Availability Reference : Chapters 11 of Ebeling(2^{nd}) Maghsoodloo

Instantaneous Availability

By instantaneous (or point) availability at time t, A(t), we mean the Pr that a repairable unit is functioning reliably at time t. Thus, if there is no repair, the availability function is simply A(t) = R(t). However, if the component is repairable, then there are two mutually exclusive possibilities: (1) The component is reliable at t, in which case $A_1(t) = R(t)$, (2) the component fails at time x, $0 < x < t$, gets renewed (or repaired) in the interval $(x, x+\Delta x)$ with Pr element $\rho(x)dx$, and then is reliable from time x to $t - x$ (Trivedi, 1982). This second Pr is given by $A_2(t) = \int_0^t \rho(x)dxR(t-x)$. Because the above two cases are mutually exclusive, then

$$A(t) = A_1(t) + A_2(t) = R(t) + \int_0^t R(t-x)\rho(x)dx \quad (95a)$$

Taking Laplace transform of the above Eq. (95a) yields

$$A^*(s) = R^*(s) + R^*(s)\,\rho^*(s) = R^*(s)\,[1+\rho^*(s)]$$

$$= R^*(s)\,[1 + \frac{f^*(s)r^*(s)}{1-f^*(s)r^*(s)}] = \frac{R^*(s)}{1-f^*(s)r^*(s)}. \quad (95b)$$

For the case when the TTF has a constant failure rate λ and time to repair is also exponential at the rate λ_r, $R^*(s) = \int_0^\infty e^{-\lambda t}e^{-st}dt = 1/(\lambda+s)$; $\bar{f}(s) = f^*(s) = \int_0^\infty \lambda e^{-\lambda t}e^{-st}dt = \lambda/(\lambda+s)$; $r^*(s) = \int_0^\infty \lambda_r e^{-\lambda_r t}e^{-st}dt = \lambda_r/(\lambda_r+s)$. Hence, the Laplace-transform of availability from 95(b) is given by $A^*(s) =$

$$\frac{1/(\lambda+s)}{1-[\lambda/(\lambda+s)][\lambda_r/(\lambda_r+s)]} = \frac{\lambda_r+s}{(\lambda+s)(\lambda_r+s)-\lambda\lambda_r} = \frac{\lambda_r+s}{s^2+(\lambda+\lambda_r)s} = \frac{\lambda_r+s}{s[s+(\lambda+\lambda_r)]} = \frac{\lambda_r}{s(\lambda+\lambda_r)} +$$

$$\frac{\lambda/(\lambda+\lambda_r)}{s+\lambda+\lambda_r} \rightarrow A(t) = L^{-1}\{A^*(s)\} = L^{-1}\{\frac{\lambda_r}{s(\lambda+\lambda_r)} + \frac{\lambda/(\lambda+\lambda_r)}{s+\lambda+\lambda_r}\} = \frac{\lambda_r}{\lambda+\lambda_r} + \frac{\lambda}{\lambda+\lambda_r}e^{-(\lambda+\lambda_r)t}.$$

For example, given that $\lambda = 0.0005$ and $\lambda_r = 0.05$ per hour, then the Pr that the unit is available (i.e., not under repair) at t = 1000 hours is given by $A(1000) = \frac{0.05}{0.0505} + \frac{0.0005}{0.0505}e^{-0.0505(1000)} =$ 0.990099009901, while R(1000 hours W/O Repair) = $e^{-0.5}$ = 0.60653066 < A(1000) = 0.9901.

More Availability Measures

(1) Steady-State (Inherent or Intrinsic) Availability

By intrinsic availability (A_I) we mean the limiting value of A(t) as $t \to \infty$, i.e., availability over the long haul. Thus $A_I = A(\infty) = \lim_{t \to \infty} A(t)$. For the case of constant failure and repair rates, $A_I = A(\infty)$

$$= \lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left[\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}\right] = \frac{\lambda_r}{\lambda + \lambda_r} = \frac{1/\mu_r}{1/\mu + 1/\mu_r}$$

, where μ_r represents mean repair time , or MTTR, and μ = MTBF. Thus, $A_I = \frac{1/\mu_r}{1/\mu + 1/\mu_r} = \frac{\mu}{\mu_r + \mu} =$

$$\frac{\text{MTBF}}{\text{MTTR} + \text{MTBF}}$$

, where MTTR includes only active repair time (no administrative or other logistic times)

(2) By interval availability (IA) we mean the proportion of the time within an interval(t_1, t_2) the component (or system) is expected (on the average) to be in the available mode. Thus,

$$\text{IA} = A(t_1, t_2) = \frac{\int_{t_1}^{t_2} A(t)dt}{t_2 - t_1}$$

For the exponential failures and repairs the interval availability $A(t_1, t_2)$=

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}\right]dt$$

$$= \frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda/(t_2 - t_1)}{(\lambda + \lambda_r)^2}\left[e^{-(\lambda+\lambda_r)t_1} - e^{-(\lambda+\lambda_r)t_2}\right], \qquad (96)$$

which is the same as Eq. (11.14) on p. 288 of Ebeling. For the interval (100, 200 hours), availability is given by A(100, 200) = $\frac{1}{100}\int_{100}^{200}\left[\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r} e^{-(\lambda+\lambda_r)t}\right]dt = \frac{\lambda_r}{\lambda + \lambda_r}$

$+ \frac{\lambda}{100(\lambda + \lambda_r)^2} \times \left[e^{-(\lambda+\lambda_r)100} - e^{-(\lambda+\lambda_r)200}\right]$; when λ = 0.0005 and λ_r = 0.05, then this IA becomes A(100, 200) = 0.99011149545. A special case of IA(t) is the average availability over the interval

[0, T] given by $\bar{A}(T) = A(0, T) = \int_0^T A(t)dt / T = \frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{T(\lambda + \lambda_r)^2}[1 - e^{-(\lambda+\lambda_r)T}]$. For the example with λ = 0.0005 and λ_r = 0.05 per hour, the average availability over the interval [0, 1000 hours] is given by $\bar{A}(1000) = \frac{1}{1000}\int_0^{1000}[\frac{\lambda_r}{\lambda + \lambda_r} + \frac{\lambda}{\lambda + \lambda_r}e^{-(\lambda+\lambda_r)t}]dt$ = 0.990295069111, which is not the same as the point availability A(1000) =0.990099009901 as expected.

