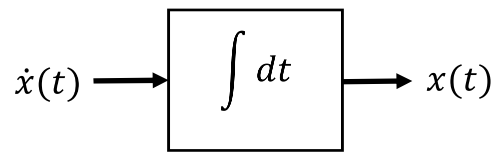
Thursday, 8/21/25

Simulating Dynamic Systems in MATLAB Simulink

Example: given $A\ddot{x} + B\dot{x} + Cx = f(t)$ (1) modelling a second order linear dynamic system

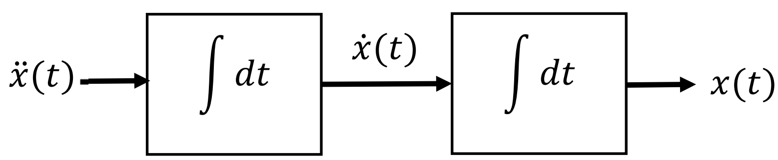
Realize that $x = \int \dot{x}\, dt$, which can be represented pictorially as:



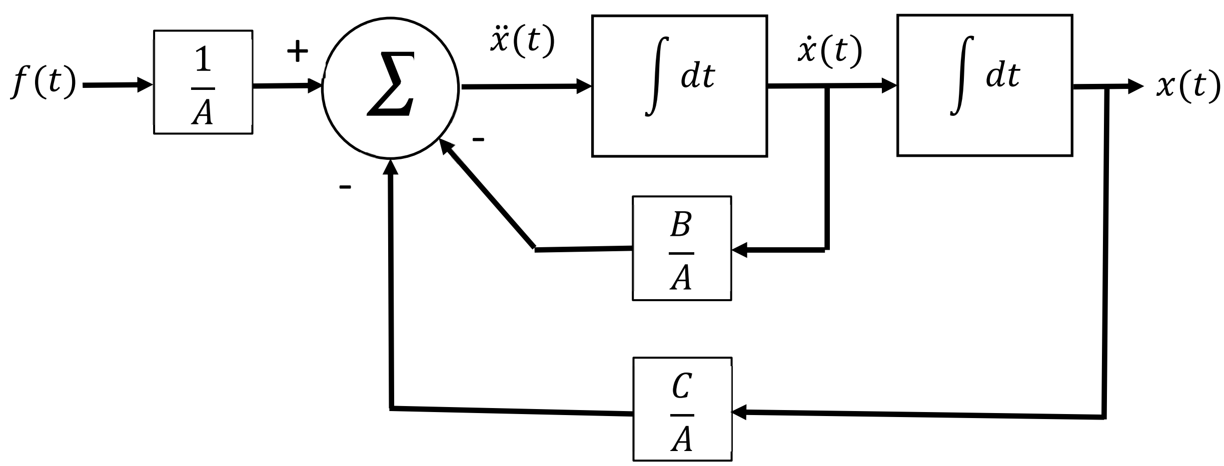
Rearrange (1) so that: $\ddot{x} = \frac{1}{A} f(t) - \frac{B}{A}\dot{x} - \frac{C}{A}x$ (2)

This form is very easy to implement in a simulation diagram using integrators

Begin with a chain of integrators representing all states:

