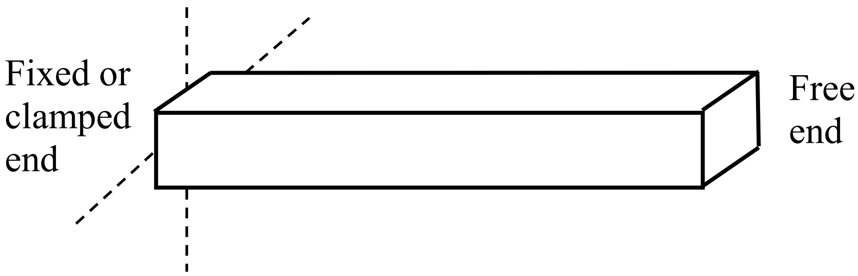
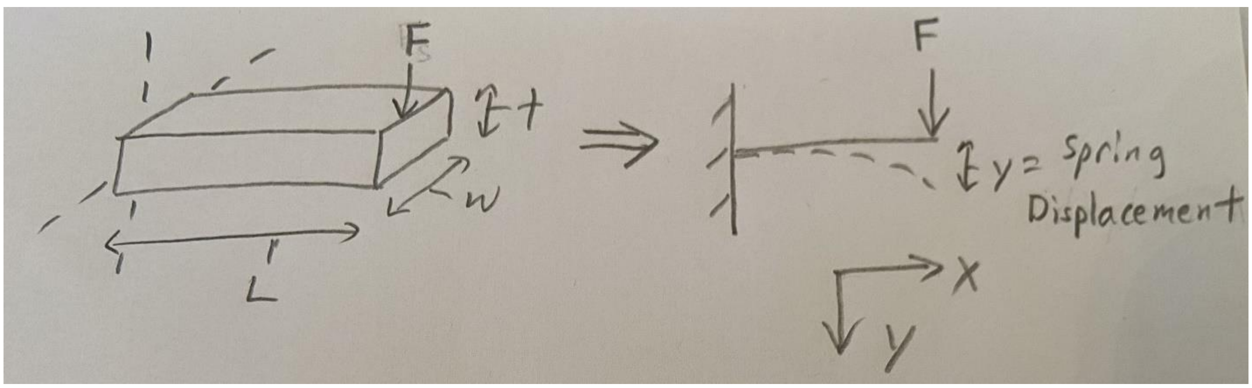
Lecture 8/24/22

Springs

Consider the simplest beam spring:



This type of beam is called a cantilever.



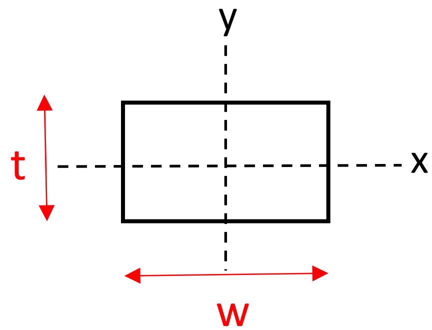
From Beam Theory:

$$y(x) = \frac{F}{6EI}(3x^2L - x^3)$$

At x = L : $y(L) = \frac{FL^3}{3EI}$

I is the 2nd moment of area or the moment of inertia.

For a rectangular cross-sectional beam, such as what we have:



$$I = I_z = \frac{wt^3}{12}$$

Definition

t (thickness): spring dimension in the direction of displacement.
w (width): spring dimension perpendicular to the direction of displacement.

Therefore: $y(L) = \frac{4FL^3}{wt^3E}$

Associated with a spring is spring force, F_s

$F_s = kd$, d = displacement, y(L)

k = spring constant, [k] = N/m

$k = \frac{F_s}{d} = \frac{Ewt^3}{4L^3}$ → spring geometry dependent

Observations about the spring constant

$k \propto w$ → if $w_{new} = 2w_{old}$: $k_{new} = 2k_{old}$

$k \propto t^3$ → if $t_{new} = 2t_{old}$: $k_{new} = 8k_{old}$

$k \propto \frac{1}{L^3} \rightarrow$ if $L_{new} = 2L_{old}$: $k_{new} = \frac{k_{old}}{8}$

