambda t}$ → h(t) = f(t)/R(t) = $\lambda(\lambda t)/(1+\lambda t)$ → H(t) = λt –ln(1+λt), which is quite different from m(t)

$= -1/4 + \frac{\lambda}{2}t + \frac{1}{4}e^{-2\lambda t}$. This paradox could be due to the fact that the renewal intensity, $\rho(t)$, is not a constant as in a HPP which is equal to λ, i.e., we are faced with a NHPP .

As yet another example, suppose the time to the 1st failure of a new machine is exponential at the rate $\lambda_1 = 0.0005$/hour, but the succeeding renewals occur at the constant rate $\lambda_2 = 0.001$; this is called a modified renewal process because the TTFF (Time To first Failure) has a different distribution from succeeding failures. We wish to compute the expected number of failures during t = 10,000 hours of operations.

$\phi_1(s) = \int_0^\infty e^{-st}\lambda_1 e^{-0.0005t}dt = \frac{\lambda_1}{s+\lambda_1}$; similarly, $f^*(s) = \int_0^\infty e^{-st}\lambda_2 e^{-\lambda_2 t}dt = \frac{\lambda_2}{s+\lambda_2} \rightarrow m^*(s) =$ $\frac{\lambda_1/(s+\lambda_1)}{s[1-\lambda_2/(s+\lambda_2)]} = \frac{\lambda_1(s+\lambda_2)}{s[(s+\lambda_1)(s+\lambda_2)-\lambda_2(s+\lambda_1)]} = \frac{\lambda_1(s+\lambda_2)}{s(s^2+\lambda_1 s)} = \frac{\lambda_1(s+\lambda_2)}{s^2(s+\lambda_1)} = \frac{\lambda_1-\lambda_2}{\lambda_1 s} + \frac{\lambda_2}{s^2}$ $+ \frac{\lambda_2-\lambda_1}{\lambda_1(s+\lambda_1)} \rightarrow m(t) = E[N(t)] = L^{-1}\{m^*(s)\} = L^{-1}\{\frac{\lambda_1-\lambda_2}{\lambda_1 s} + \frac{\lambda_2}{s^2} + \frac{\lambda_2-\lambda_1}{\lambda_1(s+\lambda_1)}\} = \frac{\lambda_1-\lambda_2}{\lambda_1} +$ $\lambda_2 t + \frac{\lambda_2-\lambda_1}{\lambda_1}e^{-\lambda_1 t} = \frac{0.0005-0.001}{0.0005} + 10 + e^{-5} = 9.00674$ failures, and $\rho(t) = \lambda_2 + (\lambda_1-\lambda_2)e^{-\lambda_1 t}$.

Some Limiting Results For Renewal Processes

The most important limiting result in renewal theory is the fact that

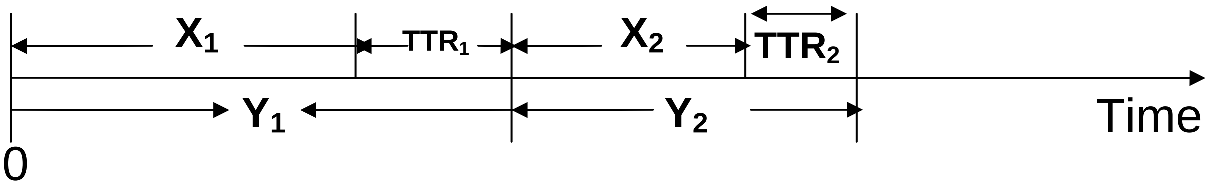
$$\lim_{t \to \infty} \rho(t) = 1/\mu$$

which states in the limit one cannot identify when the renewal process began so that over the long-term the rate at which components are replaced is inversely proportional to the average time, μ, between renewals. Further, because $dm(t)/dt = \rho(t) \cong \frac{m(t+\Delta t)-m(t)}{\Delta t} \rightarrow$ $\int_{t_1}^{t_2} dm = \int_{t_1}^{t_2} \rho(x)dx$, and also $m(t+\Delta t)-m(t) \cong \rho(t)\Delta t \rightarrow$ The expected number of renewals

during (t, t+Δt) is approximately equal to $\rho(t)\Delta t = \Delta t/\mu$. This implies that the expected number of renewals during an interval of length t is roughly given by m(t) $\cong$ t/μ, this approximation improving as t $\to \infty$. For the example on p. 154 involving the gamma pdf with λ = 0.003 and n = 2, MTBF = μ = 2/0.003 = 666.6667 hrs; thus during 1000 hours of operation, m(t) $\cong$ t/μ = 1000/666.6667 = 1.5000 (not very close to the exact m(t) = 1.2506 because 1000 hours is too short). However, at t = 10,000 hours, m(10000) = E[N(10000)] = $-\frac{1}{4} + \frac{0.003}{2}10000 + \frac{1}{4}e^{-2\lambda(10000)}$ = 14.7500 failures, and m(t) $\cong$ t/μ = 10000/666.6667 = 15 failures (much better approximation). As yet another example, see Examples 9.7 on pages 227-8 and 9.9 on p. 230 of Ebeling.