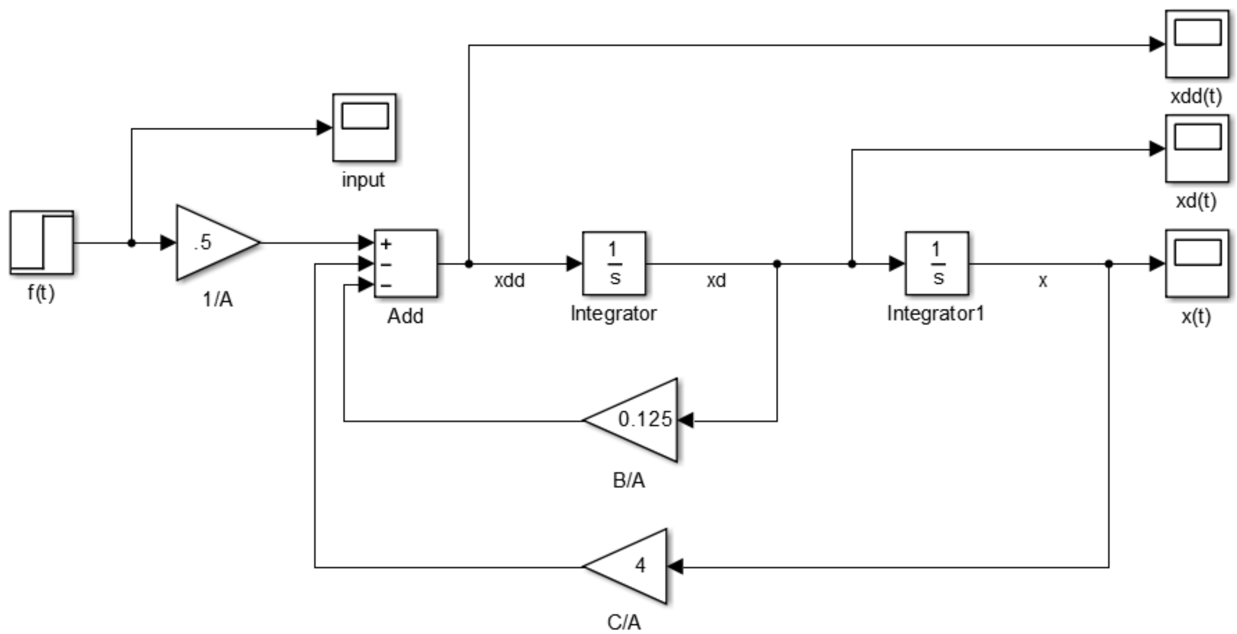


Then add a summing junction and feedback terms:

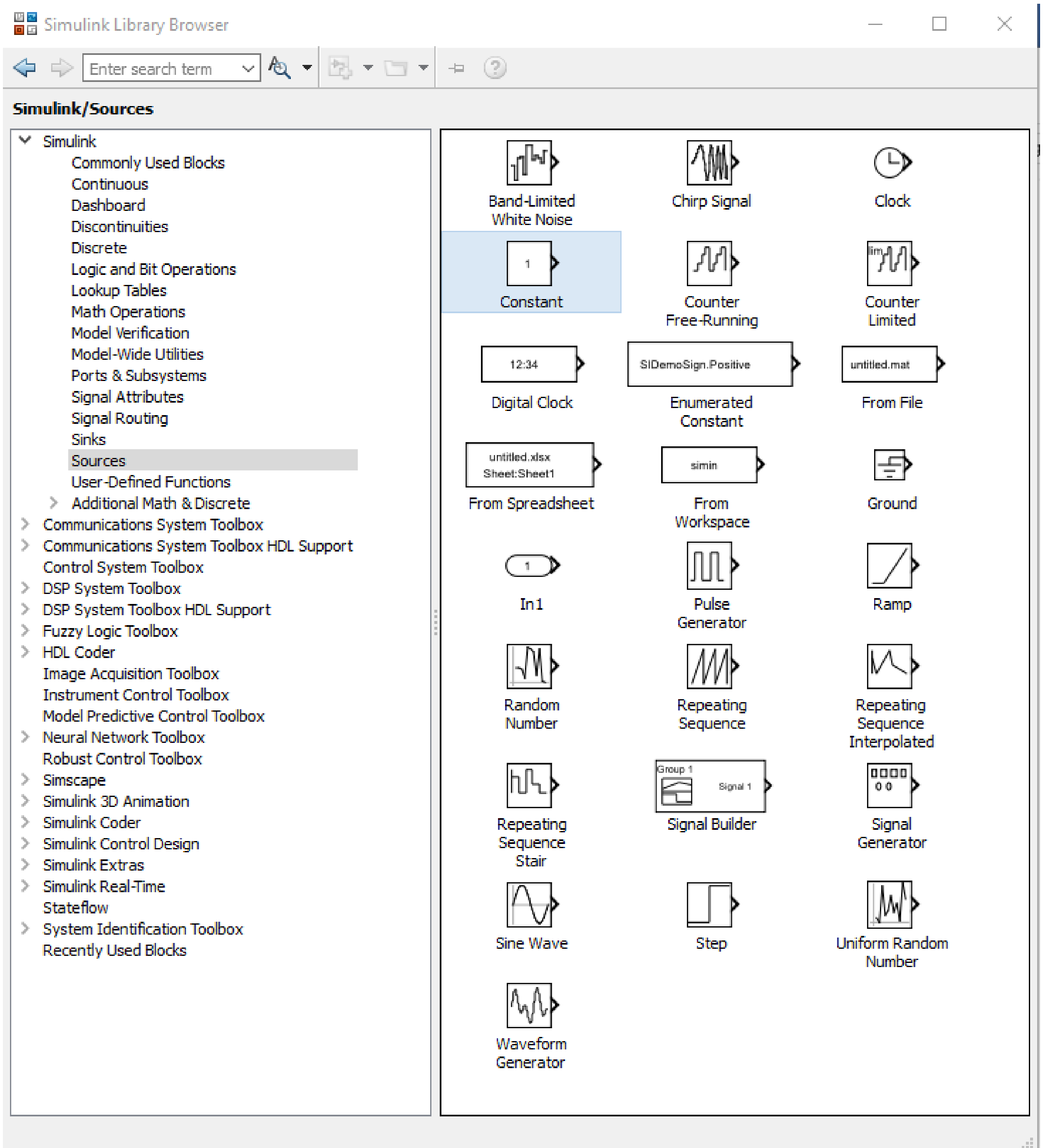


This can be built in MATLAB Simulink where the 1/s block is used for the integral block

