7.7 Introduction of the Reference Input

The controller obtained by combining the control law studied in Section 7.3 with the estimator discussed in Section 7.6 is essentially a **regulator design**. This means that the characteristic equations of the control and the estimator are chosen for good disturbance rejection, that is, to give satisfactory transients to disturbances such as $w(t)$. However, this design approach does not consider a reference input, nor does it provide for **command following**, which is evidenced by a good transient response of the combined system to command changes. In general, good disturbance rejection and good command following both need to be taken into account in designing a control system. Good command following is done by properly introducing the reference input into the system equations.

Let us repeat the plant and controller equations for the full-order estimator; the reduced-order case is the same in concept, differing only in detail:

$$\text{Plant:} \quad \dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u, \tag{7.162a}$$

$$y = \mathbf{H}\mathbf{x}; \tag{7.162b}$$

$$\text{Controller:} \quad \dot{\hat{\mathbf{x}}} = (\mathbf{F} - \mathbf{G}\mathbf{K} - \mathbf{L}\mathbf{H})\hat{\mathbf{x}} + \mathbf{L}y, \tag{7.163a}$$

$$u = -\mathbf{K}\hat{\mathbf{x}}. \tag{7.163b}$$

Figure 7.37 shows two possibilities for introducing the command input r into the system. This figure illustrates the general issue of whether the compensation should be put in the feedback or feedforward path. The response of the system to command inputs is different, depending on the configuration, because the zeros of the transfer functions are different. The closed-loop poles are identical, however, as can be easily verified by letting $r = 0$ and noting that the systems are then identical.

The difference in the responses of the two configurations can be seen quite easily. Consider the effect of a step input in r. In Fig. 7.37(a) the step will excite the estimator in precisely the same way that it excites the plant; thus the estimator error will remain zero during and after the step. This means that the estimator dynamics are not excited by the command input so the transfer function from r to y must have zeros at the estimator pole locations that cancel those poles. As a result, a step command will excite system behavior that is consistent with the control poles alone, that is, with the roots of $\det(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K}) = 0$.

In Fig. 7.37(b) a step command in r enters directly only into the estimator, thus causing an estimation error that decays with the estimator dynamic characteristics in addition to the response corresponding to the control poles. Therefore, a step command will excite system behavior consistent with both control roots and estimator roots; that is, the roots of

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K}) \cdot \det(s\mathbf{I} - \mathbf{F} + \mathbf{L}\mathbf{H}) = 0.$$

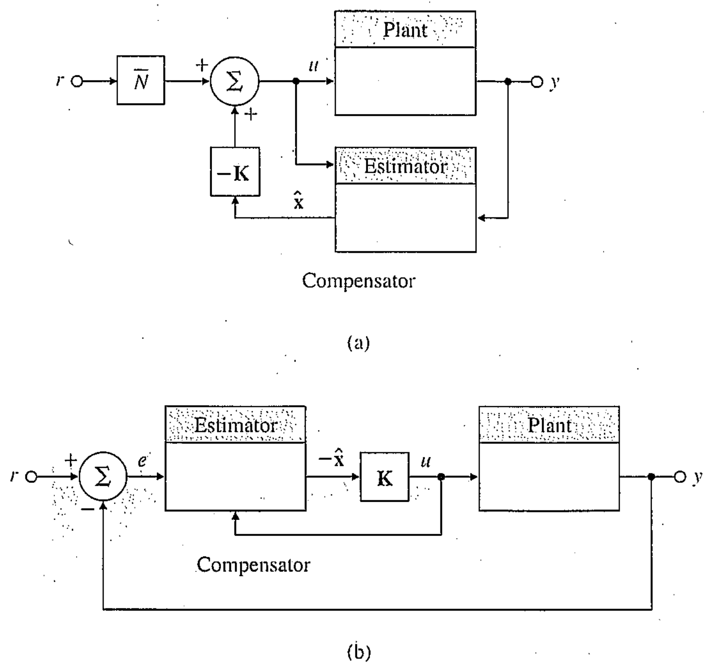


FIGURE 7.37 Possible locations for introducing the command input: (a) compensation in the feedback path; (b) compensation in the feedforward path

For this reason the configuration shown in Fig. 7.37(a) is typically the superior way to command the system where $\bar{\mathbf{N}}$ is found using Eqs. (7.79)–(7.81).

