Method of moments

Say rv $X \sim N(\mu, \sigma^2)$

1^{st} population moment $= E(X) = \mu$

Sample moment (1^{st}) $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

$$\hat{\mu} = \bar{X}$$

for σ^2

$$\sigma^2 = V(X) = E(X^2) - [E(X)]^2$$

$$\hat{\sigma}^2 = m_2 - (m_1)^2$$

sample moment

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \qquad m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\bar{x})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \cdot n \cdot (\bar{x})^2$$

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$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - n.\bar{x}.\bar{x}\right]$$

$$\underbrace{\bar{x}.n = \sum_{i=1}^{n} x_i}$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - \bar{x}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i(x_i - \bar{x})\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} (x_i - \bar{x})^2\right] \quad S^2 = \frac{\sum (x_i - \bar{x})^2}{(n-1)}$$

$$\hat{\sigma}^2 = \left[\frac{(n-1)}{n} S^2\right]$$

For any infinite population,

$$E(S^2) = \sigma^2$$

$$E(\hat{\sigma}^2) = \left[\frac{(n-1)}{n}\sigma^2\right] \quad \left(-\frac{\sigma^2}{n}\right) \rightarrow \text{Bias}$$

$$B(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2$$

$$= \frac{(n-1)}{n}\sigma^2 - \sigma^2$$

$$= \frac{n-1}{n}\sigma^2 - \frac{n}{n}\sigma^2$$

$$\underline{B(\hat{\sigma}^2) = \frac{-\sigma^2}{n}}$$

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Ex. 22.

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta} & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$\theta > -1$$

a) 1^{st} moment of the sample

$$m_1 = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{10}\sum_{i=1}^{10} x_i = 0.8$$

$$E(X) = 0.8$$

1st population moments

$$= E(X) = \int_{R_X} x . f(x;\theta)$$

$$= \int_0^1 x . (\theta+1) . x^{\theta} dx = \int_0^1 (\theta+1) x^{\theta+1} dx$$

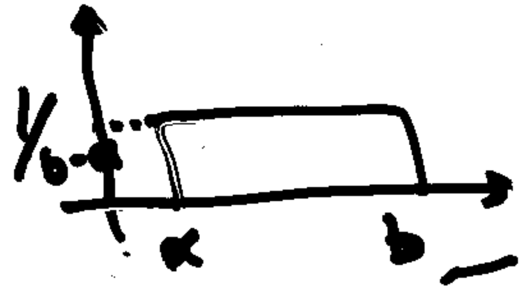
$$= \left(\frac{\theta+1}{\theta+2} . x^{\theta+2} \right) \Big|_0^1 = \frac{\theta+1}{\theta+2} - 0$$

$$E(X) = \frac{\theta+1}{\theta+2} = 0.8, \boxed{\theta = 3}$$

$$f(x;3) = (3+1) x^3$$

$$= \underline{4x^3}, \quad 0 \le x \le 1$$

let $X \sim U(a, b)$



$\bar{X} = 50.00$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \left(\neq \frac{\sum (x_i-\bar{x})^2}{(n-1)}\right)$$

$= 5.00$ Find the estimators for a & b.

$E(X)$, $V(X)$ $\Rightarrow$ for $U(a,b)$ $\Rightarrow$ $E(X) = \frac{b+a}{2}$, $V(X) = \frac{(b-a)^2}{12}$

$$\underbrace{\frac{b+a}{2}}_{1^{st}\text{ moment of population}} = \underbrace{50.00}_{1^{st}\text{ moment sample}}$$

$$V(X) = \underbrace{E(X^2) - [E(X)]^2}_{\frac{(b-a)^2}{12}} = \underbrace{m_2 - m_1^2}_{5.00}$$