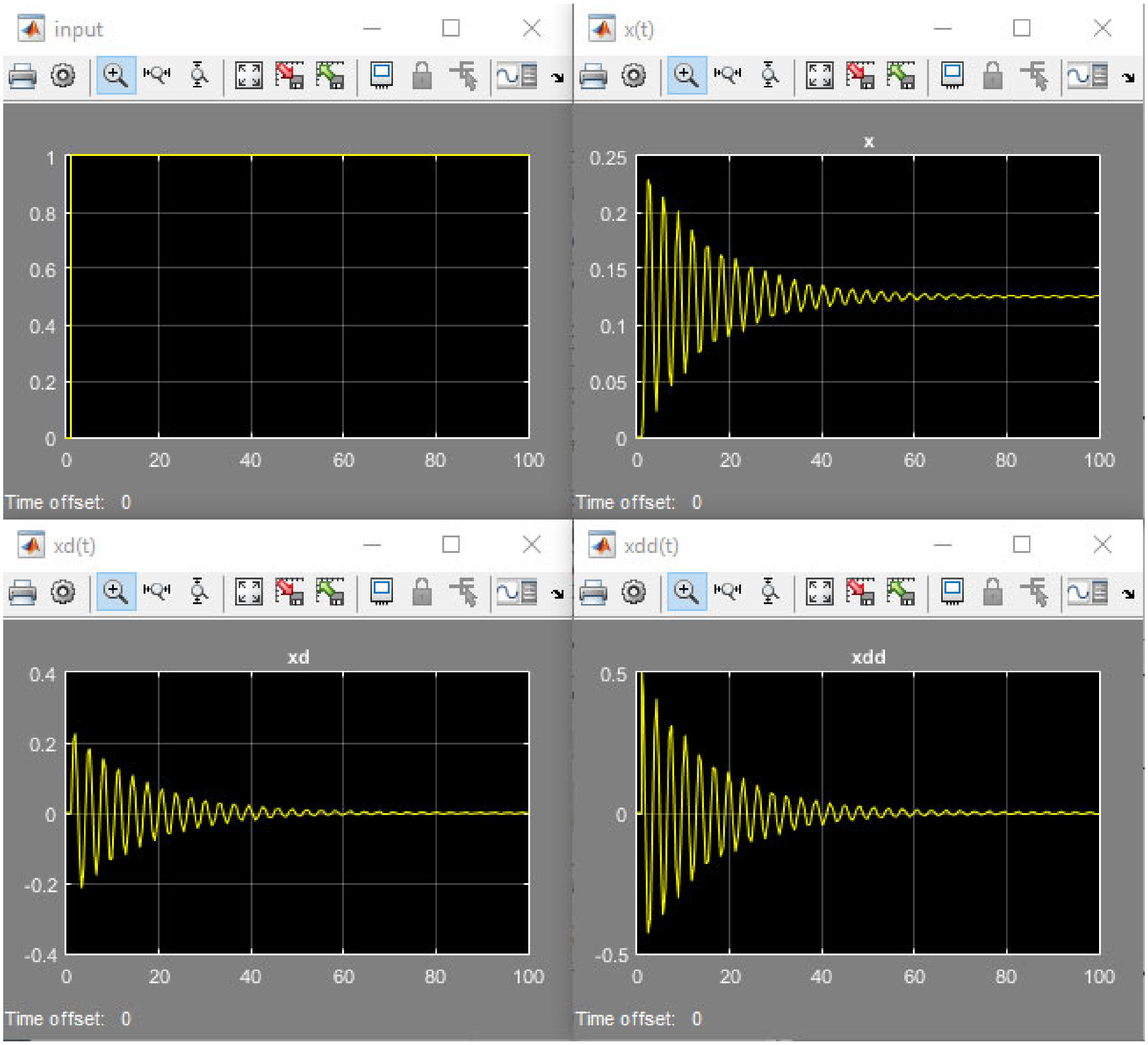
Example: $2\ddot{x} + 0.25\dot{x} + 8x = u(t-1)$