In Section 7.7.1 we will show a general structure for introducing the reference input with three choices of parameters that implement either the feedforward or the feedback case. We will analyze the three choices from the point of view of the system zeros and the implications the zeros have for the system transient response. Finally, in Section 7.7.2 we will show how to select the remaining parameter to eliminate constant errors.

7.7.1 A General Structure for the Reference Input

Given a reference input $r(t)$, the most general linear way to introduce r into the system equations is to add terms proportional to it in the controller equations. We can do this by adding $\bar{N}r$ to Eq. (7.163b) and $\mathbf{M}r$ to Eq. (7.163a). Note that in this case, $\bar{N}$ is a scalar and $\mathbf{M}$ is an $n \times 1$ vector. With these additions the controller equations become

Controller equations

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= (\mathbf{F} - \mathbf{GK} - \mathbf{LH})\hat{\mathbf{x}} + \mathbf{L}y + \mathbf{M}r, \\ u &= -\mathbf{K}\hat{\mathbf{x}} + \bar{N}r. \end{aligned} \tag{7.164}$$

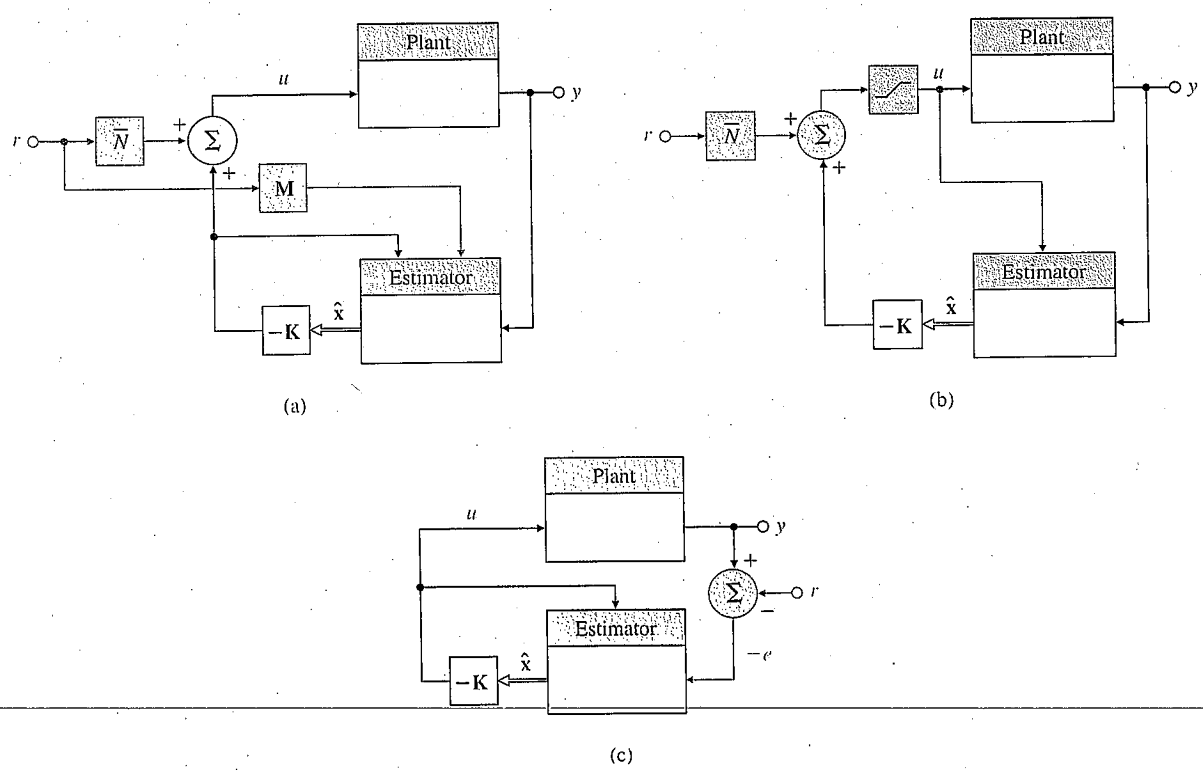
The block diagram is shown in Fig. 7.38(a). The alternatives shown in Fig. 7.37 correspond to different choices of $\mathbf{M}$ and $\bar{N}$. Since $r(t)$ is an external signal, it is clear that neither $\mathbf{M}$ nor $\bar{N}$ affects the characteristic equation of the combined controller-estimator system. In transfer-function terms, the selection of **M** and $\bar{N}$ will only affect the zeros of transmission from r to y and as a consequence can significantly affect the transient response but not the stability. How can we choose **M** and $\bar{N}$ to obtain satisfactory transient response? We should point out that we assigned the poles of the system by feedback gains **K** and **L** and we are now going to assign zeros by feedforward gains **M** and $\bar{N}$.