Alternating (or Cyclical) Renewal Processes with non-negligible Repair-time

Thus far we have assumed that the repair time has been negligible compared to operating times so that E[N(t)] is simply $\sum_{n=1}^{\infty} F_{(n)}(t)$, where $F_{(n)}(t)$ is the cdf of the time to the nth renewal measured from zero. Or, E[N(t)] = m(t) = $L^{-1}\{m^*(s)\}$ = $L^{-1}\{\frac{\phi_1(s)}{s[1-\phi(s)]}\}$. Suppose now active restoration times TTR_1, TTR_2, TTR_3, … (TTR = Time to Repair) are not negligible but are rvs with identical pdfs r(t), r for repair. Then time to the 1st cycle (or renewal) now is given by $Y_1 = X_1 + TTR_1$; time to the 2nd cycle (measured from the last) is now given by $Y_2 = X_2 + TTR_2$; time to the 3rd renewal period is given by $Y_3 = X_3 + TTR_3$, and so on, as depicted below. A network exhibits such a cyclic behavior between up and down states, but generally down states include administrative and logistic times in addition to active repair time.



Assuming that X_i's and TTR_i's are iid rvs with pdfs f(x) and r(ξ), respectively, then Y_i's are iid rvs with the cdf $G(t) = P(Y_i \leq t)$ and pdf $g_y(t)$. Clearly $g_y(t)$ is the convolution of f(x) with r(ξ), i.e.,

$g_y(t) = f(x)*r(\xi) = \int_0^t f(x)r(t-x)dx = \int_0^t f(t-\xi)r(\xi)d\xi$, where ξ = TTR stands for active repair time.

Then the Laplace transform of $g_y(t)$ is given by $L\{g_y(t)\} = L\{f(x)\} \times L\{r(\xi)\} = f^*(s)r^*(s)$. Thus the Laplace transform of the renewal function and renewal density that involves maintenance are given by $m^*(s) = \frac{\bar{f}(s)r^*(s)}{s[1-\bar{f}(s)r^*(s)]}$, and $\rho^*(s) = \frac{\bar{f}(s)r^*(s)}{1-\bar{f}(s)\bar{r}(s)}$ where is $r^*(s) = \bar{r}(s) = L\{r(t)\}$. As an example, suppose a Machine component TTF is exponential at the rate λ and its repair time is also exponential at the rate λ_r, i.e., $r(t) = \lambda_r e^{-\lambda_r t}$. Thus, $f^*(s) = \bar{f}(s) = \frac{\lambda}{s+\lambda}$ and

$$r^*(s) = \frac{\lambda_r}{s+\lambda_r} \rightarrow m^*(s) = \frac{f^*(s)r^*(s)}{s[1-f^*(s)r^*(s)]} = \frac{\lambda\lambda_r}{s[(s+\lambda)(s+\lambda_r)-\lambda\lambda_r]} = \frac{\lambda\lambda_r}{s[s^2+(\lambda+\lambda_r)s]} =$$

$$\frac{\lambda\lambda_r}{s^2[s+(\lambda+\lambda_r)]} = \frac{-\lambda\lambda_r}{s(\lambda+\lambda_r)^2} + \frac{\lambda\lambda_r}{s^2(\lambda+\lambda_r)} + \frac{\lambda\lambda_r}{(s+\lambda+\lambda_r)(\lambda+\lambda_r)^2} \rightarrow m(t) = L^{-1}\{m^*(s)\} = L^{-1}\{$$

$$\frac{-\lambda\lambda_r}{s(\lambda+\lambda_r)^2} + \frac{\lambda\lambda_r}{s^2(\lambda+\lambda_r)} + \frac{\lambda\lambda_r}{(s+\lambda+\lambda_r)(\lambda+\lambda_r)^2}\} = \frac{-\lambda\lambda_r}{(\lambda+\lambda_r)^2} + \frac{\lambda\lambda_r t}{(\lambda+\lambda_r)} + \frac{\lambda\lambda_r}{(\lambda+\lambda_r)^2}e^{-(\lambda+\lambda_r)t}$$

.

