3 Laminar Internal Flow

Exercises

1. Consider laminar flow between two parallel plates. The flow is incompressible, has constant properties, and is fully developed.

(a) Derive the velocity profile and the mean velocity.

(b) Derive the Nusselt number for the constant surface heat flux case (both surfaces are heated)

(c) Derive the Nusselt number for the case in which one surface is insulated, and the other is maintained at a constant temperature. Your should perform at least two iterations for the temperature profile.

Let $x$ and $y$ denote the flow and normal directions. The momentum equation will be

$$
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dP}{dx} + \nu \frac{\partial^2 u}{\partial y^2}
$$

(1)

The FDF assumption means that $u = u(y)$, and the convection terms drop out. The velocity is zero at $y = 0$ and $L$, and the solution is

$$
u = -\frac{1}{2\nu} \frac{dP}{dx} (y L - y^2)
$$

(2)

The mean velocity is

$$u_m = \frac{1}{L} \int_0^L u \, dy = -\frac{L^2}{12\nu} \frac{dP}{dx}
$$

(3)

and

$$\bar{u}(\bar{y}) = \frac{u}{u_m} = 6\bar{y} (1 - \bar{y}), \quad \bar{y} = \frac{y}{L}
$$

(4)

represents the dimensionless velocity profile.

For constant $q_s''$ on both surfaces,

$$\rho \, \overline{u_m} \, C \frac{\partial T}{\partial x} = \rho \, u_m \, C \frac{dT_m}{dx} = \frac{q''_m}{A} = \frac{2h(T_s - T_m)}{L}
$$

(5)

and the energy equation becomes

$$-2\pi \, Nu_L = \frac{d^2 \bar{T}}{d\bar{y}^2}, \quad \bar{T} = \frac{T - T_s}{T_m - T_s}
$$

(6)

with zero temperature at the surfaces. The solution is

$$\bar{T}(\bar{y}) = Nu_L (\bar{y} - 2 \bar{y}^3 + \bar{y}^4)
$$

The mean temperature condition is

$$1 = \int_0^1 \bar{T} \, d\bar{y} = \frac{17 \, Nu_L}{140} \quad \rightarrow Nu_L = \frac{70}{17} = 4.12
$$

The hydraulic diameter is twice $L$, and the Nusselt number based on this length is 8.24 which agrees with the book.

When one surface is insulated the factor 2 in Eq. (5) becomes 1, because the heated perimeter is only the bottom surface. Per the formulation in the notes, the DE for the constant $T_s$ case is

$$-\bar{u} \, Nu \, T = \frac{d^2 \bar{T}}{d\bar{y}^2}
$$

(7)

with

$$\bar{T}(0) = 0, \quad \bar{T}'(1) = 0
$$

An iteration strategy is used. The procedure in mathematica is as follows;
\( \text{ub}[y_] := 6(1-y) \ y \)
\( \text{t0}[y_] := 1; \)
\( \text{Clear}[\text{null}]; \)

\[
\text{Do[}
\begin{align*}
\text{de} &= \text{t''}[y] == -\text{null} \ \text{t0}[y] \ \text{ub}[y]; \\
\text{bc1} &= \text{t}[0] == 0; \\
\text{bc2} &= \text{t}'[1] == 0; \\
\text{soln} &= \text{DSolve}\{\text{de}, \text{bc1}, \text{bc2}, \text{t}[y], y\}[[1,1]]; \\
\text{tm} &= \text{Integrate}\{t[y] /\ . \ \text{soln} \ \text{ub}[y], \{y, 0, 1\}\}; \\
\text{nu} &= \text{null} /\ . \ \text{Solve}\{\text{tm} == 1, \text{null}\}[[1,1]]; \\
\text{t0}[y_] &= \text{Evaluate}[t[y] /\ . \ \text{soln} /\ . \ \text{null} -> \text{nu}];
\end{align*}
\]

\( \text{Print}[\text{N}[\text{nu}]]; \)
\( , \{i, 1, 4\}\)

2.69231
2.44793
2.432
2.43053

The converged result is \( \text{Nu}_L = 2.43 \), or 4.86 based on \( D_h \).

2. A counterflow heat exchanger consists of a simple pair of coannular tubes (i.e., one tube running inside
another). Say that the outer tube contains the hot fluid, and the inner the cold. Both fluids are
otherwise identical (the same substance) and have the same mass flow rate. Both flows are laminar
and fully developed. The outer wall of the pipe in insulated, so that heat is only exchanged between
the two fluids.

(a) Derive the velocity profile and mean velocity in the outer (annular) pipe.

(b) Show that, for this particular situation, the heat transfer problem for both the cold and hot fluids
is equivalent to the constant surface heat flux model.

(c) Derive formulas for the Nusselt numbers for the inner and outer tubes.