There are three strategies for choosing **M** and $\bar{N}$:

Three methods for selecting **M** and $\bar{N}$

1. *Autonomous estimator:* Select **M** and $\bar{N}$ so that the state estimator error equation is independent of r (Fig. 7.38b).
2. *Tracking-error estimator:* Select **M** and $\bar{N}$ so that only the tracking error, $e = (r - y)$, is used in the control (Fig. 7.38c).
3. *Zero-assignment estimator:* Select **M** and $\bar{N}$ so that n of the zeros of the overall transfer function are assigned at places of the designer's choice (Fig. 7.38a).

FIGURE 7.38
Alternative ways to introduce the reference input: (a) general case—zero assignment; (b) standard case—estimator not excited, zeros = $\alpha_e(s)$; (c) error-control case—classical compensation



CASE 1: From the viewpoint of estimator performance, the first method is quite attractive and the most widely used of the alternatives. If $\hat{\mathbf{x}}$ is to generate a good estimate of $\mathbf{x}$, then surely $\tilde{\mathbf{x}}$ should be as free of external excitation as possible; that is, $\tilde{\mathbf{x}}$ should be uncontrollable from r. The computation of $\mathbf{M}$ and $\bar{N}$ to bring this about is quite easy. The estimator error equation is found by subtracting Eq. (7.164) from Eq. (7.162), with the plant output (Eq. 7.162b) substituted into the estimator (Eq. 7.163a) and the control (Eq. 7.163b) substituted into the plant (Eq. 7.162a):

$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{F}\mathbf{x} + \mathbf{G}(-\mathbf{K}\hat{\mathbf{x}} + \bar{N}r) - [(\mathbf{F} - \mathbf{G}\mathbf{K} - \mathbf{L}\mathbf{H})\hat{\mathbf{x}} + \mathbf{L}y + \mathbf{M}r], \tag{7.165a}$$

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{F} - \mathbf{L}\mathbf{H})\tilde{\mathbf{x}} + \mathbf{G}\bar{N}r - \mathbf{M}r. \tag{7.165b}$$

If r is not to appear in Eqs. (7.165b), then we should choose

$$\mathbf{M} = \mathbf{G}\bar{N}. \tag{7.166}$$

Since $\bar{N}$ is a scalar, $\mathbf{M}$ is fixed to within a constant factor. Note that with this choice of $\mathbf{M}$ we can write the controller equations as

$$u = -\mathbf{K}\hat{\mathbf{x}} + \bar{N}r \tag{7.167a}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{F} - \mathbf{L}\mathbf{H})\hat{\mathbf{x}} + \mathbf{G}u + \mathbf{L}y, \tag{7.167b}$$

which matches the configuration in Fig. 7.38(b). The net effect of this choice is that the control is computed from the feedback gain and the reference input *before* it is applied and then the same control is input to both the plant and the estimator. In this form, if the plant control is subject to saturation, the same control limits can be applied in Eq. (7.167) to the control entering the equation for the estimate $\hat{\mathbf{x}}$, and the nonlinearity cancels out of the $\tilde{\mathbf{x}}$ equation. This behavior is essential for proper estimator performance. The block diagram corresponding to this technique is shown in Fig. 7.38(b). We will return to the selection of the gain factor on the reference input, $\bar{N}$, in Section 7.7.3 after discussing the other two methods of selecting $\mathbf{M}$.