For example, if λ = 0.0005 and λ_r = 0.05 per hour, then the expected number of cycles (or renewals) in 10000 hours is equal to m(10000) = $\frac{-(0.0005)(0.05)}{(0.0505)^2} + \frac{(0.000025)10000}{0.0505} + \frac{0.25}{(0.0505)^2}e^{-(0.0505)10000}$ = 4.9406921. The instantaneous renewal rate is given by $\rho(t) = \frac{\lambda\lambda_r}{\lambda+\lambda_r} - \frac{\lambda\lambda_r}{(\lambda+\lambda_r)}e^{-(\lambda+\lambda_r)t}$, and its limiting behavior is $\lim_{t\to\infty}\rho(t) = \frac{\lambda\lambda_r}{(\lambda+\lambda_r)} = \frac{1}{1/\lambda_r+1/\lambda} = \frac{1}{\text{MTTR}+\text{MTTF}}$, where MTTR stands for the component's mean time to repair. Thus, in the limit the mean time between cycles is given by MTBC = $1/\left(\lim_{t\to\infty}\rho(t)\right) = \text{MTTR}+\text{MTTF}$; for

the above example, MTBC = $\dfrac{1}{0.05}$ + $\dfrac{1}{0.0005}$ = 2020 hours, while if the machine component were irreparable (was just merely replaced), then MTBC = MTBF = 1/0.0005 = 2000 hours.

For yet another example, see p. 235 of Ebeling where both TTF and TTR are exponential at the rates λ and r, respectively. Thus, the density of cycle time (y= TTF +TTR) is given by

$$g_y(t) = f(x)*r(\xi)= \int_0^t f(t-\xi)r(\xi)d\xi = \int_0^t \lambda e^{-\lambda(t-\xi)}\, re^{-r\xi}d\xi = \lambda re^{-\lambda t}\int_0^t e^{-(r-\lambda)\xi}\,d\xi = \frac{\lambda re^{-\lambda t}}{-(r-\lambda)}\left[e^{-(r-\lambda)\xi}\right]_0^t =$$

$$\frac{\lambda re^{-\lambda t}}{-(r-\lambda)}\left[e^{-(r-\lambda)t}-1\right] = \frac{\lambda re^{-\lambda t}}{(r-\lambda)}\left[1-e^{-(r-\lambda)t}\right] = \frac{\lambda r}{(r-\lambda)}\left[e^{-\lambda t}-e^{-rt}\right]$$

, which is the same as Ebeling's equation atop his page 235. Now, see his example 9.13 on p. 235.

Section 9.5 of Ebeling (The Reliability function under preventive maintenance)

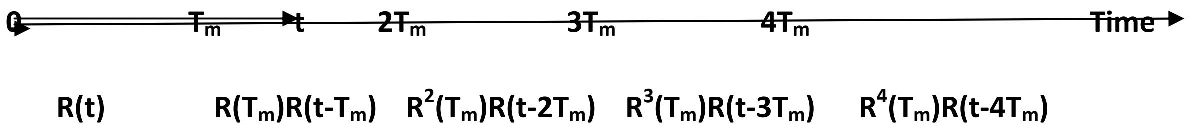
Ebeling (2^{nd} Ed. pp. 237 to 241) has an excellent discussion as how preventive maintenance (PM) can improve system RE iff the hazard function h(t) is increasing.

Suppose PM is performed periodically every T_m days. We further assume that once preventive maintenance (PM) is performed on a system, then the system is practically as-good-as-new. There are two MUEX (mutually exclusive) possibilities:

(1) The system fails by T_m , (2) the system survives beyond at least one cycle of length T_m. These 2 possibilities lead to the following RE function with PM:

$$R_m(t) = \begin{cases} R(t),\ 0 \le t < T_m \\ R(T_m)^n R(t-nT_m),\ nT_m \le t < (n+1)T_m \end{cases} . \tag{94}$$

Note that Ebeling uses T for T_m in Eq. (94) to represent the length of one PM cycle. The argument for the above RE function is depicted in the following figure.