** A unit change in w has a much smaller effect on k than the same change in t or L. **

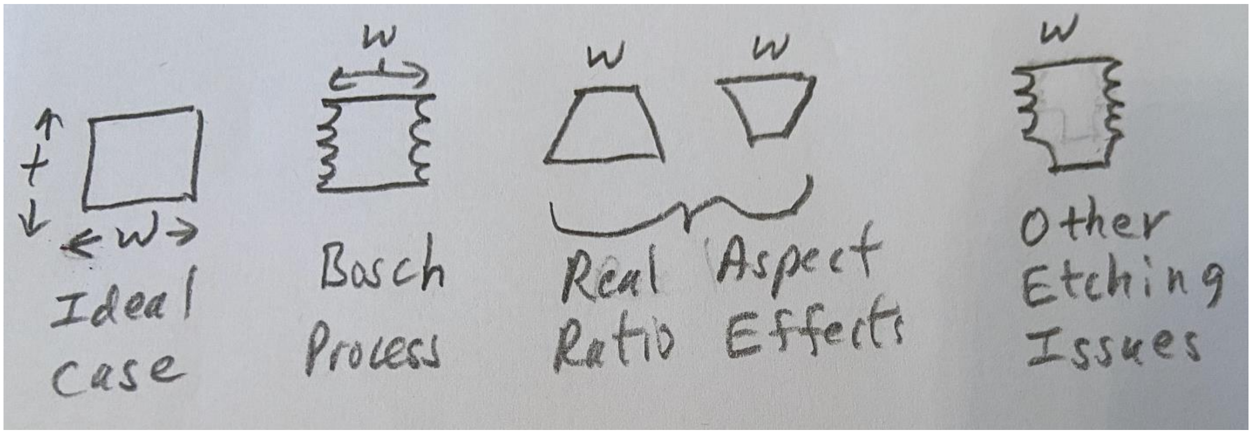
Spring Fabrication Issues in the SOI Fabrication Process

t → Device Layer thickness → very accurate

L → set by photolithography (L: usually large) → pretty accurate

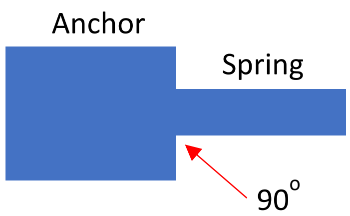
w → set by photolithography and etch process (w: usually small) → not very accurate

Cross-sectional drawings of spring elements:



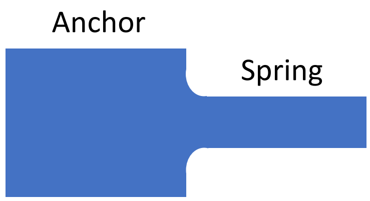
All of these non-ideal etching cases result in a non-ideal, non-constant w, resulting in the spring constant differing from the desired value. But since k is the least sensitive to changes in w, allowing w to be along the direction with the most variability minimizes the effects of fabrication tolerances on k.

Observe the anchor-spring attachment point below:



With the right angle turn at the spring-anchor attachment points, if this lines up with a Si crystal plane, it will be prone to micro cracks, which will propagate along that plane, resulting in the beam snapping off at the anchor wall.

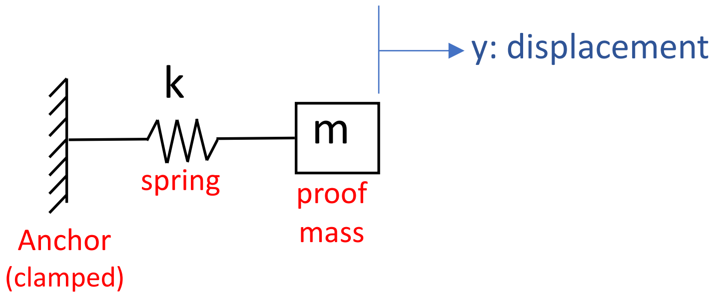
To mitigate this issue, round the corners at the spring-anchor attachments points:



Rounding the corners at the spring-anchor attachment points has minimal effect on the spring constant.

Modeling the Mass-Spring System Dynamically

Mechanical schematic diagram



$F_I \equiv$ Inertial Force = ma

$a = \frac{d^2y}{dt^2} = \ddot{y}$ (Note: $\dot{y} = \frac{dy}{dt} \equiv$ velocity)

$F_s \equiv$ Spring Force = ky

At equilibrium: $F_I + F_s = 0$

$m\ddot{y} + ky = 0 \rightarrow$ 2nd order linear differential equation with constant coefficients.

Let's pull the proof mass a displacement = y_o and let it go.

At t = 0s → initial condition: $y(t)|_{t=0} = y_o$

To solve, let's assume a solution of: $y(t) = Acos(\omega t)$

$\therefore \dot{y}(t) = -A\omega\sin(\omega t)$

And $\ddot{y}(t) = -A\omega^2\cos(\omega t)$

So plugging into $m\ddot{y} + ky = 0$ yields:

$-mA\omega^2 \cos(\omega t) + kAcos(\omega t) = 0$

Divide both sides by Acos(ωt), yielding:

$-m\omega^2 + k = 0$

Rearranging yields: $\omega = \sqrt{\frac{k}{m}} \equiv \omega_n \equiv$ natural frequency of the system

$\therefore y(t)|_{t=0} = y_o = Acos(\omega_n t)|_{t=0} = A$

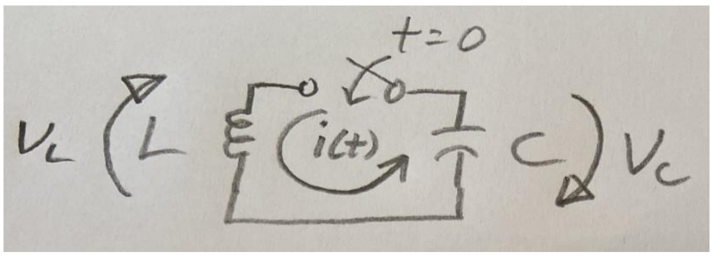
$\therefore\ y_o = A$

The solution is therefore: $y(t) = y_o \cos\left(\sqrt{\frac{k}{m}}t\right)$.