CASE 2: The second approach suggested earlier is to use the tracking error. This solution is sometimes forced on the control designer when the sensor measures only the output error. For example, in many thermostats the output is the difference between the temperature to be controlled and the setpoint temperature, and there is no absolute indication of the reference temperature available to the controller. Also, some radar tracking systems have a reading that is proportional to the pointing error, and this error signal alone must be used for feedback control. In these situations, we must select $\mathbf{M}$ and $\bar{N}$ so that Eqs. (7.164) are driven by the error only. This requirement is satisfied if we select

$$\bar{N} = 0 \quad \text{and} \quad \mathbf{M} = -\mathbf{L}. \tag{7.168}$$

Then the estimator equation is

$$\dot{\hat{\mathbf{x}}} = (\mathbf{F} - \mathbf{GK} - \mathbf{LH})\hat{\mathbf{x}} + \mathbf{L}(y-r). \tag{7.169}$$

The compensator in this case, for low-order designs, is a lead compensator in the forward path. As we have seen in earlier chapters, this design can have a considerable amount of overshoot because of the zero of the compensator. This design corresponds exactly to the compensators designed by the transform methods given in Chapters 5 and 6.

CASE 3: The third method of selecting $\mathbf{M}$ and $\bar{N}$ is to choose the values so as to assign the systems zeros to arbitrary locations of the designers choice. This method provides the designer with the maximum flexibility in satisfying transient-response and steady-state gain constraints. The other two methods are special cases of this third method. All three methods depend on the zeros. As we saw in Section 7.3.2, when there is no estimator and the reference input is added to the control, the closed-loop system zeros remain fixed as the zeros of the open-loop plant. We now examine what happens to the zeros when an estimator is present. To do so, we reconsider the controller of Eqs. (7.164). If there is a zero of transmission from r to u, then there is necessarily a zero of transmission from r to y unless there is a pole at the same location as the zero. It is therefore sufficient to treat the controller alone to determine what effect the choices of $\mathbf{M}$ and $\bar{N}$ will have on the system zeros. The equations for a zero from r to u from Eqs. (7.164) are given by

$$\det \begin{bmatrix} s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH} & -\mathbf{M} \\ -\mathbf{K} & \bar{N} \end{bmatrix} = 0. \tag{7.170}$$

(We let $y = 0$ since we care only about the effect of r.) If we divide the last column by the (nonzero) scalar $\bar{N}$ and then add to the rest the product of $\mathbf{K}$ times the last column, we find the feedforward zeros are at the values of s such that

$$\det \begin{bmatrix} s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH} - \dfrac{\mathbf{M}}{\bar{N}}\mathbf{K} & -\dfrac{\mathbf{M}}{\bar{N}} \\ \mathbf{0} & 1 \end{bmatrix} = 0,$$

or

$$\det \left(s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH} - \frac{\mathbf{M}}{\bar{N}}\mathbf{K} \right) = \gamma(s) = 0. \tag{7.171}$$

Now Eq. (7.171) is exactly in the form of Eq. (7.111) for selecting $\mathbf{L}$ to yield desired locations for the estimator poles. Here we have to select $\mathbf{M}/\bar{N}$ for a desired zero polynomial $\gamma(s)$ in the transfer function from the reference input to the control. Thus the selection of $\mathbf{M}$ provides a substantial amount of freedom to influence the transient response. We can add an arbitrary nth-order polynomial to the transfer function from r to u and hence from r to y; that is, we can assign n zeros in addition to all the poles that we assigned previously.

If the roots of $\gamma(s)$ are not canceled by the poles of the system, then they will be included in zeros of transmission from r to y.

Two considerations can guide us in the choice of $\mathbf{M}/\bar{N}$, that is, in the location of the zeros. The first is dynamic response. We have seen in Chapter 3 that the zeros influence the transient response significantly, and the heuristic guidelines given there may suggest useful locations for the available zeros. The second consideration, which will connect state-space design to another result from transform techniques, is steady-state error or velocity-constant control. In Chapter 4 we derived the relationship between the steady-state accuracy of a type I system and the closed-loop poles and zeros. If the system is type I, then the steady-state error to a step input will be zero and to a unit ramp input will be

$$e_\infty = \frac{1}{K_v}, \tag{7.172}$$

where K_v is the velocity constant. Furthermore, it was shown that if the *closed-loop* poles are at $\{p_i\}$ and the *closed-loop* zeros are at $\{z_i\}$, then (for a type I system) **Truxal's formula** gives

Truxal's formula

$$\frac{1}{K_v} = \sum \frac{1}{z_i} - \sum \frac{1}{p_i} \tag{7.173}$$

Equation (7.173) forms the basis for a partial selection of $\gamma(s)$, and hence of $\mathbf{M}$ and $\bar{N}$. The choice is based on two observations:

1. If $|z_i - p_i| \ll 1$, then the effect of this pole-zero pair on the dynamic response will be small, since the pole is almost canceled by the zero, and in any transient the residue of the pole at p_i will be very small.
2. Even though $z_i - p_i$ is small, it is possible for $1/z_i - 1/p_i$ to be substantial and thus to have a significant influence on K_v according to Eq. (7.173).

