

Interference Theory for Stress and Strength Reference: Chapter 7 of Ebeling Maghsoodloo

Let the rv Y represent the strength (or capacity) of a component (or a system) and the rv X represent the stress applied on the component. Then the component reliability is given by

$$R = P(Y > X) = \begin{cases} P(Y - X > 0), \\ \text{or } P(Y / X > 1) \end{cases} = \begin{cases} P(Y - X > 0), \\ \text{or } P(\eta > 1) \end{cases} \quad (89)$$

where the random variable $\eta = Y/X$ is called the Factor of Safety (or Safety Factor), which is clearly the ratio of two rvs. We assume that both X and Y are independent continuous rvs with pdfs $f(x)$ and $g(y)$, respectively. To obtain a general expression for RE, we can use any one of the following two approaches depending on the expressions for $f(x)$ and $g(y)$.

1. Suppose that the stress acting on a device is in the neighborhood of x_0 . Then the device incremental RE is given by

$$P[(x_0 - dx \leq x \leq x_0 + dx) \cap (Y > x_0)] =$$

$$P(x_0 - dx \leq x \leq x_0 + dx) \times P(Y > x_0 | x_0 - dx \leq x \leq x_0 + dx) = f(x_0) dx \int_{x_0}^{\infty} g(y) dy .$$

where $P(Y > x_0 | x_0 - dx \leq x \leq x_0 + dx) = P(Y > x_0)$ because Y is independent of X . Since the value of stress can range from $-\infty$ to ∞ , we can remove the condition on x_0 by integrating over all possible values of x_0 , i.e., the unconditional RE is given by

$$R = \int_{-\infty}^{\infty} [f(x_0) \int_{x_0}^{\infty} g(y) dy] dx = \int_{-\infty}^{\infty} [f(x) \int_x^{\infty} g(y) dy] dx = \int_{-\infty}^{\infty} G_y(x) f(x) dx \quad (90a)$$

where $G_y(x) = 1 - F_y(x)$ and $F_y(x)$ is the cdf of Y at the stress value x so that $G_y(x)$ is the exceeding Pr of strength at the stress value x .

2. Suppose the strength of a device (or a system) is around the value y_0 . Then the device will be RE iff the stress acting on it is less than y_0 . Letting $F_x(y)$ represent the cdf of stress at the strength y and applying similar logic as above, we deduce that

$$R = \int_{-\infty}^{\infty} F_x(y)g(y)dy \quad (90b)$$

Example 13. Suppose a component's strength, Y , $\sim N(100 \text{ MPa}, 100)$ and the stress acting on the component is exponentially distributed with mean $\mu_x = 50 \text{ MPa}$. Since the exponential cdf is directly invertible while the Gaussian is not, then it is best to use Eq. (90b) because the cdf of the exponential is given by $F_x(y) = 1 - e^{-y/50}$, while the normal cdf is an integral. Hence, from (90b)

$$\begin{aligned} R &= \int_{-\infty}^{\infty} (1 - e^{-y/50})g(y)dy = \int_{-\infty}^{\infty} (1 - e^{-y/50}) \frac{e^{-\frac{1}{2}\left(\frac{y-100}{10}\right)^2}}{10\sqrt{2\pi}} dy \\ &= 1 - \int_{-\infty}^{\infty} \frac{e^{-\frac{4y}{200} - \frac{y^2 - 200y + 10^4}{200}}}{10\sqrt{2\pi}} dy = 1 - \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2 - 196y + 10^4}{200}}}{10\sqrt{2\pi}} dy \\ &= 1 - \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2 - 196y + 10^4}{200}}}{10\sqrt{2\pi}} dy = 1 - \frac{1}{10\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2 - 196y + 10^4}{200}} dy \\ &= 1 - \frac{1}{10\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{[(y-98)^2 + 396]}{200}} dy = 1 - \frac{e^{-1.98}}{10\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{y-98}{10}\right)^2} dy = 1 - e^{-1.98} = 0.861931. \end{aligned}$$

The central factor of safety is given by $n_c = \mu_y/\mu_x = E(Y)/E(X) = 2$, while the mean factor of safety $\bar{n} = E(Y/X) = E(\eta)$ cannot be directly computed; however it can be shown that $\bar{n} \cong n_c(1 + CV_x^2)$, where $n_c = \mu_y / \mu_x$ is called the central factor of safety and CV_x is the coefficient of variation of stress. For our example, $CV_x = \sigma_x / \mu_x = 1$ (or 100%) so that the expected factor of safety is approximately $\bar{n} \cong 2(1 + 1^2) = 4$.