Simulink model of the simulation diagram

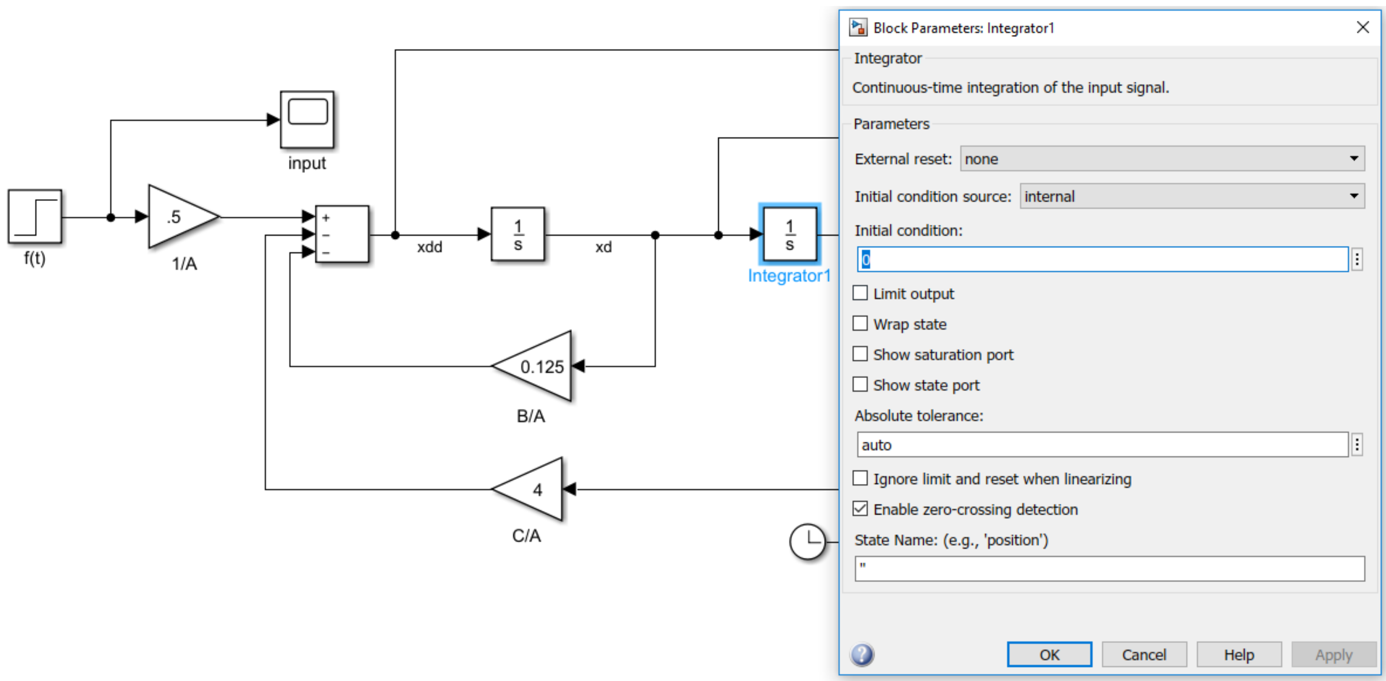


Simulink Library Browser

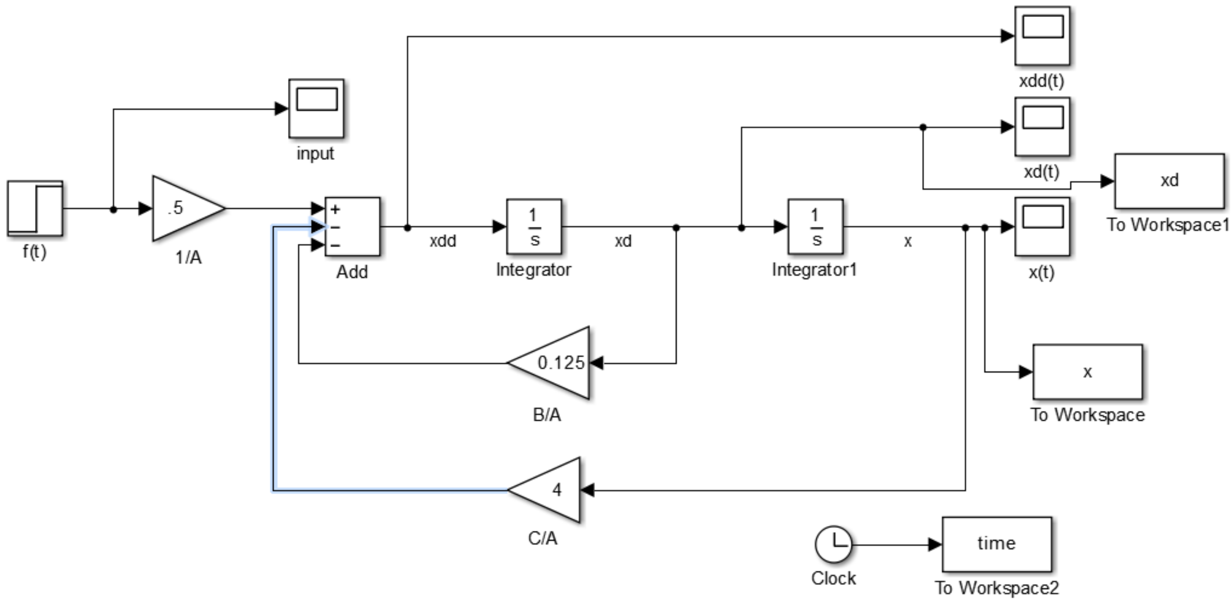

input
x(t)
x
xd(t)
xd
xdd(t)
xdd
Time offset: 0
Time offset: 0
Time offset: 0
Time offset: 0

Setting Initial Conditions:

Click on the integrator block and select the initial condition for the signal output by that integrator:

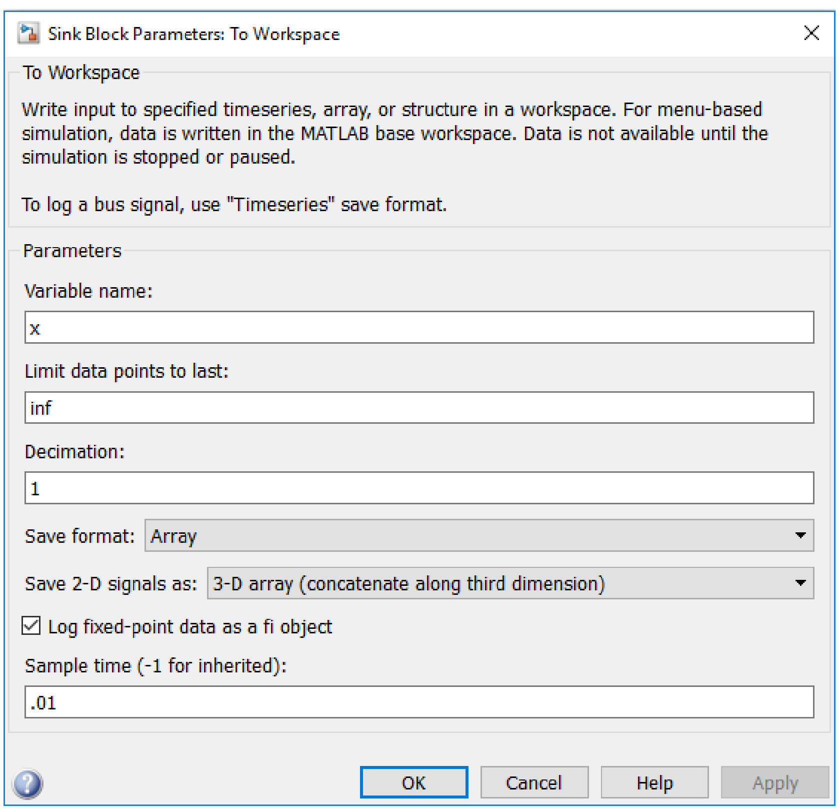


Bringing Simulink Data into the MATLAB Workspace



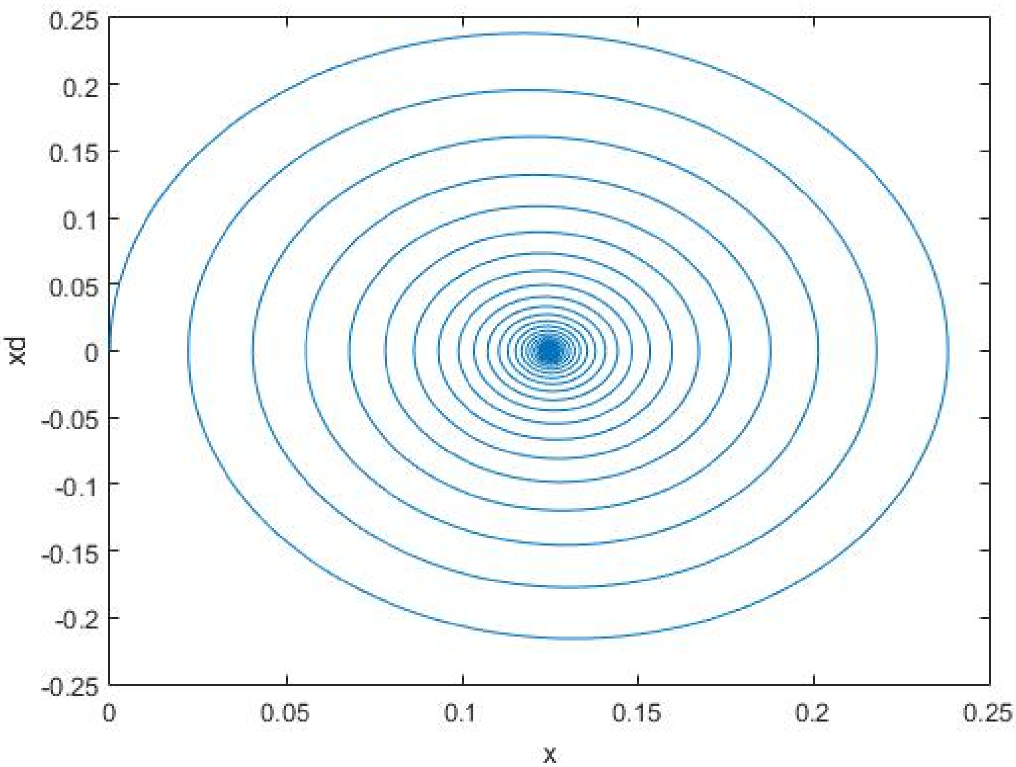
"To Workspace" blocks added to get data into the Matlab Workspace

Setting the parameters for each "To Workspace" block:



Then after running the simulation, you can process the Simulink data in the workspace or via an m-file.