In cylindrical coordinates, and assuming FDF, the momentum equation appears as

\[
\frac{1}{\rho} \frac{dP}{dx} = \nu \frac{1}{r} \frac{d}{dr} \left( \frac{du}{dr} \right) \tag{8}
\]

Define dimensionless variables via

\[
\bar{\pi} = \frac{u}{u_m}, \quad \bar{\tau} = \frac{r}{R_1}, \quad a = \frac{R_2}{R_1}
\]

and the equation becomes

\[
B = \frac{1}{\bar{\tau}} \frac{d}{d\bar{\tau}} \left( \bar{\tau} \frac{d\bar{\pi}}{d\bar{\tau}} \right) \tag{9}
\]

with

\[
B = \frac{R_1^2}{\rho \nu u_m} \frac{dP}{dx} = \text{Re}_D, \quad R_1 \frac{dP}{dx}
\]

Boundary conditions are

\[
\bar{\pi}(1) = \bar{\pi}(a) = 0
\]

and the solution is

\[
\frac{B \left( (-1 + \bar{\tau}^2) \ln(a) - (-1 + a^2) \ln(\bar{\tau}) \right)}{4 \ln(a)}
\]

The mean velocity condition is

\[
1 = \frac{2}{a^2 - 1} \int_1^a \bar{\pi} \bar{\tau} d\bar{\tau} = \frac{- (B \left( 1 - a^2 + (1 + a^2) \ln(a) \right))}{8 \ln(a)}
\]
or, solving for \( B \),
\[
B = Re_D, \quad \frac{R_1}{2 \rho u_m^2} \frac{dP}{dx} = \frac{-8 \ln(a)}{1 - a^2 + (1 + a^2) \ln(a)}
\]
which provides the relation between the pressure gradient and the mean velocity. The formula for the velocity profile is
\[
\bar{u} = \frac{2}{1 - a^2 + (1 + a^2) \ln(a)} \left[ \left( -1 + \pi^2 \right) \ln(a) + \left( 1 + a^2 \right) \ln(r) \right]
\]

For an insulated outer surface and a constant heat flux on the inner surface, the mean temperature energy balance becomes
\[
\rho u_m C \frac{\partial T}{\partial x} = \rho u_m C \frac{dT_m}{dx} = \frac{\dot{q}_i}{A} = \frac{2h R_1 (T_s - T_m)}{R_2^2 - R_1^2}
\]
Note that the perimeter refers only to the heated surface – not the insulated one – so \( P = 2\pi R_1 \). By use of the usual dimensionless variables, the energy equation for the outer annulus becomes
\[
-\pi Nu_D \frac{1}{a^2 - 1} = \frac{1}{\tau} \frac{d}{d\tau} \left( \tau \frac{dT}{d\tau} \right)
\]
with
\[
\bar{T}(1) = \bar{T}'(a) = 0
\]
and
\[
Nu_D = \frac{2h R_1}{k}
\]

The DE can be integrated – and it is most useful if you use mathematica to do the work. The mean temperature condition is
\[
1 = \frac{2}{a^2 - 1} \int_1^a T \frac{dT}{d\tau} d\tau
\]
and this provides a formula for \( Nu_D \). The formula is
\[
Nu_D = \frac{144 \left( -1 + a^2 \right)^2 \left( -1 + a^4 \right) \ln(a)}{-9 \left( -1 + a^2 \right)^3 \left( 5 + 11 a^2 \right) + \ln(a) C_1}
\]
with
\[
C_1 = 4 \left( -1 + a^2 \right)^2 \left( -19 + 8 a^2 + 62 a^4 \right) + 3 \ln(a) \left( -11 + 36 a^4 + 48 a^6 - 73 a^8 + 24 a^{10} \ln(a) \right)
\]
In the limit of \( a \rightarrow 1 \) the solution should reduce to that for two parallel plates, with one heated uniformly and one insulated. Since the formula for \( Nu_D \) is based on the inner diameter \( D_1 \), the Nusselt number based on the gap thickness will be \( Nu_D (D_2 - D_1) / D_1 = Nu_D (a - 1) \). And it can be shown that
\[
\lim_{a \rightarrow 1} Nu_D (a - 1) \rightarrow \frac{70}{13} = 5.385
\]
which is the correct parallel plate result.

The Nusselt number for the inner flow is simply that for FDF, constant \( q_i'' \): \( Nu_D = 48/11 = 4.364 \).

Denote as \( i \) and \( o \) the fluids in the inner pipe and the outer annulus. Neglect conduction resistance across the pipe wall. At some point \( x \), the energy balance for the two streams is
\[
\dot{m}_i C_i \frac{dT_{mi}}{dx} = 2\pi R_1 h_i (T_s - T_{mi}) = 2\pi R_1 h_o (T_{mo} - T_s) = \dot{m}_o C_o \frac{dT_{mo}}{dx}
\]
This statement says that the rate of heat transfer from one stream is equal to the rate of heat transfer to the other. Note that the counterflow arrangement has the flows going in opposite direction, so \( x \) is pointing in the flow direction for one fluid and opposite the flow direction for the other. The two terms on the inside can be combined to define an overall heat transfer coefficient \( U \), via
\[
U = \frac{1}{1/h_i + 1/h_o}
\]
so that
\[
\dot{m}_i C_i \frac{dT_{mi}}{dx} = \dot{m}_o C_o \frac{dT_{mo}}{dx} = 2\pi R_1 U (T_{mo} - T_{mi}) = 2\pi R_1 q_i''
\]
If the mass flow rates and the specific heats are the same, then the rate change of mean temperature will be the same for both streams. This implies that the heat flux at any point is constant.