(3) By long-term availability, $\bar{A}$, we mean the proportion of the times a system is available and considers only MTBF and MDT (mean down time) but excludes mean idle time. Thus $\bar{A}$ = $\frac{MTBF}{MTBF + MDT} \le A_I$ because MDT is generally larger than MTTR (mean active repair time). Downtime involves both administrative (or logistic) time and active repair time.

Achieved Availability

We mean the proportion of the times a system is available and considers only MTBM and MMDT (mean maintenance down time) but excludes mean idle time. The time horizon is depicted below in Figure 20. The achieved availability is defined in Eq. (11.6) on page 284 of Ebeling as

$$\bar{A}_a = \frac{MTBM}{MTBM + MMDT}, \quad (11.6Ep284)$$

where MMDT = $\overline{M}$ stands for mean maintenance downtime that includes both corrective and preventive maintenance downtimes. For example, suppose a certain maintenance on a car occurs every 5000 miles and the value of MMDT is 8 hours. Then, assuming that the car averages

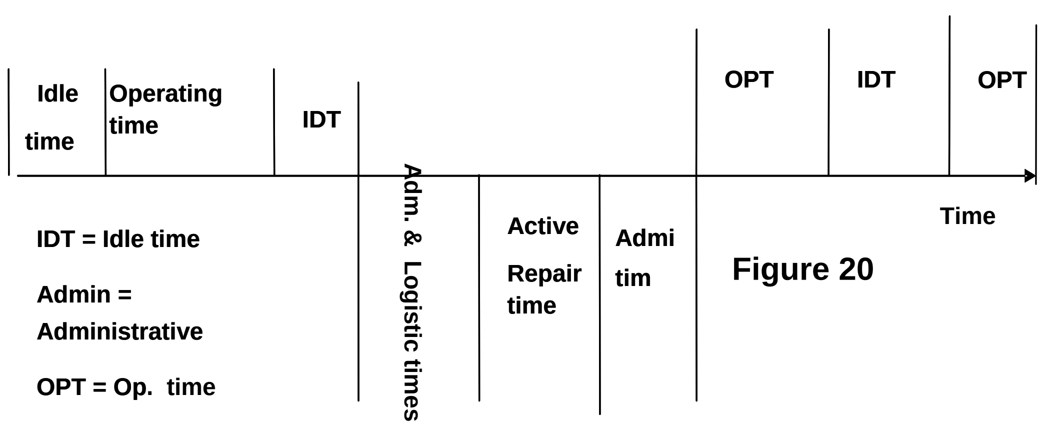


Figure 20

12000 miles/year, then the MTBM = $\frac{\textbf{5000 miles}}{\textbf{12000 / year}}$ = 0.4167 year = 0.4167year×(365×24 hours/year) = 3650 hours → $\bar{A}_a$ = 3650/(3650 +8) = 0.9978130126.

Example 19. Suppose a system of 5 components fails a total of 6 times and goes under repair a total of 6 times in 300 hours of operations. The first failure occurs at TTF_1 = 55.3 days and repair on it starts immediately with TTR_1 = 1.3 days; the 2^{nd} failure occurs at TTF_2 = 85 days (measured from zero) with TTR_2 = 0.90 days; the 3^{rd} failure occurs at TTF_3 = 165 days with TTR_3 = 1.4 days; the 4^{th} failure occurs at 205 days with TTR_4 = 1.5 days; the 5^{th} failure occurs at 220 days with TTR_5 = 1.8, and the 6^{th} failure occurs at 260 days (from zero) with TTR_6 = 1.6 days. Note that Our objective is to estimate the MTBF and MTTR.

$$\widehat{MTBF} = [55.3+(85-55.3-1.3)+(165-85-0.9)+(205-165-1.4)+(220-205-1.5)+(260-220-1.8)]/6 = 42.1833 \text{ days}$$

and $\widehat{MTTR}$ = (1.3+0.9+1.4+1.5+1.8+1.6)/6 = 1.4167 days. Assuming roughly constant failure and repair rates, then $\hat{\lambda}_f = 1/(42.1833*24)$ = 0.000987752/hour and $\hat{r} = \hat{\lambda}_r = 1/(1.4167*24)$ = 0.029411765/hour, and $A_I = \frac{\lambda_r}{\lambda+\lambda_r} \approx 0.96751$.

To further illustrate the exact difference between R(t) and A(t), consider a system of N = 12 new components that are placed in service at time 0. Table 8 shows their TTFs and TTRs and the corresponding estimates of R(t) and A(t). Note that for a repairable system A(t) ≥ R(t) for all t. Further, in Table 8 $N_s(t)$ stands for number of components surviving at t and $N_A(t)$ stands for number of components available for service at time t, and the last two columns give $\hat{R}(t)$ and $\hat{A}(t)$.