$$\frac{(b-a)^2}{12} = 5.00$$

$$\frac{b+a}{2} = 50.00$$

$$b+a = 100.00$$

$$(b-a)^2 = 60$$

$$b-a = 7.746$$

$$b+a = 100.00$$

$$2b = 107.746$$

$$b = 53.87$$

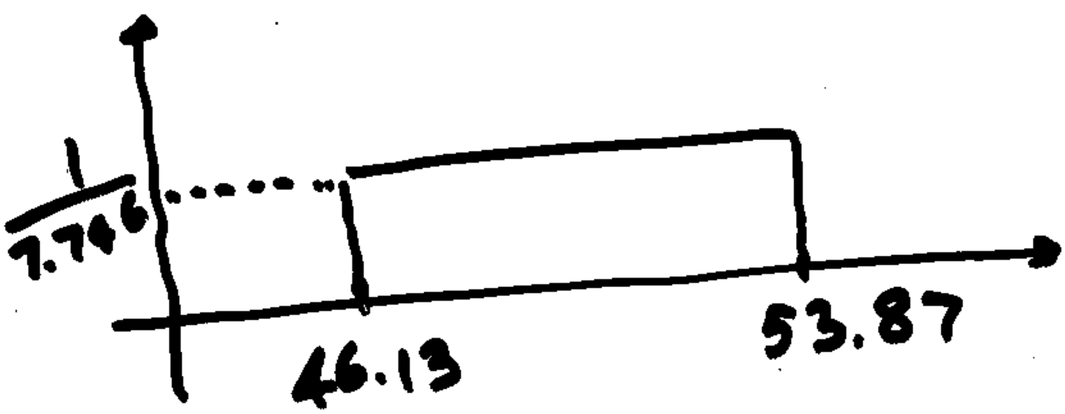
$$b-a = 7.746$$

$$53.87 - a = 7.746$$

$$a = 46.124$$

$$a = 46.13$$

$$X \sim U(46.13, 53.87)$$



For $\mathcal{U}(a,b)$

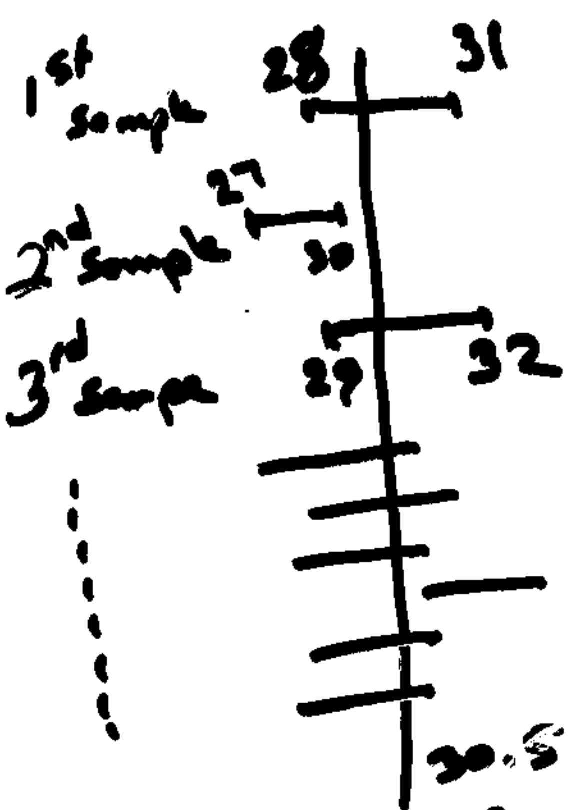
$$V(X) = \frac{(b-a)^2}{12}$$

$$= \underbrace{E(X^2)}_{\int_a^b x^2 \cdot \frac{1}{b-a}} - [E(X)]^2$$

$$\int_a^b x^2 \cdot \frac{1}{b-a} - \left(\frac{a+b}{2}\right)^2$$

Confidence Interval

$\Rightarrow$ 95% CIs



real μ

unknown.