Example: plot(x,xd)



This is a plot of x-dot vs. x. It is called a <u>phase plot</u> and is very useful in analyzing dynamical systems.

<u>Hand out Homework 1</u>:

Review of Second Order Dynamic Systems

Consider systems of the form: $A\ddot{x} + B\dot{x} + Cx = f(t)$

Example: mechanical spring-mass-damper system: $m\ddot{x} + c\dot{x} + kx = f(t)$

Often convenient to analyze using Laplace Transforms: $ms^2X(s) + csX(s) + kX(s) = F(s)$

Then $X(s) = \frac{F(s)/m}{s^2+s\frac{c}{m}+\frac{k}{m}} = \frac{F(s)/m}{s^2+s2\zeta\omega_o+\omega_o^2} = \frac{F(s)/m}{s^2+s\frac{\omega_o}{Q}+\omega_o^2}$, where:

ω_o is the natural frequency

Q is the quality factor

ζ is the damping ratio

finally, $x(t) = L^{-1}\left[\frac{F(s)/m}{s^2+s\frac{c}{m}+\frac{k}{m}}\right]$

$\zeta = 0$ or $Q \to \infty$: undamped system

$0 < \zeta < 1$ or $Q \to \infty > Q > ½$: underdamped system

$\zeta = 1$ or $Q = ½$: critically damped system

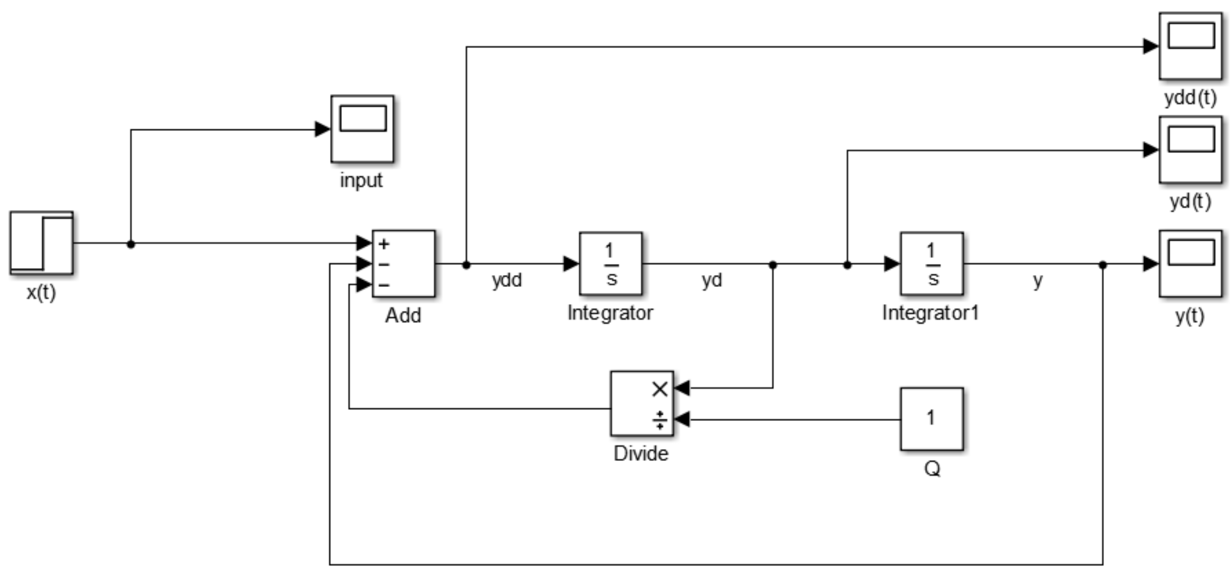
$\zeta > 1$ or $Q < ½$: overdamped system