This mechanical system will oscillate forever with $f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ with an amplitude of y_o.

This system is lossless → all real systems have energy losses.

Analogous Electrical System



$V_L = L\frac{di}{dt}$ and $V_C = \frac{1}{C}\int_0^\infty i(t)dt$

At $t = 0^+$: $V_L + V_C = 0$

$\therefore\ L\frac{di}{dt} + \frac{1}{C}\int_0^\infty i(t)dt = 0$: An integro-differential equation

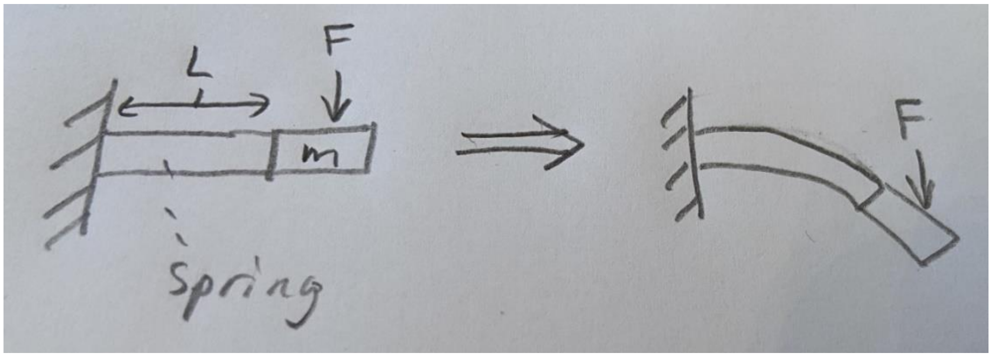
Therefore, differentiate both sides to realize a differential equation:

$L\ddot{i} + \frac{1}{C}i = 0$ → same form as: $m\ddot{y} + ky = 0$

Electrical – Mechanical System Equivalence

Electrical Parameters	**Mechanical Parameters**
L	m
1/C	k
R	mechanical losses

Other Spring (Suspension System) Considerations



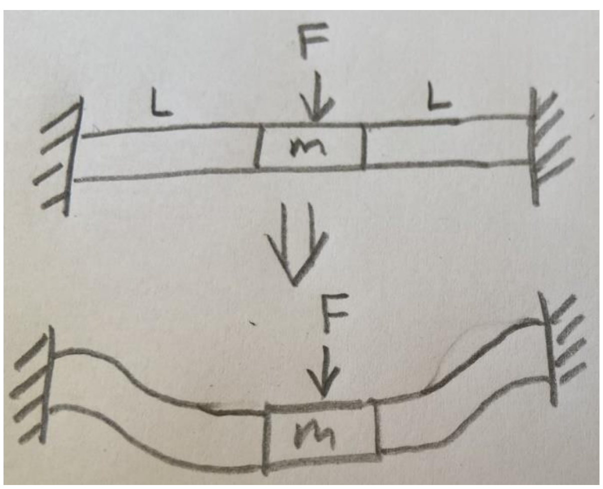
Observe that the proof mass has tilted while the spring has deformed due to the applied force, F.

This may or may not be desirable:

Capacitive detection: not desirable

Piezoresistive detection: OK

Consider this two-beam suspension system:



The proof mass, m, does not tilt now.

However, each spring element (beam) is now in tension AND is bent: this beam structure is now statically indeterminate (i.e. we cannot solve for the displacement with a simple beam equation). So, we must use a different approach.

Assume that the deflections are small compared to the spring length.

Therefore, use this approximation for the system spring constant:

$$k \approx \frac{N_{Leg}}{N_{Zig}} \frac{Ewt^3}{L^3}$$

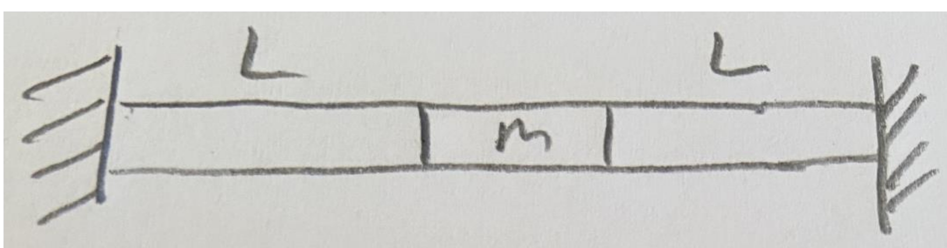
Where N_{Leg} = # Legs or spring elements

and N_{Zig} = # cutbacks (straight beam = 1, folded beam = 2, etc.)

Note: this CANNOT be used with the simple cantilever:

For the simple cantilever: $k = \frac{Ewt^3}{4L^3}$ → the multi-beam suspension system is stiffer.

<u>Example multi-beam suspension system:</u>



2 beams: $N_{Leg} = 2$, $N_{Zig} = 1$

Therefore $k \approx \frac{2Ewt^3}{L^3}$