The MTTF of a PM system is given by (m standing for with maintenance)

$$\text{MTTF}_m = \int_0^\infty R_m(t)dt = \sum_{n=0}^{\infty} \int_{nT_m}^{(n+1)T_m} R(T_m)^n R(t-nT_m)dt = \sum_{n=0}^{\infty} R(T_m)^n \int_{nT_m}^{(n+1)T_m} R(t-nT_m)dt$$

In the above integral, put $x = t-nT_m$. This leads to

$$\text{MTTF}_m = \sum_{n=0}^{\infty} R(T_m)^n \int_0^{T_m} R(x)dx = \frac{\int_0^{T_m} R(t)dt}{1-R(T_m)} \qquad \text{(Eq. 9.27 of Ebeling)}$$

Ebeling provides a good example of an exponential lifetime with PM in his Example 9.17 on p. 238 in which he illustrates that PM does not alter R(t) iff h(t) = λ is a constant. Note that if we compute the MTTF$_m$ using Eq. (9.27) under constant h(t), we obtain MTTF = $\dfrac{\int_0^{T_m} e^{-\lambda t}dt}{1-e^{-\lambda T_m}}$ **=**

$\dfrac{\frac{-1}{\lambda}\left[e^{-\lambda t}\right]_0^{T_m}}{1-e^{-\lambda T_m}}$ **=** $\dfrac{\frac{-1}{\lambda}\left[e^{-\lambda T_m}-1\right]}{1-e^{-\lambda T_m}}$ **= 1/λ, i.e., for an exponential TTF the MTTF with and without PM are identically equal to 1/λ. For another excellent example, see the Example 9.18 on p. 238 of Ebeling. For this Example, I am changing the value of t_c for the compressors from 100 days to 120 days so that now TTF ~ W(0, θ =120 days, shape= β =2); further, I am changing the maintenance cycle to one month = 30 days (25% of θ). Thus, the RE function with PM is given by R$_m$(t) =** $\begin{cases} R(t),\ 0 \le t < 30 \text{ days} \\ R(30)^n R(t-30n),\ \ 30n \le t < 30(n+1) \end{cases}$ **, n = 0, 1, 2, 3, 4, …**

The value of RE at 160 days from time zero is computed first by recognizing that n = 5 PM cycles and that 165 days lies within the interval (5×30, 6×30), and hence t = 10 days; thus, R$_m$(160) = $R(30)^5 R(160-150),\ 150 \le t < 180 \text{ days}$ **. I am providing an Excel file that computes the preventive-maintenance RE functions for all parameters of the Weibull, namely t_0, θ, β, and**

n. My Excel file gives $R_m(160 \text{ days}) = 0.726553$ with PM versus 0.1690133 for the RE value W/O PM, i.e., PM improves RE by 329.88%. The MTTF(W/O PM) = 106.347 days versus

MTTF(with PM) =MTTF_m= $\dfrac{\int_0^{T_m} R(x)dx}{1-R(T_m)} = \dfrac{\int_0^{30} e^{-(x/120)^2}dx}{1-0.9394131} = \dfrac{\int_0^{\sqrt{2}/4} e^{-z^2/2}(120dz/\sqrt{2})}{0.0605869}$ **, where x/120**

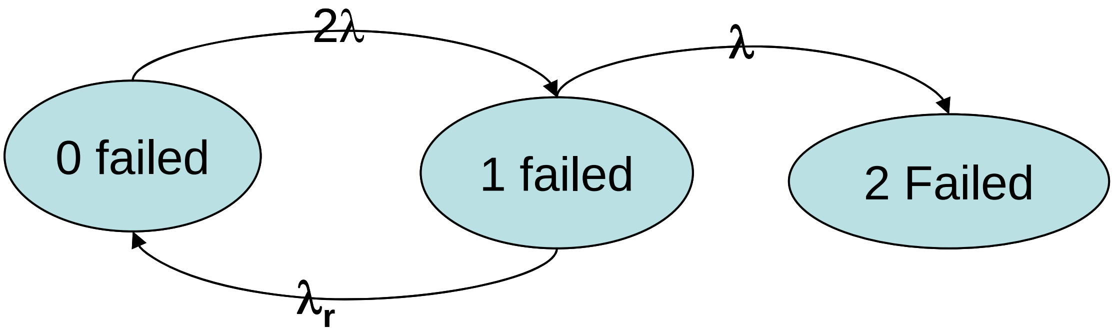
= z/$\sqrt{2}$**. Thus, MTTF_m =** $\dfrac{120\sqrt{\pi}\int_0^{\sqrt{2}/4} e^{-z^2/2}(dz/\sqrt{2\pi})}{0.0605869} = \dfrac{120\sqrt{\pi}[\Phi(\sqrt{2}/4)-0.50]}{0.0605869} =$

$\dfrac{120\sqrt{\pi}(0.1381632)}{0.0605869}$ **= 485.03136, which is an improvement of 356.083% in MTTF due to PM.**

Ebeling also discusses the case when PM induces failure into the system with a Pr, p, and the RE function with PM is given atop page 240 in his Eq. (9.28). See his Example 9.19 on p. 240.