Suppose now in the above problem the strength Y was deterministic at a constant value of 100 MPa but stress was still exponential. Then the component RE reduces to $R = P(X < 100) = F_x(100) = 1 - e^{-100/50} = 0.864665$. Conversely, if stress were

deterministic at the constant value of $x = 50$ MPa, then $R = P(Y > 50) = P(Z > (50 - 100) / 10) = P(Z > -5) = 0.99999971334843$.

Normal Stress and Strength

Suppose the strength of a device $Y \sim N(\mu_y, \sigma_y^2)$ and is subject to the stress X which is also $N(\mu_x, \sigma_x^2)$. Then from Eq. (89), we deduce that $R = P(Y > X) = P(Y - X > 0) = P(W > 0)$, where $W = Y - X$ is a LC of normally and independently distributed rvs and hence itself is Gaussian with $\mu = E(W) = \mu_y - \mu_x$ and variance $V(W) = V(Y) + V(X) = \sigma_y^2 + \sigma_x^2$,

depicted in Figure 18. Figure 18 shows that $R = P(W > 0) = P(Y - X > 0) = P(Z > \frac{0 - \mu}{\sigma_w})$

$$= P(Z > \frac{-\mu}{\sqrt{\sigma_y^2 + \sigma_x^2}}) = P(Z > \frac{-(\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}}) = P(Z > -Z_0) = \Phi(Z_0), \text{ where } Z_0 =$$

$$(\mu_y - \mu_x) / \sqrt{\sigma_y^2 + \sigma_x^2}.$$

Example14. The strength of a component is $N(\mu_y, 400 \text{ MPa}^2)$ and the stress acting on it is also Gaussian $N(\mu_x, 625 \text{ MPa}^2)$. Determine the value of central factor of safety, $n_c = \mu_y / \mu_x$, such that the component RE is at least 0.999 if $CV(Y) = 0.10$.

Solution: $CV(Y) = 0.10 \rightarrow 0.10 = \sigma_y / \mu_y = 20 / \mu_y \rightarrow \mu_y = 200 \text{ MPa}$. $\sigma_w = \sqrt{\sigma_y^2 + \sigma_x^2} =$

$$\sqrt{1025} = 32.10562; \Phi(Z_0) = 0.999 \rightarrow Z_{0.001} = \frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}} \rightarrow 3.090232 = \frac{200 - \mu_x}{32.10562} \rightarrow \mu_x$$

$$= 101.0643 \rightarrow n_c = 200 / 101.0643 = 1.9789383 \text{ and } CV_x = 25 / 101.0643 = 0.2473673.$$

Thus, on the average the mean strength has to be nearly at least twice the mean stress to attain a RE of at least 0.999. In the normal case, the device RE can also be expressed in terms of CV_y and CV_x as follows:

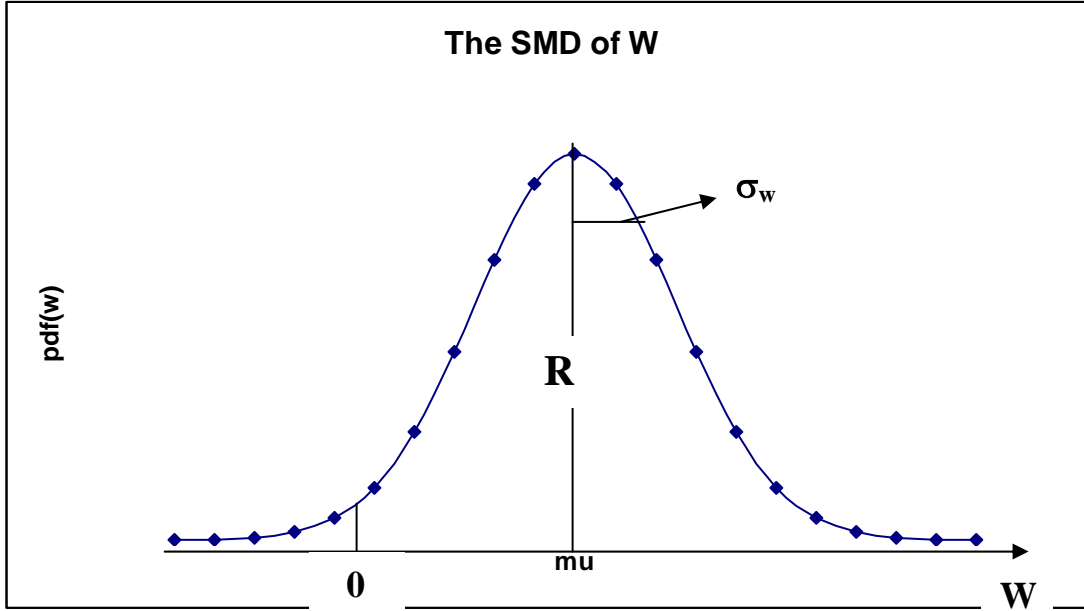


Figure 18. The SMD of $W = Y - X$

$$\begin{aligned}
 Z_0 &= \frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}} = \frac{\mu_y / \mu_x - 1}{\sqrt{\sigma_y^2 / \mu_x^2 + \sigma_x^2 / \mu_x^2}} = \frac{n_c - 1}{\sqrt{(\sigma_y^2 / \mu_y^2)(\mu_y^2 / \mu_x^2) + \sigma_x^2 / \mu_x^2}} = \\
 &= \frac{n_c - 1}{\sqrt{CV_y^2(n_c^2) + CV_x^2}} \rightarrow R = P(Z > -Z_0) = \Phi(Z_0).
 \end{aligned}$$

Further, it can be shown that given the values of $\bar{n} = E(Y/X) = E(\eta)$ and σ_η , the maximum UNRE value, regardless of the underlying distributions, is given by

$$Q = \bar{R} \leq \frac{(\bar{n})^2 CV_\eta^2}{(\bar{n})^2 CV_\eta^2 + (\bar{n} - 1)^2}$$

where $CV_\eta = \sigma_\eta / \bar{n}$. Conversely, if the desired value of R and the CV_η are given, then the minimum required mean factor of safety, \bar{n} , is given by

$$\bar{n} \geq \frac{1}{1 - CV_\eta \sqrt{R/Q}}$$

where $Q = \bar{R}$ and $CV_{\eta} \cong \frac{\sqrt{CV_y^2 + CV_x^2}}{1 + CV_x^2}$. For example, if $r = 0.999$ is required and it is

known that $CV(\eta = Y/X) = 0.03$, then the minimum required factor of safety is given by

$$\bar{n} \geq \frac{1}{1 - 0.03\sqrt{0.999 / 0.001}} = 19.3081. \text{ Note that this required } \bar{n} \geq 19.3081 \text{ makes no}$$

assumptions about $f(x)$ and $g(y)$; for example, if both underlying distributions are

normal as in Example 14, then $\bar{n} \cong n_c(1 + CV_x^2) = 1.9789383 \times (1 + 0.2473673^2) = 2.100031$.

On the other hand, if no assumptions are made about $f(x)$ and $g(y)$, and the design value of R is given along with CV_y and CV_x , then the minimum required value of central safety factor is given by

$$n_c = \frac{\mu_y}{\mu_x} \geq \frac{1}{1 + CV_x^2 - \sqrt{R(CV_y^2 + CV_x^2) / \bar{R}}}$$

For example, suppose the design value of $R = 0.999$, and it is known that $CV_y = 2.4\%$ while $CV_x = 1.8\%$, then the minimum central safety factor is given by $n_c \geq$

$$\frac{1}{1 + (0.018)^2 - \sqrt{0.999(0.024^2 + 0.018^2) / 0.001}} = 19.1883.$$

The Exponential Strength and Uniform Stress

As an example suppose $Y \sim \text{Exp}(\mu_y = 1000 \text{ MPa})$ and stress acting on the device is $U(200, 500 \text{ MPa})$. Our objective is to compute device (or system) RE.

$$\begin{aligned} R &= \int_{-\infty}^{\infty} g(y)dy \times F_x(y) = \int_{-\infty}^{\infty} 0.001e^{-0.001y} \frac{y-200}{300} dy = \\ &= \int_{-\infty}^0 0 \times 0 dy + \int_0^{200} 0.001e^{-0.001y} \times 0 dy + \int_{200}^{500} 0.001e^{-0.001y} \frac{y-200}{300} dy + \\ &\qquad\qquad\qquad \int_{500}^{\infty} 0.001e^{-0.001y} (1) dy \end{aligned}$$

$$= \frac{1}{300} \left[-(y - 200)e^{-0.001y} \right]_{200}^{500} + \int_{200}^{500} e^{-0.001y} dy + \left[-e^{-0.001y} \right]_{500}^{\infty}$$

$$= \frac{1}{300} \left[-300e^{-0.5} + (e^{-0.20} - e^{-0.50}) / 0.001 \right] + 0.60653066 = 0.100803 + 0.60653066 =$$