System Availability (Section 11.3 of Ebeling)

Because A(t) can stand for availability of a system, then the laws of Pr will prevail. For example, for a serial system consisting of n units each with availability of $A_i(t)$ and its own server, the system availability is given by

$$A_{Sys}(t) = \prod_{i=1}^{n} A_i(t) \le \text{Minimum}[A_i(t)] \qquad \text{(11.16E p. 289)}$$

Table 8. [t = time; $\hat{R}(t) = N_s(t)/N$; $\hat{A}(t) = N_A(t)/N$]

t in days	N	TTF_i	TTR_i	$N_s(t)$	$N_A(t)$	$\hat{R}(t)$	$\hat{A}(t)$
0	12			12	12	1	1
200	12	250 days	3 days	12	12	1	1
400	12	598	5	11	12	11/12	12/12
600	11	795	8	10	11	10/12	11/12
800	11	980	4	9	11	9/12	11/12
1000	12	1130	6	8	12	8/12	12/12
1200	12			7	12	7/12	12/12

Similarly, for a pure redundant parallel system, the system will be unavailable iff all n units are under repair and hence $U_{Sys}(t) = \prod_{i=1}^{n} U_i(t) = \prod_{i=1}^{n}[1 - A_i(t)]$. Therefore,

$$A_{Sys}(t) = 1 - U_{Sys}(t) = 1 - \prod_{i=1}^{n}[1 - A_i(t)] \ge \text{Maximum}[A_i(t)] \qquad \text{(11.17E, p. 289)}$$

where the above Eq. (11.17E) of Ebeling is valid only if there are n servers. If there is a single repair-crew, then $A_{Sys}(t) < 1 - \prod_{i=1}^{n}[1 - A_i(t)]$, as will be shown later (See Example 11.4 on page 289 of Ebeling).

Note that we can easily use Markov analysis to obtain the long-term inherent availability of both a series and parallel system. For the example 11.4 on p. 289 of Ebeling, the TRD for the 2-identiacl-unit series system is given in Figure 22. Note that because there are 2 crews repairing when the system is in state "2", still the transition from 2 to 0 is not assessable during Δt, because in Markov analysis only one event is allowed during Δt. Figure 22 clearly shows that $2\lambda\,\pi_0 = \lambda_r\,\pi_1 \rightarrow \pi_1 = (2\lambda/\lambda_r)\pi_0$; $\lambda\pi_1 = 2\lambda_r\,\pi_2$, $\rightarrow \pi_2 = \dfrac{\lambda\,\pi_1}{2\lambda_r} \rightarrow \pi_2 = (\lambda/\lambda_r)^2\pi_0 \rightarrow \pi_0 + (2\lambda/\lambda_r)\pi_0 +$

$(\lambda/\lambda_r)^2\pi_0 = 1 \rightarrow \pi_0 = \dfrac{1}{1+(2\lambda/\lambda_r)+(\lambda/\lambda_r)^2} = \dfrac{\lambda_r^2}{\lambda_r^2+2\lambda\lambda_r+\lambda^2} = \dfrac{\lambda_r^2}{(\lambda+\lambda_r)^2} = [\lambda_r/(\lambda_r+\lambda)]^2$.

Once π_0 is computed, the other two steady-state Prs can easily be obtained. For the example 11.4 of Ebeling, λ = 0.10 and λ_r = 0.20 per hour, then $\pi_0 = (0.2/0.3)^2 = 0.444\overline{4}$, which agrees with

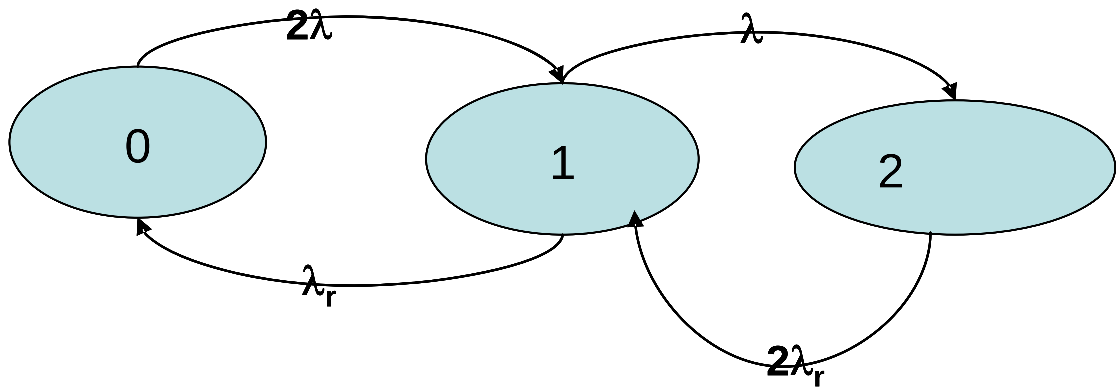


Figure 22. The TRD for a 2-unit Serial system with 2 servers

Ebeling's answer to 3 decimals. Then for a series system the intrinsic system availability is $A_I = \pi_0$ = $0.444\overline{4}$, and U = $0.555\overline{5}$. However, if the system is a pure parallel one, then $A_I = \pi_0 + \pi_1$ = $0.444\overline{4}$ + $(0.2/0.2)\pi_0 = 0.888\overline{8}$ and U_I = $0.111\overline{1}$, which again aggress with Ebeling's answer on page 289 to 3 decimals.

The transient solution for availability of a 2-identical-unit serial system (with 2 crews) can be obtained also through solving the following system of differential equations.

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t) \\ dP_1(t)/dt = 2\lambda P_0(t) - (r+\lambda)P_1(t) + 2rP_2(t) \end{cases}$$

Because $P_2(t) = 1 - P_0(t) - P_1(t)$, the above system reduces to

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t) \\ dP_1(t)/dt = (2\lambda - 2r)P_0(t) - (3r+\lambda)P_1(t) + 2r \end{cases}$$