Section 9.6 of Ebeling (Repairable Systems)

Figure 9.5 on page 241 of Ebeling describes one cycle of a 2-component repairable pure-parallel system where state 3 is absorbing. For convenience I have modified his Figure 9.5 below, where r = λ_r represents repair rate and state 2 represents system failure. From the figure



we can deduce that $\begin{cases} P_0(t+\Delta t) = P_0(t)(1-2\lambda\Delta t) + P_1(t)\lambda_r\Delta t \\ P_1(t+\Delta t) = P_0(t)(2\lambda\Delta t) + P_1(t)(1-\lambda_r\Delta t - \lambda\Delta t) \\ P_2(t+\Delta t) = P_1(t)(\lambda\Delta t) \end{cases}$.

After transposing the pertinent Pr functions from the RHS to the LHS, dividing by Δt and taking limit as $\Delta t \to 0$ result in the following system of differential equations:

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + \lambda_r P_1(t) \\ dP_1(t)/dt = 2\lambda P_0(t) - (\lambda_r + \lambda)P_1(t) \\ dP_2(t)/dt = \lambda P_1(t) \end{cases}$$

Because $P_0(t) + P_1(t) + P_2(t) = 1$, then $dP_2(t)/dt = -\,dP_1(t)/dt - dP_0(t)/dt \to$ the above system of differential equations can be reduced to

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + \lambda_r P_1(t) \\ dP_1(t)/dt = 2\lambda P_0(t) - (\lambda_r + \lambda)P_1(t) \end{cases}$$

Taking Laplace transforms, we obtain

$$\begin{cases} sP_0{}^*(s) - P_0(0) = -2\lambda P_0{}^*(s) + \lambda_r P_1{}^*(s) \\ sP_1{}^*(s) - P_1(0) = 2\lambda P_0{}^*(s) - (\lambda_r + \lambda)P_1{}^*(s) \end{cases}$$

Applying the boundary conditions $P_0(0) = 1$ and $P_1(0) = 0$, we obtain

$$\begin{cases} sP_0{}^*(s) - 1 = -2\lambda P_0{}^*(s) + \lambda_r P_1{}^*(s) \\ sP_1{}^*(s) = 2\lambda P_0{}^*(s) - (\lambda_r + \lambda)P_1{}^*(s) \end{cases} \to \begin{cases} (s+2\lambda)P_0{}^*(s) - \lambda_r P_1{}^*(s) = 1 \\ 2\lambda P_0{}^*(s) - (s+\lambda_r+\lambda)P_1{}^*(s) = 0 \end{cases} \to$$

$$P_0{}^*(s) = \frac{\begin{vmatrix} 1 & -\lambda_r \\ 0 & -(s+\lambda_r+\lambda) \end{vmatrix}}{\begin{vmatrix} (s+2\lambda) & -\lambda_r \\ 2\lambda & -(s+\lambda_r+\lambda) \end{vmatrix}} = \frac{-(s+\lambda_r+\lambda)}{-(s+\lambda_r+\lambda)(s+2\lambda)+2\lambda\lambda_r} = \frac{(s+\lambda_r+\lambda)}{(s+\lambda_r+\lambda)(s+2\lambda)-2\lambda\lambda_r}$$

$$= \frac{s+\lambda_r+\lambda}{s^2+(\lambda_r+3\lambda)s+2\lambda^2} = \frac{s+\lambda_r+\lambda}{(s-u_1)(s-u_2)} = \frac{A}{s-u_1} + \frac{B}{s-u_2}$$

where u_1 and u_2 are the roots of the quadratic $s^2 + (\lambda_r + 3\lambda)s + 2\lambda^2 = 0$, u_i =

$\frac{-(\lambda_r + 3\lambda) \pm \sqrt{(\lambda_r + 3\lambda)^2 - 8\lambda^2}}{2}$ = $\frac{-(\lambda_r + 3\lambda) \pm \sqrt{\lambda_r^2 + 6\lambda\lambda_r + \lambda^2}}{2}$, i = 1, 2. → A =