Application of these two guidelines to the selection of $\gamma(s)$, and hence of $\mathbf{M}$ and $\bar{N}$, results in a lag-network design. We illustrate this with an example.

◆ EXAMPLE 7.25 *Increasing the Velocity Constant through Zero Assignment*

Consider the second-order system described by

Lag compensation by a state space method

$$H(s) = \frac{1}{s(s+1)}$$

and with state description

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_2 + u.$$

Design a controller using pole placement so that it has control characteristic equation

$$\alpha_c(s) = (s+2)^2 + 4 = s^2 + 4s + 8,$$

and a velocity constant $K_v = 10$.

Solution. For this problem the state feedback gain

$$\mathbf{K} = [8 \quad 3]$$

results in the desired control poles. However, with this gain, $K_v = 2$, and we need $K_v = 10$. What effect will using estimators designed according to the three methods for $\mathbf{M}$ and $\bar{N}$ selection have on our design? Using the first strategy (the autonomous estimator), we find that the value of K_v does not change. If we use the second method (error control), we introduce a zero at a location unknown beforehand, and the effect on K_v will not be under direct design control. However, if we use the third option (zero placement) along with Truxal's formula (Eq. 7.173), we can satisfy both the dynamic response and the steady-state requirements.

First we must select the estimator pole p_3 and the zero z_3 to satisfy Eq. (7.173) for $K_v = 10$. We want to keep $z_3 - p_3$ small so that there is little effect on the dynamic response and yet have $1/z_3 - 1/p_3$ be large enough to increase the value of K_v. To do this, we arbitrarily set p_3 small compared with the control dynamics. For example, we let

$$p_3 = -0.1.$$

Notice that this approach is opposite to the usual philosophy of estimation design, where fast response is the requirement. Now using Eq. (7.173) to get

$$\frac{1}{K_v} = \frac{1}{z_3} - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3},$$

where $p_1 = -2 + 2j$, $p_2 = -2 - 2j$, and $p_3 = -0.1$. We solve for z_3 such that $K_v = 10$:

$$\frac{1}{K_v} = \frac{4}{8} + \frac{1}{0.1} + \frac{1}{z_3} = \frac{1}{10}$$

or

$$z_3 = -\frac{1}{10.4} = -0.096.$$

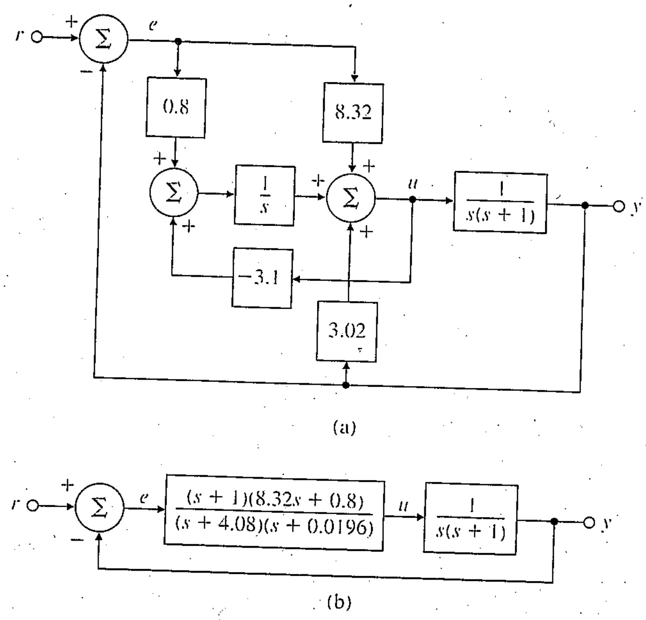
We thus design a reduced-order estimator to have a pole at -0.1 and choose $\mathbf{M}/\bar{N}$ such that $\gamma(s)$ has a zero at -0.096. A block diagram of the resulting system is shown in Fig. 7.39(a). You can readily verify that this system has the overall transfer function

$$\frac{Y(s)}{R(s)} = \frac{8.32(s+0.096)}{(s^2+4s+8)(s+0.1)}, \tag{7.174}$$

for which $K_v = 10$, as specified.