. Taking the Laplace transform of this system results in

$$\begin{cases} sP_0{}^*(s) - 1 = -2\lambda P_0{}^*(s) + rP_1{}^*(s) \\ sP_1{}^*(s) - 0 = (2\lambda - 2r)P_0{}^*(s) - (3r+\lambda)P_1{}^*(s) + 2r/s \end{cases} \rightarrow$$

$$\begin{cases}(s+2\lambda)P_0^*(s) - \qquad r\,P_1^*(s) = 1 \\ (2r-2\lambda)P_0^*(s) + (s+3r+\lambda)P_1^*(s) = 2r/s\end{cases} \rightarrow P_0^*(s) = \frac{\begin{vmatrix}1 & -r \\ 2r/s & (s+3r+\lambda)\end{vmatrix}}{\begin{vmatrix}(s+2\lambda) & -r \\ 2r-2\lambda & (s+3r+\lambda)\end{vmatrix}}$$

$$P_0^*(s) = \frac{(s+3r+\lambda)+2r^2/s}{(s+2\lambda)(s+3r+\lambda)+r(2r-2\lambda)}$$

$$= \frac{s(s+3r+\lambda)+2r^2}{s[s^2+(3r+3\lambda)s+(2\lambda^2+2r^2+4\lambda r)]} = \frac{s(s+3r+\lambda)+2r^2}{s[(s-u_1)(s-u_2)]}$$

where $u_i = \dfrac{-(3\lambda+3r)\pm\sqrt{(3\lambda+3r)^2-4(2\lambda^2+2r^2+4\lambda r)}}{2} = \dfrac{-(3\lambda+3r)\pm\sqrt{\lambda^2+r^2+2\lambda r}}{2}$

$= \dfrac{-(3\lambda+3r)\pm(\lambda+r)}{2}$ $\rightarrow$ $u_1 = -2(\lambda+r)$ and $u_2 = -(\lambda+r)$ $\rightarrow$

$$P_0^*(s) = \frac{s+3r+\lambda}{(s+\lambda+r)(s+2\lambda+2r)} + \frac{2r^2}{s(s+\lambda+r)(s+2\lambda+2r)}$$

$$P_0^*(s) = \frac{s}{(s+\lambda+r)(s+2\lambda+2r)} + \frac{3r+\lambda}{(s+\lambda+r)(s+2\lambda+2r)} + \frac{2r^2}{s(s+\lambda+r)(s+2\lambda+2r)} \rightarrow$$

$$P_0(t) = L^{-1}\{P_0^*(s)\} = \frac{1}{\lambda+r}[2(\lambda+r)e^{-2(\lambda+r)t}-(\lambda+r)e^{-(\lambda+r)t}] + \frac{3r+\lambda}{\lambda+r}[e^{-(\lambda+r)t}-e^{-2(\lambda+r)t}] +$$

$$\frac{2r^2}{2(\lambda+r)^2}\left\{1+\frac{1}{\lambda+r}[(\lambda+r)e^{-2(\lambda+r)t}-2(\lambda+r)e^{-(\lambda+r)t}]\right\} \rightarrow$$

$$A_{Sys}(t) = P_0(t) = \frac{r^2}{(\lambda+r)^2} + \frac{2\lambda r}{(\lambda+r)^2}e^{-(\lambda+r)t} + \frac{\lambda^2}{(\lambda+r)^2}e^{-2(\lambda+r)t} \qquad (97)$$

Eq. (97) gives A_{Sys}(10 hours) = 0.467674, which is in agreement with that of Ebeling's on his page 289 for $A_{Sys}(10) = (0.684)^2 = 0.468$ to 3 decimals. Further, Eq. (97) shows that the availability of a serial system is simply the product of the individual availabilities, i.e., $A_{Sys}(t) = \prod_{i=1}^{n} A_i(t)$ (see Eqs. (11.13) and (11.15) on p. 289 of Ebeling).

System Availability For a 2-identical-unit Parallel System with a Single Server and Identical repair rates r for both states "1-failed" and "2-failed"

It can easily be verified that the system of differential equations is given by

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t) \\ dP_1(t)/dt = 2\lambda P_0(t) - (r+\lambda)P_1(t) + rP_2(t) \end{cases}$$

Because $P_2(t) = 1 - P_0(t) - P_1(t)$, the above system reduces to

$$\begin{cases} dP_0(t)/dt = -2\lambda P_0(t) + rP_1(t) \\ dP_1(t)/dt = (2\lambda - r)P_0(t) - (2r+\lambda)P_1(t) + r \end{cases}$$

. Taking the Laplace transform of this system results in

$$\begin{cases} sP_0^*(s) - 1 = -2\lambda P_0^*(s) + rP_1^*(s) \\ sP_1^*(s) - 0 = (2\lambda - r)P_0^*(s) - (2r+\lambda)P_1^*(s) + r/s \end{cases} \rightarrow$$

$$\begin{cases} (s+2\lambda)P_0^*(s) - \quad rP_1^*(s) = 1 \\ (r-2\lambda)P_0^*(s) + (s+2r+\lambda)P_1^*(s) = r/s \end{cases} \rightarrow$$

$$P_0^*(s) = \frac{\begin{vmatrix} 1 & -r \\ r/s & (s+2r+\lambda) \end{vmatrix}}{\begin{vmatrix} (s+2\lambda) & -r \\ r-2\lambda & (s+2r+\lambda) \end{vmatrix}} = \frac{(s+2r+\lambda) + r^2/s}{(s+2\lambda)(s+2r+\lambda) + r(r-2\lambda)}$$

$$= \frac{s(s+2r+\lambda) + r^2}{s[s^2 + (3\lambda + 2r)s + (2\lambda^2 + r^2 + 2\lambda r)]}$$

$$\rightarrow \quad P_0^*(s) = \frac{s(s+2r+\lambda) + r^2}{s[(s-u_1)(s-u_2)]}, \quad \text{where}$$

$$u_i = \frac{-(3\lambda+2r) \pm \sqrt{(3\lambda+2r)^2 - 4(2\lambda^2 + r^2 + 2\lambda r)}}{2} = \frac{-(3\lambda+2r) \pm \sqrt{\lambda^2 + 4\lambda r}}{2}, \quad u_1 - u_2 = -\sqrt{\lambda^2 + 4\lambda r},$$

and $u_1 u_2 = 2\lambda^2 + 2\lambda r + r^2 \rightarrow P_0^*(s) = \dfrac{s+2r+\lambda}{(s-u_1)(s-u_2)} + \dfrac{r^2}{s(s-u_1)(s-u_2)}$