$\frac{\lambda + \lambda_r + u_1}{\sqrt{\lambda_r^2 + 6\lambda\lambda_r + \lambda^2}}$ = $\frac{\lambda + \lambda_r + u_1}{u_1 - u_2}$; B = $-\frac{\lambda + \lambda_r + u_2}{u_1 - u_2}$, and note that both u_1 and u_2 < 0, and u_1

$-u_2$ = $-\sqrt{\lambda_r^2 + 6\lambda\lambda_r + \lambda^2}$. Thus, $P_0(t)$ = $L^{-1}\{P_0{*}(s)\}$ = $L^{-1}\{\frac{A}{s-u_1}$ + $\frac{B}{s-u_2}\}$ = $Ae^{u_1 t}$ + $Be^{u_2 t}$ →

$dP_0(t)/dt$ = $Au_1e^{u_1 t}$ + $Bu_2e^{u_2 t}$; because $dP_0(t)/dt + 2\lambda P_0(t) = \lambda_r P_1(t)$ → $Au_1e^{u_1 t}$ + $Bu_2e^{u_2 t}$ +

$2\lambda(Ae^{u_1 t} + Be^{u_2 t}) = \lambda_r P_1(t)$ → $A(u_1 + 2\lambda)e^{u_1 t}$ + $B(u_2 + 2\lambda)e^{u_2 t}$ = $\lambda_r P_1(t)$; but A =

$\frac{(-\lambda + \lambda_r - \sqrt{\lambda_r^2 + 6\lambda\lambda_r + \lambda^2}\ )/2}{u_1 - u_2}$ and B = $\frac{(\lambda - \lambda_r - \sqrt{\lambda_r^2 + 6\lambda\lambda_r + \lambda^2}\ )/2}{u_1 - u_2}$. Further, $u_1 + 2\lambda$ =

$\frac{\lambda - \lambda_r - \sqrt{\lambda_r^2 + 6\lambda\lambda_r + \lambda^2}}{2}$ → A×$(u_1 + 2\lambda)$ = $\frac{2\lambda\lambda_r}{u_1 - u_2}$; similarly, $(u_2 + 2\lambda)$ =

$\frac{\lambda - \lambda_r + \sqrt{\lambda_r^2 + 6\lambda\lambda_r + \lambda^2}}{2}$ → B×$(u_2 + 2\lambda)$ = $-\frac{2\lambda\lambda_r}{u_1 - u_2}$ → $P_1(t)$ = $\frac{2\lambda}{u_1 - u_2}e^{u_1 t}$ − $\frac{2\lambda}{u_1 - u_2}e^{u_2 t}$ →

R(t) = $P_0(t)$ + $P_1(t)$ = $\frac{-u_2}{u_1 - u_2}e^{u_1 t}$ + $\frac{u_1}{u_1 - u_2}e^{u_2 t}$. The system MTTF can be obtained by integrating

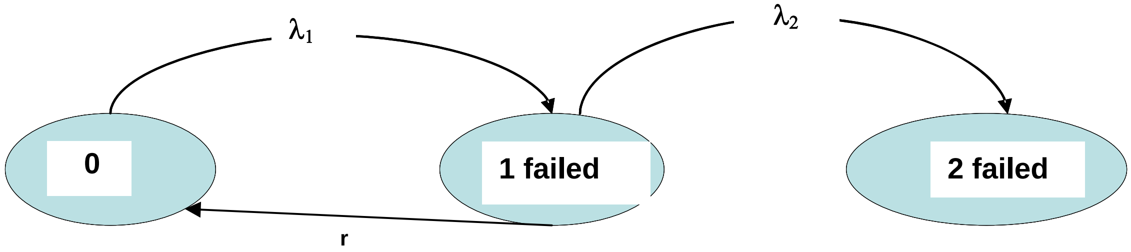
the RE function from 0 to ∞, which Ebeling does near the top of page 242 in his Eq.

(9.34) as $MTTF_{Sys}$ = $\frac{3\lambda + \lambda_r}{2\lambda^2}$. If λ = 0.0005 and λ_r = 0.05 per hour, $MTTF_{Sys}$ = $\frac{0.0015 + 0.05}{2(0.0005)^2}$ = 103000

hours so that the system's effective failure rate is λ_{Sys} = 1/$MTTF_{Sys}$ = 0.00000970874 per hour.

Ebeling also obtains the transient solution for the 2-unit standby system (with perfect sensor and switching) using TRD 9.6 and 9.7 on pp. 243 and quiescent failure rate of roughly zero. For your convenience, I have reproduced his Figure 9.7 atop the next page.

