Joint distribution for 2 discrete rvs.

In-class example.

X_1	mpmf(X_1)
1	.10
2	.25
3	.35
4	.30

$$E(X_1) = (1)(.10) + (2)(.25) + (3)(.35) + (4)(.30)$$
$$= 2.85 \text{ units/day.}$$

X_2	mpmf(X_2)
1	.20
2	.45
3	.35

$$E(X_2) = (1)(.20) + (2)(.45) + (3)(.35)$$
$$= 2.15 \text{ units/day.}$$

$$V(X_1) = E(X_1^2) - [E(X_1)]^2$$

X_1^2	
1	.10
4	.25
9	.35
16	.30

$$E(X_1^2) = 9.05$$
$$V(X_1) = 9.05 - (2.85)^2$$
$$V(X_1) = 0.9275$$

Conditional Pr.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$p_2(x_2 \mid x_1) = \frac{p(x_1, x_2)}{p_2(x_2)}$$

From our example:

$$p_2(X_2 \mid X_1 = 1) = \frac{p(1, X_2)}{0.10}$$

$$p_1(1) \rightarrow 0.10$$

$$p_2(X_2 \mid X_1 = 1) = \begin{cases} X_2 = 1 & p_2(1 \mid X_1 = 1) = \dfrac{p(1,1) = 0.01}{0.10} \\ X_2 = 2 & = \dfrac{p(1,2)}{0.10} \rightarrow 0.05 \\ X_2 = 3 & = \dfrac{p(1,3)}{0.10} \rightarrow 0.04 \end{cases}$$

$$p_2(X_2 \mid X_1 = 1) = \begin{cases} 0.01/0.1 & X_2 = 1 \\ 0.05/0.1 & X_2 = 2 \\ 0.04/0.1 & X_2 = 3 \end{cases}$$

$$p_2(X_2 \mid X_1 = 1) = \begin{cases} 0.1 & X_2 = 1 \\ 0.5 & X_2 = 2 \\ 0.4 & X_2 = 3 \end{cases}$$

$$+ \quad \overline{1.0}$$

$$p_1(x_1 \mid X_2 = 3) = \frac{p(x_1, 3)}{p_2(3)} = \frac{p(x_1, 3)}{0.35}$$

$$p_1(x_1 \mid X_2 = 3) = \begin{cases} \frac{p(1,3)}{.35} = \frac{0.04}{0.35} & x_1 = 1 \\ .10/0.35 & x_1 = 2 \\ .10/0.35 & x_1 = 3 \\ .11/0.35 & x_1 = 4 \end{cases}$$

$$+ \overline{\quad 1.0 \quad}$$

Conditional Expectation

Given $X_1 = 1$ $\quad E(X_2 \mid X_1 = 1) = ?$

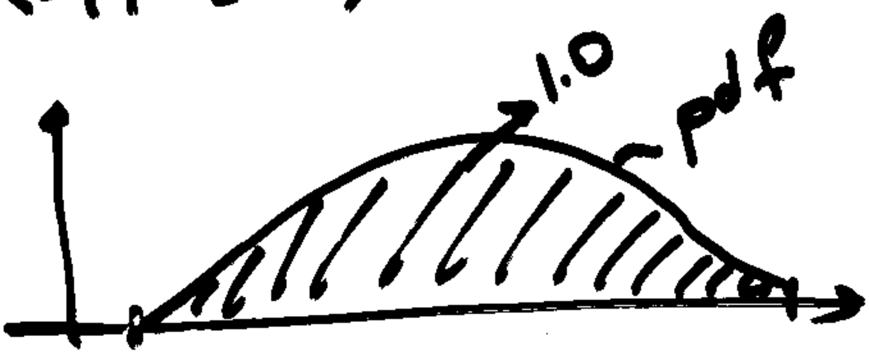
$$E(X_2 \mid X_1 = 1) = (1)(0.10) + (2)(0.50) + (3)(0.40)$$

$$E(X_2 \mid X_1 = 1) = 2.30$$

Given $X_2 = 3$ $\quad E(X_1 \mid X_2 = 3) = ?$

$$(1)(4/35) + (2)(10/35) + (3)(10/35) + (4)(11/35)$$

$$E(X_1 \mid X_2 = 3) = 2.80$$



Continuous Bivariate RV

$$f(x_1, x_2) = C(6 - x_1 - x_2) \quad \begin{array}{l} 0 \leq x_1 \leq 2 \\ 2 \leq x_2 \leq 4 \end{array}$$

X_1 = surface tension
X_2 = acidity

$$\int_{x_2=2}^{4} \int_{x_1=0}^{2} C(6 - x_1 - x_2)\, dx_1\, dx_2$$

$$C \int_{x_2=2}^{4} \left(6x_1 - \frac{x_1^2}{2} - x_1 x_2\right)\Bigg|_0^2 dx_2$$

$$C \int_2^4 (\overbrace{12 - 2}^{10} - 2x_2)\, dx_2 = C\left(10x_2 - x_2^2\right)\Big|_2^4$$

$$= C\left[(24) - (16)\right] = 8C = \text{Volume.}$$

for $f(x_1, x_2) = C(6 - x_1 - x_2)$ to be a pdf (a joint pdf), Volume needs to be 1. $C = 1/8$ $\quad f(x_1, x_2) = 1/8\,(6 - x_1 - x_2)$