$$P_0^*(s) = \frac{s}{(s-u_1)(s-u_2)} + \frac{2r+\lambda}{(s-u_1)(s-u_2)} + \frac{r^2}{s(s-u_1)(s-u_2)} \quad \rightarrow$$

$$P_0(t) = \mathcal{L}^{-1}\{P_0^*(s)\} = \frac{1}{u_2-u_1}\left(u_2e^{u_2t} - u_1e^{u_1t}\right) + \frac{\lambda+2r}{u_2-u_1}\left(e^{u_2t}-e^{u_1t}\right) +$$

$$\frac{r^2}{u_1u_2}\left[1+\frac{1}{u_2-u_1}\left(u_1e^{u_2t}-u_2e^{u_1t}\right)\right]$$

$$P_0(t) = \frac{r^2}{r^2+2\lambda^2+2\lambda r} + \frac{u_1^2+u_1(\lambda+2r)+r^2}{u_1(u_1-u_2)}e^{u_1t} + \frac{u_2^2+u_2(\lambda+2r)+r^2}{u_2(u_2-u_1)}e^{u_2t} \rightarrow$$

$$P_0(t) = \frac{r^2}{r^2+2\lambda^2+2\lambda r} + \frac{u_1^2+u_1(\lambda+2r)+r^2}{u_1(u_1-u_2)}e^{u_1t} + \frac{u_2^2+u_2(\lambda+2r)+r^2}{u_2(u_2-u_1)}e^{u_2t} \qquad \text{(98a)}$$

Similarly, $P_1^*(s) = \frac{2r\lambda}{s(s-u_1)(s-u_2)} + \frac{2\lambda}{(s-u_1)(s-u_2)}$ **, and**

$$P_1(t) = \frac{2\lambda r}{r^2+2\lambda^2+2\lambda r} + \frac{2\lambda u_1+2\lambda r}{u_1(u_1-u_2)}e^{u_1t} + \frac{2\lambda r+2\lambda u_2}{u_2(u_2-u_1)}e^{u_2t} \qquad \text{(98b)}$$

Combining Eqs. (98 a &b) yields

$$A(t) = P_0(t) + P_1(t) = \frac{r^2+2\lambda r}{r^2+2\lambda^2+2\lambda r} + \frac{u_1^2+u_1(3\lambda+2r)+(r^2+2\lambda r)}{u_1(u_1-u_2)}e^{u_1t} +$$

$$\frac{u_2^2+u_2(3\lambda+2r)+(r^2+2\lambda r)}{u_2(u_2-u_1)}e^{u_2t}, \quad u_2 < u_1 < 0. \rightarrow$$

$$A(t) = \frac{r^2+2\lambda r}{r^2+2\lambda^2+2\lambda r} + \frac{2\lambda^2}{u_1(u_2-u_1)}e^{u_1t} + \frac{2\lambda^2}{u_2(u_1-u_2)}e^{u_2t}$$

$$= \frac{r^2+2\lambda r}{r^2+2\lambda^2+2\lambda r} + \frac{4\lambda^2e^{u_1t}}{(3\lambda+2r)\sqrt{\lambda^2+4\lambda r}-(\lambda^2+4\lambda r)} - \frac{4\lambda^2e^{u_2t}}{\lambda^2+4\lambda r+(3\lambda+2r)\sqrt{\lambda^2+4\lambda r}} \qquad \text{(99)}$$

At $\lambda = \lambda_f = 0.10$ and $\lambda_r = r = 0.20$, the value of A(10 hours) from Eq. (99) is A(10) = 0.844213368. This availability is less than $A_s(10) = 0.900$, as expected, given by Ebeling on his page 289 for the case of 2-identical parallel units with 2 servers because the system availability given in Eq. (11.16) on page 289 of Ebeling is valid only for the case of n servers, while availability function of Eq. (99) developed above is valid for a 2-uint parallel system with only one server.

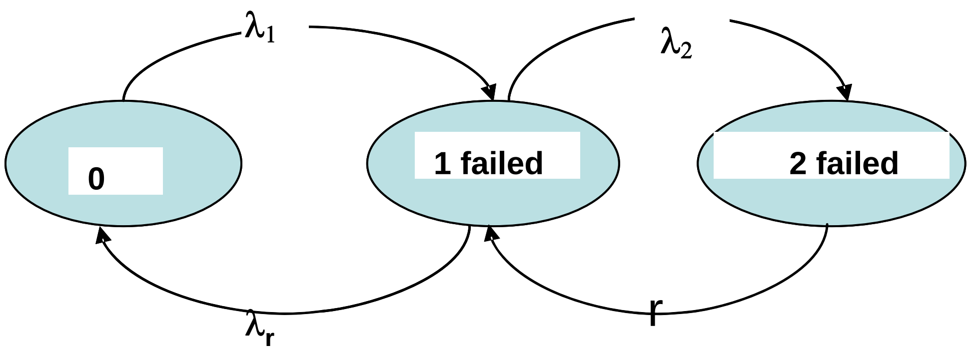
System Availability For a 2-unit Standby System

Figure 11.4 on page 290 of Ebeling describes one cycle of a 2-component repairable

Standby system where the quiescent failure rate of the standby unit λ_f^- is negligible relative to its active failure rate λ_2. For convenience I have modified his Figure 11.4 as follows, where λ_r = r represents repair rate, and the failed state 2 represents system failure with also repair rate r. From the modified Figure 11.4 (with one-repair at-a-time) we can deduce that

$$\begin{cases} dP_0(t)/dt = -\lambda_1 P_0(t) + rP_1(t) \\ dP_1(t)/dt = \lambda_1 P_0(t) - (r+\lambda_2)P_1(t) + rP_2(t) \end{cases}$$