The compensation shown in Fig. 7.39(a) is nonclassical in the sense that it has two inputs (e and y) and one output. If we resolve the equations to provide pure error compensation by finding the transfer function from e and u, which would give Eq. (7.174), we obtain the system shown in Fig. 7.39(b). This compensation is a classical

FIGURE 7.39
Servomechanism with assigned zeros (a lag network): (a) the two-input compensator; (b) equivalent unity feedback system



lag-lead network. The root locus of the system in Fig. 7.39(b) is shown in Fig. 7.40. Note the pole-zero pattern near the origin that is characteristic of a lag network. The Bode plot in Fig. 7.41 shows the phase lag at low frequencies and phase lead at high frequencies. The step response of the system is shown in Fig. 7.42.

FIGURE 7.40
Root locus of lag-lead compensation

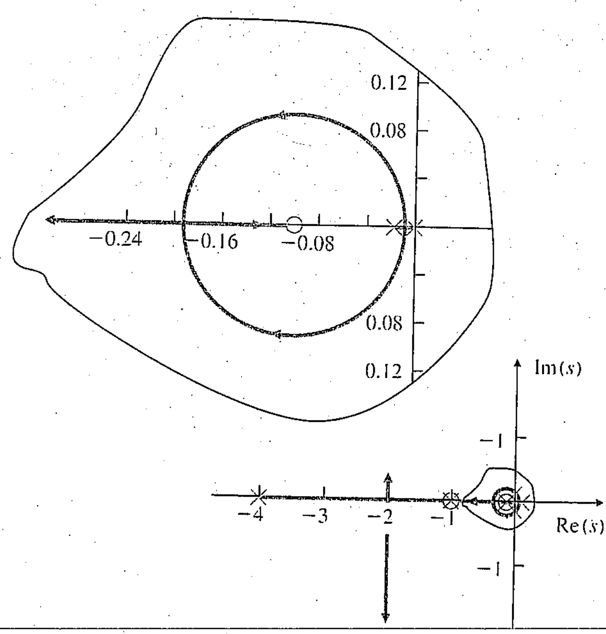


FIGURE 7.41
Frequency response of lag-lead compensation

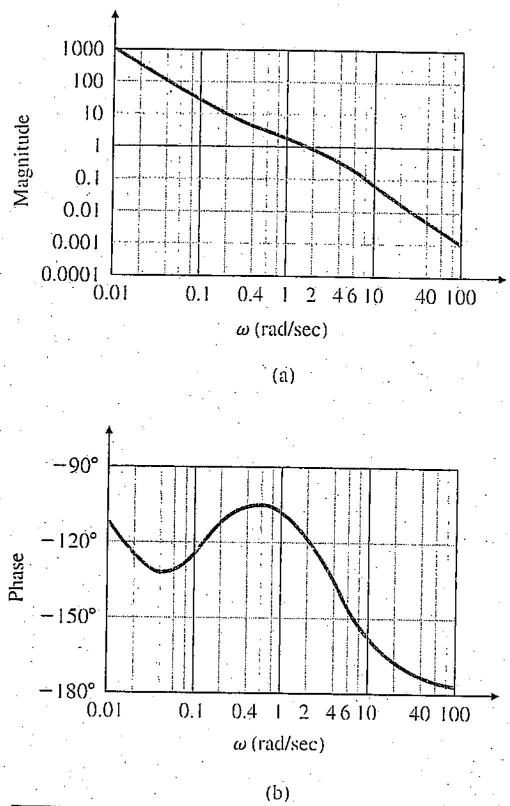
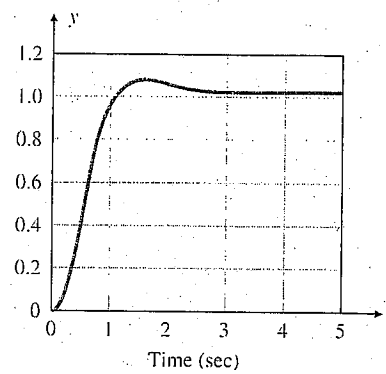