The TRD shows leads to the ddeqs: $\begin{cases} dP_0(t)/dt = -\lambda_1 P_0(t) + rP_1(t) \\ dP_1(t)/dt = \lambda_1 P_0(t) - (\lambda_2 + r)P_1(t) \end{cases}$



The TRD for a 2-unit Standby with On-line Repair rate r

Taking Laplace transforms of both sides of the two equations leads to

$$\begin{cases} sP_0^*(s) - P_0(0) = -\lambda_1 P_0^*(s) + rP_1^*(s) \\ sP_1^*(s) - P_1(0) = \lambda_1 P_0^*(s) - (\lambda_2 + r)P_1^*(s) \end{cases}$$

Applying the boundary conditions $P_0(0) = 1$ and $P_1(0) = 0$, we obtain

$$\begin{cases} sP_0^*(s) - 1 = -\lambda_1 P_0^*(s) + rP_1^*(s) \\ sP_1^*(s) - 0 = \lambda_1 P_0^*(s) - (\lambda_2 + r)P_1^*(s) \end{cases} \rightarrow \begin{cases} (s+\lambda_1)P_0^*(s) - \quad rP_1^*(s) = 1 \\ \lambda_1 P_0^*(s) - (s + \lambda_2 + r)P_1^*(s) = 0 \end{cases}$$

$$P_0^*(s) = \frac{\begin{vmatrix} 1 & -r \\ 0 & -(s+\lambda_2+r) \end{vmatrix}}{\begin{vmatrix} (s+\lambda_1) & -r \\ \lambda_1 & -(s+\lambda_2+r) \end{vmatrix}} = \frac{-(s+\lambda_2+r)}{-(s+\lambda_2+r)(s+\lambda_1)+r\lambda_1} = \frac{(s+\lambda_2+r)}{(s+\lambda_2+r)(s+\lambda_1)-r\lambda_1}$$

$$= \frac{s+\lambda_2+r}{s^2+(\lambda_1+\lambda_2+r)s+\lambda_1\lambda_2} = \frac{s+\lambda_2+r}{(s-u_1)(s-u_2)} = \frac{A}{s-u_1} + \frac{B}{s-u_2}$$

where u_1 and u_2 are the roots of the quadratic $s^2 + ks + \lambda_1\lambda_2 = 0$, where $k = \lambda_1+\lambda_2+r$; thus,

u_i = $\frac{-k \pm \sqrt{k^2 - 4\lambda_1\lambda_2}}{2}$, i = 1, 2. → Because $P_0^*(s) = \frac{A(s-u_2)+B(s-u_1)}{(s-u_1)(s-u_2)}$, then A+B = 1 and

$-Au_2 - Bu_1 = \lambda_2 + r$. Solving these last two equations simultaneously, we obtain A = $\frac{r+\lambda_2+u_1}{u_1-u_2}$; B =

$-\frac{r+\lambda_2+u_2}{u_1-u_2}$, and note that both u_1 and $u_2 < 0$ and $u_1 - u_2 = -\sqrt{k^2 - 4\lambda_1\lambda_2}$. When $\lambda_1 = \lambda_2 = \lambda$, u_1=

$-(\lambda + r/2) - \sqrt{\lambda r + (r^2/4)}$, $u_1 - u_2 = -\sqrt{4\lambda r + r^2}$, A = $\frac{\sqrt{4\lambda r + r^2} - r}{2\sqrt{4\lambda r + r^2}}$ and B = $\frac{\sqrt{4\lambda r + r^2} + r}{2\sqrt{4\lambda r + r^2}}$. Thus,

$P_0(t) = L^{-1}\{P_0^*(s)\} = L^{-1}\{\frac{A}{s-u_1} + \frac{B}{s-u_2}\} = Ae^{u_1 t} + Be^{u_2 t} \rightarrow P_0(t) = \frac{r+\lambda_2+u_1}{u_1-u_2}e^{u_1 t} - \frac{r+\lambda_2+u_2}{u_1-u_2}$

$e^{u_2 t}$. Because, $dP_0(t)/dt = -\lambda_1 P_0(t) + rP_1(t)$, then it follows that $P_1(t) = [P_0'(t) + \lambda_1 P_0(t)]/r$ =