Because $P_0(t) + P_1(t) + P_2(t) = 1$, then $P_2(t) = 1 - P_1(t) - P_0(t)$ $\rightarrow$ the above system of differential



A 2-unit standby system with on-Line Repair

equations reduces to

$$\begin{cases} dP_0(t)/dt = -\lambda_1 P_0(t) + rP_1(t) \\ dP_1(t)/dt = (\lambda_1 - r)P_0(t) - (2r+\lambda_2)P_1(t) + r \end{cases}$$

Taking Laplace transforms, we obtain

$$\begin{cases} sP_0^*(s) - P_0(0) = -\lambda_1 P_0^*(s) + rP_1^*(s) \\ sP_1^*(s) - P_1(0) = (\lambda_1 - r)P_0^*(s) - (2r+\lambda_2)P_1^*(s) + r/s \end{cases}$$

Applying the boundary conditions $P_0(0) = 1$ and $P_1(0) = 0$, we obtain

$$\begin{cases} sP_0^*(s) - 1 = -\lambda_1 P_0^*(s) + rP_1^*(s) \\ sP_1^*(s) = (\lambda_1 - r)P_0^*(s) - (2r+\lambda_2)P_1^*(s) + r/s \end{cases} \rightarrow$$

$$\begin{cases} (s+\lambda_1)P_0^*(s) - rP_1^*(s) = 1 \\ (r-\lambda_1)P_0^*(s) + (s+2r+\lambda_2)P_1^*(s) = r/s \end{cases} \rightarrow$$

$$P_0^*(s) = \frac{\begin{vmatrix} 1 & -r \\ r/s & (s+2r+\lambda_2) \end{vmatrix}}{\begin{vmatrix} (s+\lambda_1) & -r \\ r-\lambda_1 & (s+2r+\lambda_2) \end{vmatrix}} = \frac{(s+2r+\lambda_2)+r^2/s}{(s+2r+\lambda_2)(s+\lambda_1)+r(r-\lambda_1)} =$$

$$= \frac{s(s+2r+\lambda_2)+r^2}{s[(s+2r+\lambda_2)(s+\lambda_1)+r(r-\lambda_1)]} = \frac{s(s+2r+\lambda_2)+r^2}{s[s^2+(\lambda_1+2r+\lambda_2)s+(r^2+r\lambda_1+\lambda_1\lambda_2)]}$$

$= \dfrac{s(s+2r+\lambda_2)+r^2}{s(s-u_1)(s-u_2)}$, where u_1 and u_2 are the roots of the quadratic

$s^2+(\lambda_1+2r+\lambda_2)s+(r^2+r\lambda_1+\lambda_1\lambda_2)=0$, i.e., $u_i = \dfrac{-(\lambda_1+2r+\lambda_2)\pm\sqrt{(\lambda_2-\lambda_1)^2+4r\lambda_2}}{2}$,

and $u_1u_2 = \lambda_1\lambda_2 + r^2 + r\lambda_1$. Thus, $P_0^*(s) = \dfrac{s+2r+\lambda_2}{(s-u_1)(s-u_2)} + \dfrac{r^2}{s(s-u_1)(s-u_2)} =$

$\dfrac{s}{(s-u_1)(s-u_2)} + \dfrac{2r+\lambda_2}{(s-u_1)(s-u_2)} + \dfrac{r^2}{s(s-u_1)(s-u_2)}$; inverting this Laplace transform yields

$P_0(t) = = L^{-1}\{P_0^*(s)\} = \dfrac{1}{u_1-u_2}\left(u_1e^{u_1t}-u_2e^{u_2t}\right) + \dfrac{\lambda_2+2r}{u_1-u_2}\left(e^{u_1t}-e^{u_2t}\right) +$

$$\frac{r^2}{u_1u_2}\left[1+\frac{1}{u_1-u_2}\left(u_2e^{u_1t}-u_1e^{u_2t}\right)\right] = \frac{r^2}{r^2+\lambda_1\lambda_2+\lambda_1r} + \frac{u_1^2+u_1(\lambda_2+2r)+r^2}{u_1(u_1-u_2)}e^{u_1t} +$$

$$\frac{u_2^2+u_2(\lambda_2+2r)+r^2}{u_2(u_2-u_1)}e^{u_2t}. \qquad \textbf{(100)}$$

$$\text{Similarly, } P_1^*(s) = \frac{\begin{vmatrix} (s+\lambda_1) & 1 \\ (r-\lambda_1) & r/s \end{vmatrix}}{\begin{vmatrix} (s+\lambda_1) & -r \\ r-\lambda_1 & (s+2r+\lambda_2) \end{vmatrix}} = \frac{(s+\lambda_1)r/s - (r-\lambda_1)}{(s+2r+\lambda_2)(s+\lambda_1)+r(r-\lambda_1)}$$

$$= \frac{r(s+\lambda_1)-(r-\lambda_1)s}{s[(s+2r+\lambda_2)(s+\lambda_1)+r(r-\lambda_1)]} = \frac{r\lambda_1+s\lambda_1}{s[s^2+(\lambda_1+2r+\lambda_2)s+(r^2+r\lambda_1+\lambda_1\lambda_2)]}$$

$$= \frac{r\lambda_1+\lambda_1 s}{s(s-u_1)(s-u_2)} = \frac{r\lambda_1}{s(s-u_1)(s-u_2)} + \frac{\lambda_1}{(s-u_1)(s-u_2)} \rightarrow$$

$P_1(t) = L^{-1}\{P_1^*(s)\} = \dfrac{r\lambda_1}{u_1u_2}\left[1+\dfrac{1}{u_1-u_2}\left(u_2e^{u_1t}-u_1e^{u_2t}\right)\right] + \dfrac{\lambda_1}{u_1-u_2}\left(e^{u_1t}-e^{u_2t}\right) \rightarrow$

$$P_1(t) = \frac{r\lambda_1}{r^2+\lambda_1\lambda_2+\lambda_1 r} + \frac{r\lambda_1+\lambda_1 u_1}{u_1(u_1-u_2)}e^{u_1 t} + \frac{r\lambda_1+\lambda_1 u_2}{u_2(u_2-u_1)}e^{u_2 t} \qquad (101)$$