$P(0 \leq X_1 \leq 1 \text{ and } 2 \leq X_2 \leq 3) = ?$

$$\int_2^3 \int_0^1 \frac{1}{8}(6 - x_1 - x_2)\,dx_1\,dx_2 = \frac{1}{8}\int_2^3 \left(6x_1 - \frac{x_1^2}{2} - x_1 x_2\right)\Big|_0^1 dx_2$$

$$= \frac{1}{8}\int_2^3 \left(6 - \frac{1}{2} - x_2 - 0\right) dx_2 = \frac{1}{8}\left[\frac{11}{2}x_2 - \frac{x_2^2}{2}\right]_2^3$$

$$= \frac{1}{8}\left[\left(\frac{33}{2} - \frac{9}{2}\right) - \left(\frac{22}{2} - \frac{4}{2}\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{11}{2} - \frac{5}{2}\right)\right] = \left(\frac{1}{\cancel{8}_4}\right)\left(\frac{\cancel{6}^3}{2}\right) = \frac{3}{8} = 0.375$$

$P(X_1 + X_2 \leq 4) = ?$ $\quad P(X_2 \leq 4 - X_1)$

$P(X_1 \leq 4 - X_2)$

$$\int_{x_2=2}^{4} \int_{x_1=0}^{4-x_2} \frac{1}{8}(6 - x_1 - x_2)\,dx_1\,dx_2$$

$$= \frac{1}{8}\int_2^4 \left(6x_1 - \frac{x_1^2}{2} - x_1 x_2\right)\Big|_0^{4-x_2}$$

$$\frac{1}{8}\int_2^4 \left(24 - 6x_2 - \frac{\cancel{16} - \cancel{8x_2} + x_2^2}{2} - \cancel{4x_2} + x_2^2\right)$$

$$\frac{1}{8}\int_2^4 \left(16 - 6x_2 + \frac{x_2^2}{2}\right)dx_2$$

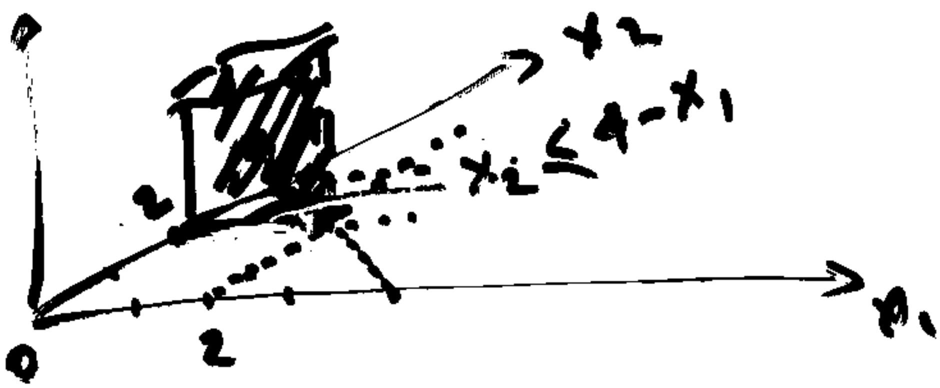
$$\frac{1}{8}\left(16x_2 - 3x_2^2 + \frac{x_2^3}{6}\right)\Big|_2^4$$

$$\frac{1}{8}\left[\left(64 - 48 + \frac{64}{6}\right) - \left(32 - 12 + \frac{8}{6}\right)\right]$$

$$\frac{1}{8}\left[\left(16 + \frac{64}{6}\right) - \left(20 + \frac{8}{6}\right)\right]$$

$$= \frac{1}{8}\left[\frac{160}{6} - \frac{128}{6}\right] = \frac{1}{8}\left(\frac{32}{6}\right) = \frac{32}{48} = \underline{0.6667}$$

correct



Marginal pdf

$$f_1(x_1) = \int_{R_{X_2}} f(x_1, x_2)\, dx_2$$

$$= \int_2^4 \tfrac{1}{8}(6 - x_1 - x_2)\, dx_2 = \tfrac{1}{8}\left(6x_2 - x_1 x_2 - \frac{x_2^2}{2}\right)\Bigg|_2^4$$

$$= \tfrac{1}{8}\left[\left(24 - 4x_1 - \frac{16}{2}\right) - (12 - 2x_1 - 2)\right]$$

$$= \frac{6 - 2x_1}{8} = \frac{3 - x_1}{4} \quad 0 \le x_1 \le 2$$

$$\int_0^2 \underbrace{\frac{3 - x_1}{4}}_{\text{mpdf for } x_1} dx_1 = \left(\frac{3}{4}x - \frac{x_1^2}{8}\right)\Bigg|_0^2 = \frac{6}{4} - \frac{2}{4} = \frac{4}{4}$$

$$= 1 \checkmark$$

$$E(X_1) = \int_0^2 x_1 \cdot \left(\frac{3 - x_1}{4}\right) dx_1$$

$$\left(\frac{3}{8}x_1^2 - \frac{x_1^3}{12}\right)\Bigg|_0^2 = \frac{3}{2} - \frac{8}{12}$$

$$= 10/12 = \underline{\underline{5/6}}$$