FIGURE 7.42
Step response of the system with lag compensation



We now reconsider the first two methods for choosing $\mathbf{M}$ and $\bar{N}$, this time to examine their implications in terms of zeros. Under the first rule (for the autonomous estimator), we let $\mathbf{M} = \mathbf{G}\bar{N}$. Substituting this into Eq. (7.171) yields, for the controller feedforward zeros,

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{LH}) = 0. \tag{7.175}$$

This is exactly the equation from which $\mathbf{L}$ was selected to make the characteristic polynomial of the estimator equation equal to $\alpha_e(s)$. Thus we have created n zeros in exactly the same locations as the n poles of the estimator. Because of this pole-zero cancellation (which causes "uncontrollability" of the estimator modes), the overall transfer function poles consist only of the state-feedback controller poles.

The second rule (for a tracking-error estimator) selects $\mathbf{M} = -\mathbf{L}$ and $\bar{N} = 0$. If these are substituted into Eq. (7.170), then the feedforward zeros are given by

$$\det \begin{bmatrix} s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH} & \mathbf{L} \\ -\mathbf{K} & 0 \end{bmatrix} = 0. \tag{7.176}$$

If we postmultiply the last column by $\mathbf{H}$ and subtract the result from the first n columns, and then premultiply the last row by $\mathbf{G}$ and add it to the first n rows, Eq. (7.176) then reduces to

$$\det \begin{bmatrix} s\mathbf{I} - \mathbf{F} & \mathbf{L} \\ -\mathbf{K} & 0 \end{bmatrix} = 0. \tag{7.177}$$

If we compare Eq. (7.177) with the equations for the zeros of a system in a state description, Eq. (7.54), we see that the added zeros are those obtained by replacing the input matrix with $\mathbf{L}$ and the output with $\mathbf{K}$. Thus, if we wish to use error control, we have to accept the presence of these compensator zeros that depend on the choice of $\mathbf{K}$ and $\mathbf{L}$ and over which we have no direct control. For low-order cases this results, as we said before, in a lead compensator as part of a unity feedback topology.

Let us now summarize our findings on the effect of introducing the reference input. When the reference input signal is included in the controller, the overall transfer function of the closed-loop system is

Transfer function for the closed-loop system when reference input is included in controller

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_s \gamma(s) b(s)}{\alpha_e(s)\alpha_c(s)}, \tag{7.178}$$

where K_s is the total system gain and $\gamma(s)$ and $b(s)$ are monic polynomials. The polynomial $\alpha_c(s)$ results in a control gain $\mathbf{K}$ such that $\det[s\mathbf{I} - \mathbf{F} + \mathbf{GK}] = \alpha_c(s)$. The polynomial $\alpha_e(s)$ results in estimator gains $\mathbf{L}$ such that $\det[s\mathbf{I} - \mathbf{F} + \mathbf{LH}] = \alpha_e(s)$. Because as designers we get to choose $\alpha_c(s)$ and $\alpha_e(s)$, we have complete freedom in assigning the poles of the closed-loop system. There are three ways to handle the polynomial $\gamma(s)$: We can select it so that $\gamma(s) = \alpha_e(s)$ by using the implementation of Fig. 7.38(b), in which case $\mathbf{M}/\bar{N}$ is given by Eq. (7.166); we may accept $\gamma(s)$ as given by Eq. (7.177), so that error control is used; or we may give $\gamma(s)$ arbitrary coefficients by selecting $\mathbf{M}/\bar{N}$ from Eq. (7.171). It is important to point out that the plant zeros represented by $b(s)$ are not moved by this technique and remain as part of the closed-loop transfer function unless α_c or α_e are selected to cancel some of these zeros.

7.7.2 Selecting the Gain

We now turn to the process of determining the gain $\bar{N}$ for the three methods of selecting $\mathbf{M}$. If we choose method 1, the control is given by Eq. (7.167a) and $\hat{x}_{ss} = x_{ss}$. Therefore, we can use either $\bar{N} = N_u + \mathbf{K}\mathbf{N}_x$, as in Eq. (7.81), or