Combining Eqs. (100) and (101), the point (or instantaneous) availability for a 2-unit standby system is given by A(t) = $P_0(t)+P_1(t) = \frac{r^2+r\lambda_1}{r^2+\lambda_1\lambda_2+\lambda_1 r} + \frac{u_1^2+u_1(\lambda_2+2r+\lambda_1)+r^2+r\lambda_1}{u_1(u_1-u_2)}e^{u_1 t} +$

$$\frac{u_2^2+u_2(\lambda_2+2r+\lambda_1)+r^2+r\lambda_1}{u_2(u_2-u_1)}e^{u_2 t} \qquad (102a)$$

Substituting $u_i = \frac{-(\lambda_1+2r+\lambda_2)\pm\sqrt{(\lambda_2-\lambda_1)^2+4r\lambda_2}}{2}$ **into Eq. (102a) reduces the above availability function to**

$$A(t) = \frac{r^2+r\lambda_1}{r^2+\lambda_1\lambda_2+\lambda_1 r} + \frac{\lambda_2\lambda_1}{u_1(u_2-u_1)}e^{u_1 t} + \frac{\lambda_2\lambda_1}{u_2(u_1-u_2)}e^{u_2 t} \qquad (102b)$$

The intrinsic availability is given by $A_I = \lim_{t\to\infty}[P_0(t)+P_1(t)] = \pi_0+\pi_1 \rightarrow$

$$A_I = \frac{r(r+\lambda_1)}{r^2+r\lambda_1+\lambda_1\lambda_2} \qquad (103)$$

Eq. (103) clearly shows that for an irreparable 2-unit standby system, i.e., r = 0, the value of $A_I = 0$ **while for the same system with r = 0 the RE function from Eq. (102b) is given by**

$$R(t) = \frac{\lambda_2\lambda_1}{u_1(u_2-u_1)}e^{u_1 t} + \frac{\lambda_2\lambda_1}{u_2(u_1-u_2)}e^{u_2 t} \qquad (104a)$$

However, when r = 0, $u_1 = -\lambda_2$ **and** $u_2 = -\lambda_1$ **so that the Eq. (104a) reduces to**

$$R(t) = \frac{\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1-\lambda_2}e^{-\lambda_2 t} \qquad (104b)$$

Eq. (104b) is identical to the RE function for a 2-unit standby system given in Eq. (6.27) on page 131 of Ebeling where we are assuming that the quiescent failure rate of the standby unit $\lambda_2^- = 0$**.**

For the Example 11.5 on page 290 of Ebeling, $\lambda_1 = 0.002$, $\lambda_2 = 0.001$, **and r = 0.01 per hour. Substituting these into Eq. (103) results in** $A_I = 0.9836065574$**, which agrees with that of Ebeling's to 3 decimals near the bottom of his page 290. Ebeling's failure rates in this example are unrealistic because the primary unit in a 2-unit standby should hardly ever have a failure rate twice that of the**

standby unit. Therefore, the more realistic failure rates are λ_1 = 0.001, λ_2 = 0.002, and r = 0.01 per hour. Substituting these into Eq. (103) results in A_I = 0.98214286, which should be a bit smaller than the case of $\lambda_1 > \lambda_2$. The value of RE at t = 500, λ_1 = 0.001, λ_2 = 0.002, and r = 0.01 from Eq. (104b) is given by $R(500) = \frac{0.002}{0.002-0.001}\, e^{-0.5} + \frac{0.001}{0.001-0.002}\, e^{-1}$ **= 1.21306131942527 − 0.36787944117144 = 0.845181878254. While the value of the point availability A(t) at t = 500 hours from Eq. (102b) is A(500) = 0.9830968, where u_2 =– 0.007000 and u_1 = – 0.01600000. Note that u_2 always exceeds u_1.**

Chapter 11 Summary

1. Exponential Failures and Repairs of a Single unit (or Component)

(a) For the exponential failures and no repairs (r =0), the RE function is $R(t) = e^{-\lambda t}$.

The renewal function is $m(t) = E[N_f(t)] = \lambda t$ **and the intensity function** $\rho = \lambda$ **is a constant.**

(b) For the exponential failures with exponential repairs (constant r > 0), then

$$A(t) = \frac{\lambda_r}{\lambda+\lambda_r} + \frac{\lambda}{\lambda+\lambda_r}\, e^{-(\lambda+\lambda_r)t}$$

$$A(t_1,\ t_2) = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \left[\frac{\lambda_r}{\lambda+\lambda_r} + \frac{\lambda}{\lambda+\lambda_r}\, e^{-(\lambda+\lambda_r)t}\right] dt$$

$$= \frac{\lambda_r}{\lambda+\lambda_r} + \frac{\lambda/(t_2-t_1)}{(\lambda+\lambda_r)^2}\left[e^{-(\lambda+\lambda_r)t_1} - e^{-(\lambda+\lambda_r)t_2}\right]; \quad A_I = \frac{\lambda_r}{\lambda+\lambda_r};$$

The renewal function is $m(t) = E[N_C(t)] = \frac{-\lambda\lambda_r}{(\lambda+\lambda_r)^2} + \frac{\lambda\lambda_r t}{(\lambda+\lambda_r)} + \frac{\lambda\lambda_r}{(\lambda+\lambda_r)^2}\, e^{-(\lambda+\lambda_r)t}$.

Note that the above expected number of cycles seems troublesome because when r = λ_r = 0, its value does not become equal to the mean number of failures m(t) = λt for the case of W/O repair. This is probably due to the fact that the renewal function for a cyclic process does not reduce to the corresponding expected number of failures for an irreparable system. E. A. Elsayed provides the same expression as above near the bottom of his page 426 W/O detailed proof.