$\frac{Au_1e^{u_1t} + Bu_2e^{u_2t} + \lambda_1(Ae^{u_1t} + Be^{u_2t})}{r} = \frac{Ae^{u_1t}(u_1+\lambda_1) + Be^{u_2t}(u_2+\lambda_1)}{r}$. I have provided an Excel file on my website that obtains Both $P_0(t)$ and $P_1(t)$ for any given values of λ_1, λ_2 and r. Thus, R(t) = $P_0(t)+P_1(t)$ so that the MTTF = $-A/u_1 - B/u_2 - A(u_1+\lambda_1)/(u_1r) - B(u_2+\lambda_1)/(u_2r)$ =

$-\frac{A(r+\lambda_1+u_1)}{ru_1} - \frac{B(r+\lambda_1+u_2)}{ru_2}$. For the Example 9.20 on p. 244 of Ebeling, we have $\lambda_1 = 0.0005$, $\lambda_2 = 0.002$, and r = 0.10 so that the $MTTF_{Sys}$ = 102,500 hrs $\lambda_{Sys} = 1/MTTF_{Sys}$ = 0.0000097561 per hour. While without repair, the MTTF = $1/\lambda_1 + 1/\lambda_2$ = 2500 hours with R(t) = $e^{-\lambda_1 t}$ +

$\frac{\lambda_1}{\lambda_1-\lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t})$. Note that if $\lambda_1 = \lambda_2 = \lambda$, then this last RE function (W/O) repair reduces to

$e^{-\lambda t}(1+\lambda t)$, which is a Poisson process.

Chapter Summary

1. Minimal Repair

The renewal function m(t) = E[N$_f$(t)] = $\sum_{n=1}^{\infty} n \times P_n(t)$ = $\sum_{n=1}^{\infty} F_{(n)}(t)$ because Pr{N$_f$(t) ≥ n} = Pr(T$_n$ ≤ t).

Further, m(t) = $L^{-1}\{m^*(s)\}$, where $m^*(s) = \dfrac{\phi_1(s)}{s[1-\bar{f}(s)]}$ $= \dfrac{f_1{}^*(s)}{s[1-f^*(s)]}$, and f*(s) = $\int_0^{\infty} e^{-st} f(t)dt$.

As a result, the renewal intensity function is ρ(t) = dm/dt, its Laplace transform is $\rho^*(s) =$ $\dfrac{\phi_1(s)}{1-f^*(s)}$, and MTTF = $\lim_{t\to\infty}[1/\rho(t)]$.

2. Non-negligible Repair Time

$m^*(s) = \dfrac{\bar{f}(s)r^*(s)}{s[1-\bar{f}(s)r^*(s)]}$, **$\rho^*(s) = \dfrac{\bar{f}(s)r^*(s)}{1-\bar{f}(s)\bar{r}(s)}$, m(t) = $L^{-1}\{m^*(s)\}$, and MTBC = $\lim_{t\to\infty}[1/\rho(t)]$.**

(a) The case of two-identical-unit redundant parallel system with on-line repair and one crew only when the system is in state "1": MTTF$_{Sys}$ = $\dfrac{3\lambda+\lambda_r}{2\lambda^2}$ = $\dfrac{3\lambda+r}{2\lambda^2}$, and R(t) = R(t) = P$_0$(t) + P$_1$(t) = $\dfrac{u_2}{u_2-u_1} e^{u_1 t}$ + $\dfrac{u_1}{u_1-u_2} e^{u_2 t}$, where u$_1$ = $\dfrac{-(\lambda_r+3\lambda)-\sqrt{\lambda_r^2+6r\lambda+\lambda^2}}{2}$, where r = λ$_r$.

(b) The case of two-unit standby system with on-line repair only on the primary-unit when the system is in state "1": MTTF$_{Sys}$ $= -\dfrac{A(r+\lambda_1+u_1)}{ru_1} - \dfrac{B(r+\lambda_1+u_2)}{ru_2}$, and R(t) = R(t) = P$_0$(t) + P$_1$(t) where P$_0$(t) = $\dfrac{r+\lambda_2+u_1}{u_1-u_2} e^{u_1 t}$ $-$ $\dfrac{r+\lambda_2+u_2}{u_1-u_2} e^{u_2 t}$, and P$_1$(t) = $\dfrac{Ae^{u_1 t}(u_1+\lambda_1)+Be^{u_2 t}(u_2+\lambda_1)}{r}$

Where u$_i$ = $\dfrac{-k \pm \sqrt{k^2-4\lambda_1\lambda_2}}{2}$, k = k = λ$_1$+λ$_2$+r, A = $\dfrac{r+\lambda_2+u_1}{u_1-u_2}$, and B = $\dfrac{r+\lambda_2+u_2}{u_2-u_1}$.