0.707333645. We could also arrive at the same RE as follows:

$$R = \int_{-\infty}^{\infty} G_y(x)f(x)dx = \int_{-\infty}^{\infty} e^{-0.001x} dx / 300 = \int_{200}^{500} e^{-0.001x} dx / 300 = \frac{-1}{0.30} e^{0.001x} \Big|_{200}^{500} =$$

$$\frac{1}{0.30} (e^{-0.20} - e^{-0.50}) = 0.707333645, \text{ as before.}$$

Exercise 25. Suppose stress acting on a component has an exponential distribution with mean $\mu_x = 100$ MPa and its strength is Weibull with minimum strength $\delta = 600$ MPa, characteristic strength $\theta = 900$ MPa and shape $\beta = 2.00$. Compute the device RE. ANS: $R = R_1 + R_2 = P(X < 600) \times P(Y \geq 600) + P(X \geq 600) \times P(Y \geq x) = 0.9996405635$. Hint: Eventually a change-of-variable like $(x-150)/300 = Z/\sqrt{2}$ has to be made.

The Lognormal Stress and Strength

Suppose X and Y are independent and lognormally distributed with parameters $x_{0.50} = e^{\mu_x}$, σ_x , $y_{0.50} = e^{\mu_y}$ and σ_y . Then, $R = P(Y/X > 1) = P(\ln Y - \ln X > 0) = P(W > 0)$, where $W = \ln Y - \ln X$ is normally distributed with mean $E(W) = E(\ln Y) - E(\ln X) = \mu_y - \mu_x = \ln(y_{0.50}) - \ln(x_{0.50}) = \ln(y_{0.50}/x_{0.50})$ and $V(W) = V(\ln Y - \ln X) = \sigma_y^2 + \sigma_x^2$. Hence, $R = P(W > 0) = P\left[Z > \frac{0 - \ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right] = 1 - \Phi\left[\frac{-\ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right] = \Phi\left[\frac{\ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right]$

As an example consider the Example 7.11 on page 154 of Ebeling where $y_{0.50} = 8.1$, σ_y

= 0.07, $x_{0.50} = 5.5$, and $\sigma_x = 0.15$. Then, $R = \Phi[\ln(8.1/5.5)/\sqrt{0.0049 + 0.0225}] = 0.990323303$, which match's that of Ebeling's to 2 decimal accuracy.

As Ebeling notes atop his p. 156, there are several other underlying stress/strength distributions for which analytical solution for R_{Sys} may be obtained. If analytical solution is impossible, then simulation must be used to generate random data from the two underlying distributions and the following approximation can be applied to approximate the system reliability.

RE Estimation from Strength/Stress Realized (or simulated) Data

Recall from Eq. (90a) that

$$R = \int_{-\infty}^{\infty} f(x)dxG_y(x); \text{ making the transformation } f(x) = dF_x(x)/dx \text{ in this integral results}$$

$$\text{in } f(x)dx = dF_x(x) \text{ and } R = \int_0^1 G_y(x)dF_x; \text{ thus, if we graph } G_y(x) \text{ versus } F_x(x), \text{ the area}$$

under $G_y(x)$ and the abscissa $F_x(x)$ -axis from zero to 1 will give the approximate reliability.

As an example suppose that stress and strength analysis performed on a randomly selected component resulted in the following simulated data.

Stress (MPa): 8 15 12 13 14 17 15

Strength (MPa): 14 10 17 18 20 19 23 22 25 19

We first tabulate the estimates of $F_x(x)$ and $G_y(x)$ as shown below.

KPa	0	8	10	12	13	14	15
$F_x(x)$	0	1/7	1/7	2/7	3/7	4/7	6/7
$G_y(x)$	1	1	9/10	9/10	9/10	8/10	8/10
KPa	17	18	19	20	22	23	25
$F_x(x)$	7/7	1	1	1	1	1	1
$G_y(x)$	7/10	6/10	4/10	3/10	2/10	1/10	0

The Figure atop the next page shows that the approximate value of RE is given by

$$\hat{R} = \frac{1}{7} \left[\frac{1+0.95}{2} + \frac{0.95+0.90}{2} + 0.90 + 3(0.8) + 0.8/2 \right] = 0.80.$$