95% CIs

A lower 1-sided CI $[\theta_L, \infty]$

upper 1-sided CI $[-\infty, \theta_U]$

LSL = 1200 psi

$n = 25$
sample size

$\sum_{i=1}^{25} x_i = 31{,}500$ psi.

The real variance of the process σ^2 is assumed to be known

$\sigma^2 = 625$

$X \rightarrow$ breaking strength $\rightarrow$ LTB

A lower 1-sided CI for the population mean, μ.

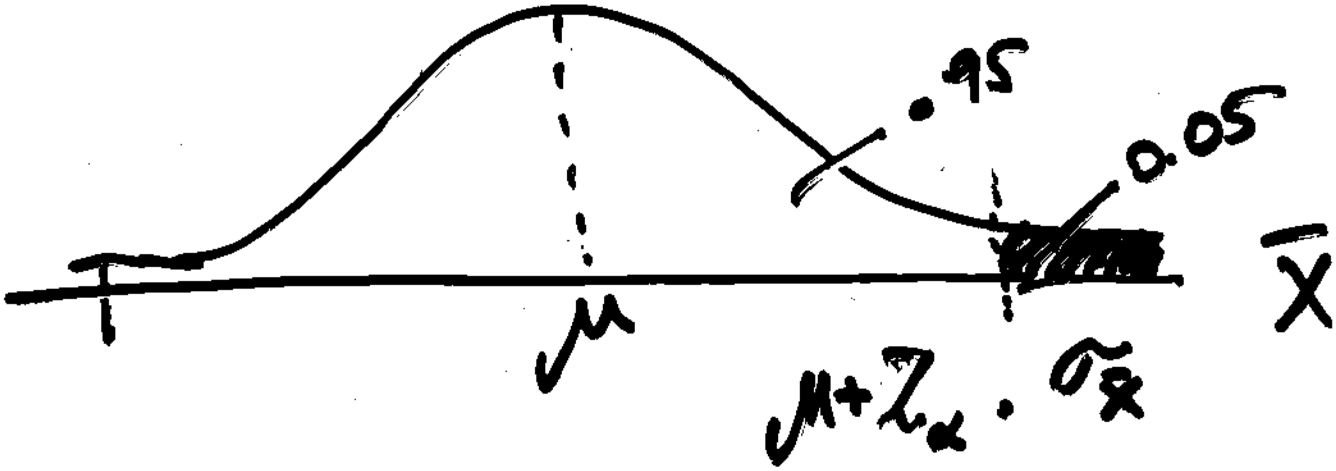
$1 - \alpha = 0.95$

$\alpha = 0.05$

$N(\mu, 625 \text{ psi}^2)$

$\bar{x} = 1260$ psi.

$\bar{X} \sim N(\mu, \frac{625}{25})$



$$\underbrace{P(\bar{X} \le \mu + Z_\alpha \cdot \sigma_{\bar{x}}) = 0.95}$$

$$P\left(\frac{\bar{X} - \mu}{\sigma_{\bar{x}}} \le Z_{\alpha=0.05}\right) = 0.95$$

$$\bar{X} - \mu \le Z_{.05} \cdot \sigma_{\bar{x}}$$

$$\rightarrow P(\bar{X} - Z_{.05} \cdot \sigma_{\bar{x}} \le \mu) = 0.95$$

$$P(\mu \ge \bar{X} - Z_{.05} \cdot \sigma_{\bar{x}}) = 0.95$$

$\bar{X}$: From the sample. $Z_{.05}$: From table. $\sigma_{\bar{x}}$: known.