2. Two-Identical-Unit with constant failure rates Serial System

(a) **No Repairs ($r = \lambda_r = 0$): $R(t) = e^{-2\lambda t}$,**

The renewal function is $m(t) = E[N_f(t)] = 2\lambda t$.

(b) **If $r > 0$, then $A(t) = [\frac{\lambda_r}{\lambda+\lambda_r} + \frac{\lambda}{\lambda+\lambda_r} e^{-(\lambda+\lambda_r)t}]^2$; $A_I = (\frac{r}{\lambda+r})^2$**

3. Two-Identical-Unit (Constant FRs) Active Redundant System

(a) No Repairs ($r = 0$) : $R(t) = 2e^{-\lambda t} - e^{-2\lambda t}$. In order to obtain the exact renewal function $E[N_f(t)] = m(t)$, one must first derive the Laplace transform of the renewal function $m^*(s) = \frac{f^*(s)}{s[1-f^*(s)]}$, and then $m(t) = L^{-1}\{m^*(s)\}$. The approximate value of $m(t) = E[N_f(t)] \cong t/MTTF$;

$MTTF = \frac{2}{\lambda} - \frac{1}{2\lambda} = 1.5/\lambda \rightarrow m(t) = E[N_f(t)] \cong \lambda t/1.5$. Because $f(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}$, then $f^*(s)$

$= \frac{2\lambda}{s+\lambda} - \frac{2\lambda}{s+2\lambda}$ resulting in $m^*(s) = (\frac{2\lambda}{s+\lambda} - \frac{2\lambda}{s+2\lambda}) / \{s[1 - \frac{2\lambda}{s+\lambda} + \frac{2\lambda}{s+2\lambda}]\}$ =

$$\frac{2\lambda(s+2\lambda) - 2\lambda(s+\lambda)}{s[(s+\lambda)(s+2\lambda) - 2\lambda(s+2\lambda) + 2\lambda(s+\lambda)]} = \frac{2\lambda^2}{s[(s+\lambda)(s+2\lambda) - 2\lambda^2]} =$$

$$= \frac{2\lambda^2}{s[s^2+3\lambda s]} = \frac{2\lambda^2}{s^2(s+3\lambda)} \rightarrow m(t) = \frac{2\lambda^2}{(3\lambda)^2}(e^{-3\lambda t} + 3\lambda t - 1) = 2(e^{-3\lambda t} + 3\lambda t - 1)/9.$$

To check the validity of this last expected number of failures for a length of time t, we note that $E[N_f(t=0)] = 2(e^0 - 1)/9 = 2(1-1)/9 = 0$, as expected. Secondly, the value renewal intensity is $\rho(t) = dm(t)/dt = 2(-3\lambda e^{-3\lambda t} + 3\lambda - 0)/9 = -2/3\lambda e^{-3\lambda t} + 2\lambda/3$. Because $\mu = 1.5/\lambda$, then the $\lim_{t\rightarrow\infty} \rho(t) = 1/\mu = 2\lambda/3$.

(b) **The case of $r >> 0$. $A(t) = 1 - [1 - \frac{\lambda_r}{\lambda+\lambda_r} - \frac{\lambda}{\lambda+\lambda_r} e^{-(\lambda+\lambda_r)t}]^2$, assuming two servers, i.e., each unit has its own repair-crew.**

$A_l = \frac{2\lambda_r}{\lambda + \lambda_r} - (\frac{\lambda_r}{\lambda + \lambda_r})^2$ **(assuming two servers)**

For one server, $A(t) = \frac{r^2 + 2\lambda r}{r^2 + 2\lambda^2 + 2\lambda r} + \frac{2\lambda^2}{u_1(u_2 - u_1)} e^{u_1 t} + \frac{2\lambda^2}{u_2(u_1 - u_2)} e^{u_2 t}$

where $u_i = \frac{-(3\lambda + 2r) \pm \sqrt{\lambda^2 + 4\lambda r}}{2}$ **and** $A_l = \frac{r^2 + 2\lambda r}{r^2 + 2\lambda r + 2\lambda^2}$**. In order to obtain m(t) = E[N_C(t)], one must obtain** $m^*(s) = \frac{\bar{f}(s) r^*(s)}{s[1 - \bar{f}(s) r^*(s)]}$ **, and then E[N_C(t)] =** $\mathcal{L}^{-1}\{m^*(s)\}$**. This is probably too difficult to accomplish because** $\bar{f}(s)$ **and** $r^*(s)$ **now must represent the system Laplace transforms of TTF and TTR. The system TTF distribution can easily be obtained, but the system TTR distribution,** $r_{Sys}(t)$**, is difficult to obtain. The approximate value of m(t) = E[N_C(t)] is given by t/MTBC, where MTBC represents mean time between cycles.**

4. Two-Unit Standby Redundant System

(a) No or Minimal Repair (r = 0, $\lambda_1 = \lambda_2 = \lambda$**). R(t) = (1+** λ**t)** $e^{-\lambda t}$ **, m(t)** $\cong$ **t/(2/**λ**)=** λ**t/2.**

(b) r >> 0, $A(t) = \frac{r^2 + r\lambda_1}{r^2 + \lambda_1\lambda_2 + \lambda_1 r} + \frac{\lambda_2\lambda_1}{u_1(u_2 - u_1)} e^{u_1 t} + \frac{\lambda_2\lambda_1}{u_2(u_1 - u_2)} e^{u_2 t}$,

where $u_i = \frac{-(\lambda_1 + 2r + \lambda_2) \pm \sqrt{(\lambda_2 - \lambda_1)^2 + 4r\lambda_2}}{2}$.

$A_l = \frac{r(r + \lambda_1)}{r^2 + r\lambda_1 + \lambda_1\lambda_2}$ **because both** u_1 **and** $u_2 < 0$**.**