INSY 7310 Spring 2003 Cramer's Rule Maghsoodloo

As an example, consider the following non-homogenous (or heterogeneous) system of 3 equations with 3 unknowns x_1, x_2, and x_3.

$$\begin{cases} 2x_1 - x_2 + x_3 = 4 \\ -x_1 + 4x_2 - x_3 = 10 \\ x_1 - x_2 + 3x_3 = 6 \end{cases}$$

In matrix form, this system can be represented as

$\mathbf{A}\,\mathbf{X} = \mathbf{B}$, where the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 3 \end{bmatrix} = [\,\mathbf{C}_1 \quad \mathbf{C}_2 \quad \mathbf{C}_3\,]$, the column vectors $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 \\ 10 \\ 6 \end{bmatrix}$, and the definition of the 3 column vectors **$\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$** are self-explanatory. Obviously, we 1st need to ask the question " is there a unique and single solution to the above system of equations?". The answer is that a unique solution exists **if and only if** the determinant of A, denoted by det(**A**) = $|\mathbf{A}|$, is **different from zero**. If the **det (A) = 0**, then there will not exit any solution, and the heterogeneous system of equations is said to be inconsistent.

One way to solve the above system of 3 equations with 3 unknowns is to premultiply both sides of the system of equations $\mathbf{A}\,\mathbf{X} = \mathbf{B}$ **by** $\mathbf{A}^{-1}$ which yields $\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$ **, i.e,** the unique solution, $\mathbf{X}_0$**,** to the above heterogeneous system is given by

$$\mathbf{X}_0 = \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \end{bmatrix} = \begin{bmatrix} \mathbf{0.647059} & \mathbf{0.117647} & \mathbf{-0.176471} \\ \mathbf{0.117647} & \mathbf{0.294118} & \mathbf{0.0588235} \\ \mathbf{-0.176471} & \mathbf{0.0588235} & \mathbf{0.4117647} \end{bmatrix} \times \begin{bmatrix} \mathbf{4} \\ \mathbf{10} \\ \mathbf{6} \end{bmatrix} = \begin{bmatrix} \mathbf{2.705882} \\ \mathbf{3.764706} \\ \mathbf{2.352941} \end{bmatrix}.$$

Note that since the 3×3 matrix **A** is symmetric, then by necessity its inverse $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{0.647059} & \mathbf{0.117647} & \mathbf{-0.176471} \\ \mathbf{0.117647} & \mathbf{0.294118} & \mathbf{0.0588235} \\ \mathbf{-0.176471} & \mathbf{0.0588235} & \mathbf{0.4117647} \end{bmatrix}$ is also symmetric. This means that if you ever invert a symmetric matrix and obtain an asymmetric matrix as the inverse, then your inverse is for certain wrong!

To understand the Cramer Rule, we will 1st expand the det(**A**) in terms of the cofactors of column 1 of **A**, denoted by C_{i1}, i.e., C_{i1} is the cofactor of $\mathbf{C_1}$ and

Det(**A**) = $|\,\mathbf{C_1} \quad \mathbf{C_2} \quad \mathbf{C_3}\,| = 2\times C_{11} + (-1) \times C_{21} + 1\times C_{31}$, where $C_{11} = (-)^{1+1} \times \begin{vmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{3} \end{vmatrix} = 11$, $C_{21} = (-1)^{2+1} \times \begin{vmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{3} \end{vmatrix} = 2$, and $C_{31} = (-1)^{3+1} \times \begin{vmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{4} & \mathbf{-1} \end{vmatrix} = -3$. Hence, the det(**A**) $= 2\times 11 + (-1) \times 2 + 1\times(-3) = 17$.

It can proven that, in general, if the cofactors of column $\mathbf{C}_j$ (j = 1, 2, 3 for our example) of any square matrix are multiplied by any other column $\mathbf{C_k}$ ($k \neq j$) of that matrix, then the resultant product is always identically zero. For our example, if we multiply C_{i1} (i = 1, 2, 3) by the elements of columns $\mathbf{C_2}$ **or** $\mathbf{C_3}$, we will get zero for the resulting product as shown below:

$$(-1)\times C_{11} + 4\times C_{21} + (-1)\times C_{31} = (-1)\times 11 + 4\times 2 + (-1)\times(-3) \equiv 0,$$

and

$$1\times C_{11} + (-1)\times C_{21} + 3\times C_{31} = 1\times 11 + (-1)\times 2 + 3\times(-3) \equiv 0.$$

We now use the above properties of cofactors to derive the Cramer Rule for the example under consideration. To this end, we multiply the 1st equation of the system at the top of page 1 throughout by C_{11}, the 2nd equation by C_{21}, and we multiply the 3rd equation by C_{31} and then we will sum the 3 resulting equations in order to obtain the following result:

$$C_{11}[\, 2x_1 - x_2 + x_3 \,] + C_{21}\, [- x_1 + 4\, x_2 - x_3\,] + C_{31}\, [\, x_1 - x_2 + 3\, x_3\,] =$$

$$= 4\times C_{11} + 10\times C_{21} + 6\times C_{31}.$$

Clearly, the LHS of this last equation reduces to $C_{11}(2x_1) + C_{21}\,(- x_1) + C_{31}\,(\, x_1) =$ $[2C_{11} + (-1)C_{21} + C_{31}\,](\, x_1) = \det(\mathbf{A}) \times (x_1) = |\,\mathbf{A}\,| \times (x_1)$. Secondly, the RHS of the above equation can be rewritten as

$$4\times\begin{vmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{3} \end{vmatrix} + 10 \times (-1)^{2+1} \begin{vmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{3} \end{vmatrix} + 6\times(-1)^{3+1} \begin{vmatrix} \mathbf{-1} & \mathbf{1} \\ \mathbf{4} & \mathbf{-1} \end{vmatrix} = \begin{vmatrix} \mathbf{4} & \mathbf{-1} & \mathbf{1} \\ \mathbf{10} & \mathbf{4} & \mathbf{-1} \\ \mathbf{6} & \mathbf{-1} & \mathbf{3} \end{vmatrix} = |\mathbf{A_1}|.$$

Therefore, $|\,\mathbf{A}\,| \times (x_1) = |\mathbf{A_1}|$, which yields $x_1 = |\mathbf{A_1}| / |\,\mathbf{A}\,| = 46/17$.

Note that the matrix $\mathbf{A_1}$ is obtained by replacing the entire 1st column of the matrix $\mathbf{A}$ by the column vector $\mathbf{B^T = [\;4 \quad 10 \quad 6]' = \begin{bmatrix} 4 \\ 10 \\ 6 \end{bmatrix}}$.

To solve for the unique value of x_2, we follow the same procedure but multiply the system of 3 equations with 3 unknowns by the cofactors of the 2nd column of the matrix $\mathbf{A}$, C_{12}, C_{22}, and C_{32}, respectively, and sum the resulting 3 equations to obtain $x_2 = |\mathbf{A_2}| / |\,\mathbf{A}\,| = 64/17$. Similarly, $x_3 = |\mathbf{A_3}| / |\mathbf{A}| = 40/17$.