$Z_{.05} = 1.645$

$$P\left(\mu \ge \bar{X} - (1.645)\left(\sqrt{\frac{625}{25}}\right)\right.$$

$$P(\mu \geq \bar{X} - (1.645)(5)) = 0.95$$

$$P(\mu \geq \bar{X} - 8.225) = 0.95$$

For the given sample,

$$P(\mu \geq 1260 - 8.225)$$

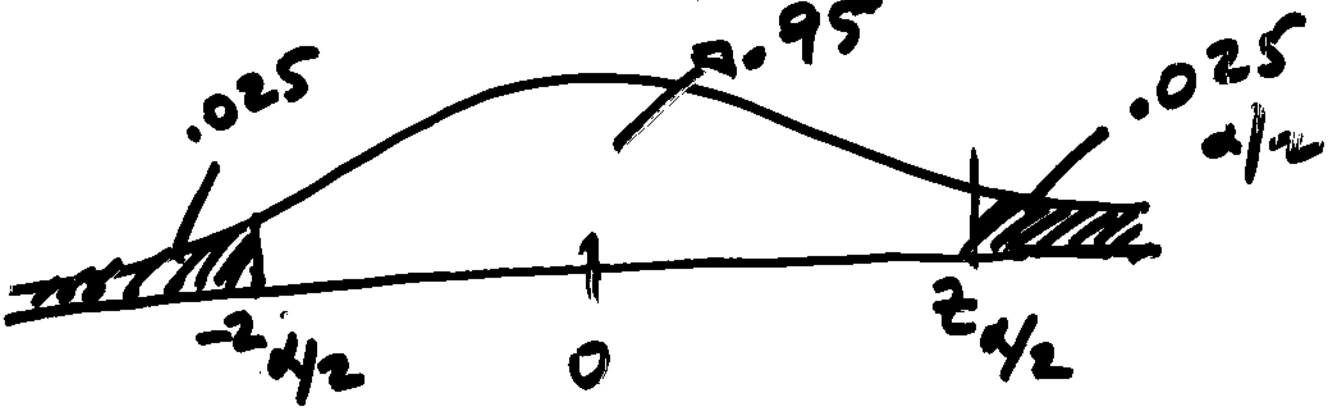
$$P(\mu \geq 1251.775) = 0.95$$

Lets take the same sample, and assume X is NTB

$\Rightarrow$ a 2-sided CI is needed for the mean μ.

$\alpha = 0.05$

$$P\left(-Z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} \leq Z_{\alpha/2}\right)$$



$Z_{.025} = 1.96$

$$P\left(-1.96 \le \frac{\bar{x}-\mu}{\sigma_{\bar{x}}} \le 1.96\right) = 0.95$$

$$P\left(-1.96\sigma_{\bar{x}} - \bar{x} \le -\mu \le 1.96\sigma_{\bar{x}} - \bar{x}\right)$$

$$P\left(1.96\sigma_{\bar{x}} + \bar{x} \ge \mu \ge -1.96\sigma_{\bar{x}} + \bar{x}\right)$$

$$P\left(\underbrace{-1.96}_{z_{\alpha/2}}\sigma_{\bar{x}} + \bar{x} \le \mu \le \underbrace{1.96}_{z_{\alpha/2}}\sigma_{\bar{x}} + \bar{x}\right)$$

The probability = 0.95

A two-sided $\underset{(1-\alpha)}{CI}$ for μ:

$$\left(\bar{x} \pm Z_{\alpha/2} \cdot \sigma_{\bar{x}}\right)$$

For our example:

$\bar{X} = 1260, \quad \sigma_{\bar{X}} = 5$

A 95% two-sided CI for μ

$$1260 \pm Z_{.025} \cdot 5$$

$$1250.2 \leq \mu \leq 1269.8$$

$1250.2 \downarrow 1260 - Z_{.025} \cdot 5$

$1269.8 \downarrow 1260 + Z_{.025} \cdot 5$

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if we don't know the population σ^2 we estimate by $S^2 = \frac{CSS}{(n-1)}$

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

$n > 40$

$\Rightarrow$ for $n > 40$ still practically equal to standard normal.

if $n < 40$

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

if $n < 40$, has the t distribution with $\nu = n-1$ degrees of freedom.