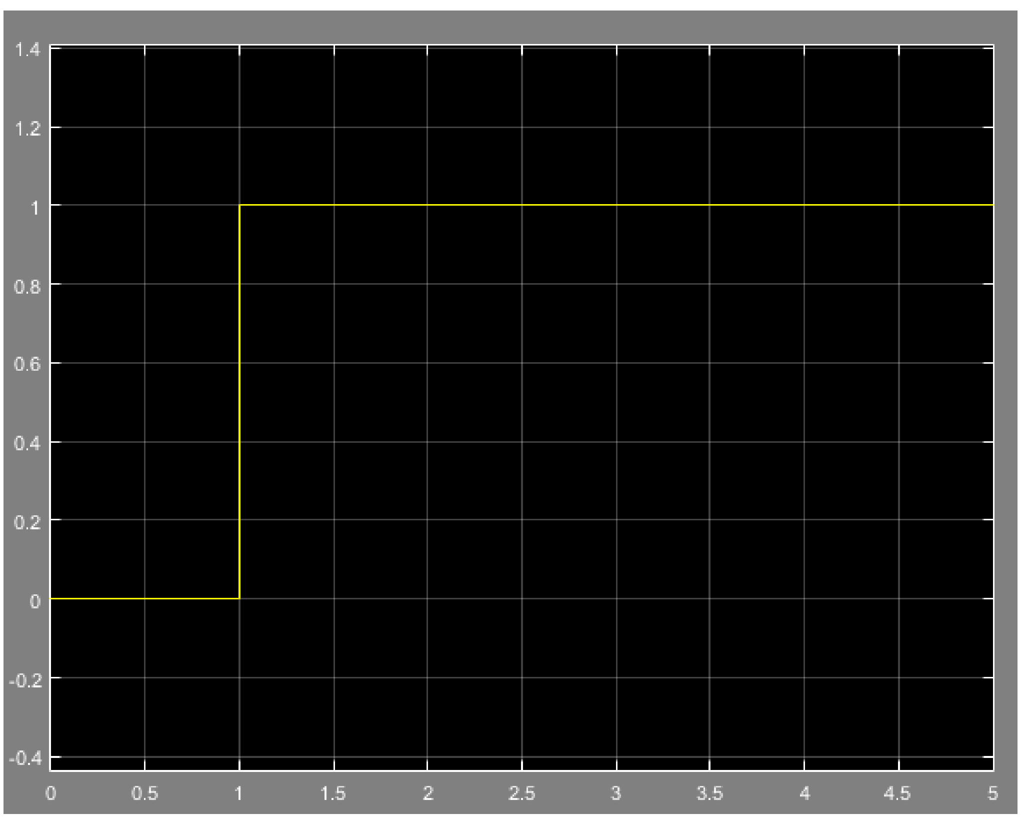
$\zeta = Q = 0.707$: maximally flat response (no resonant peak in the frequency domain)

Example. Consider this system with $\omega_o = 1$ rad/s:

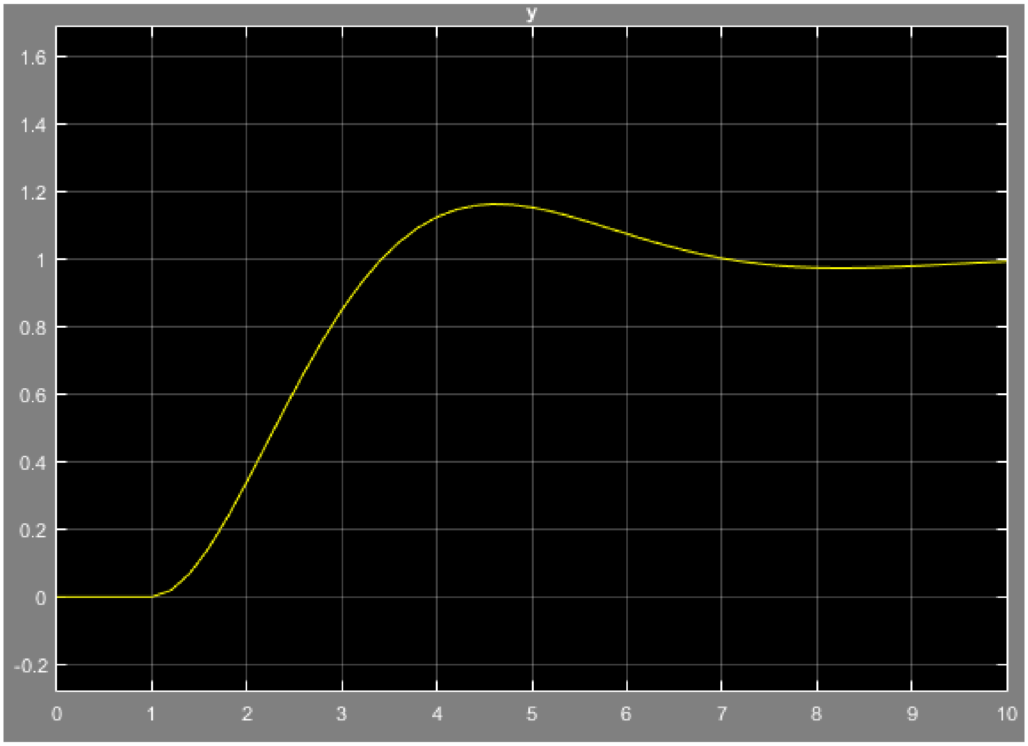
$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + s\frac{1}{Q} + 1}$$

Consider the Simulink model shown on next page with a step response:

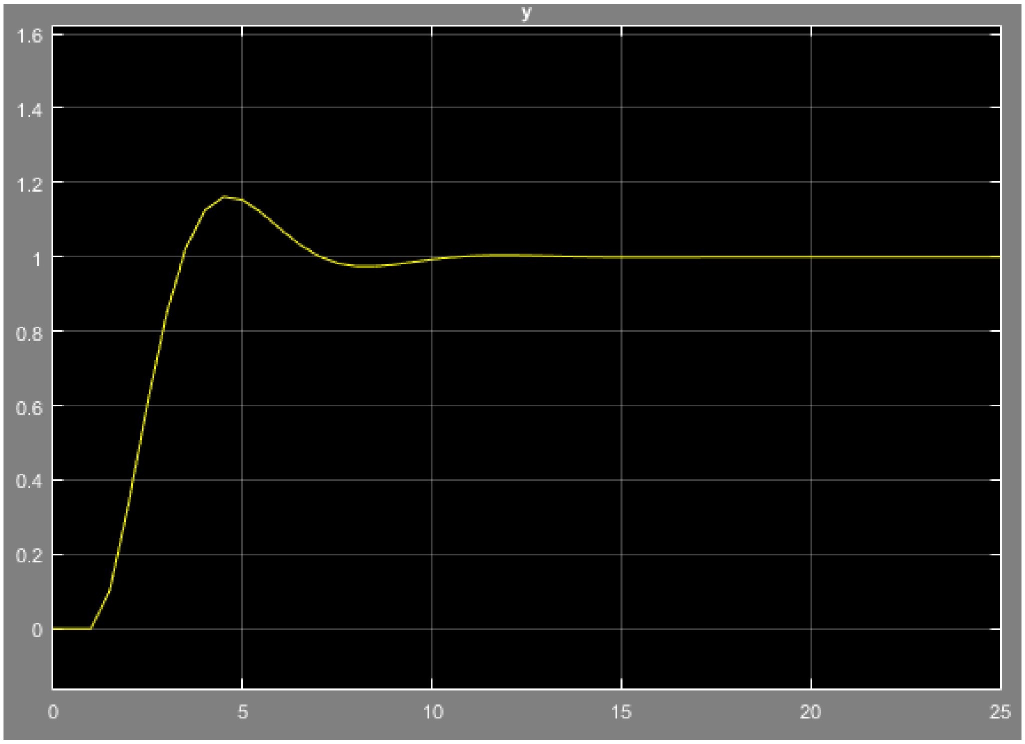

ydd(t)
input
yd(t)
x(t)
Add
ydd
1
s
Integrator
yd
1
s
Integrator1
y
y(t)
Divide
1
Q


1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

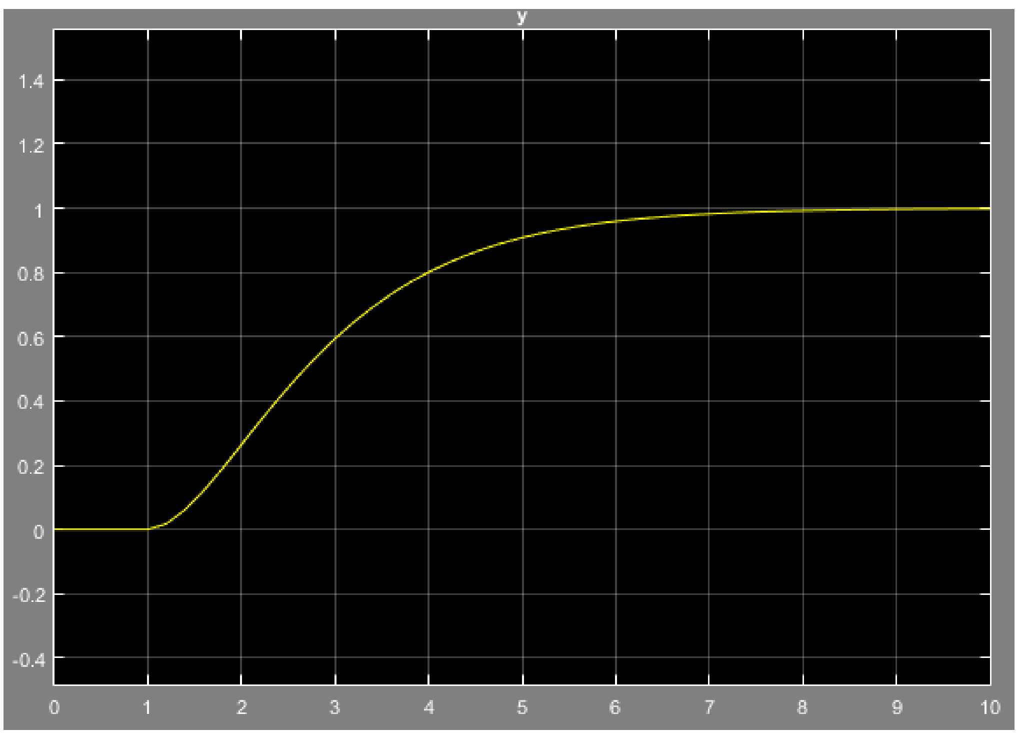
Input: x(t)



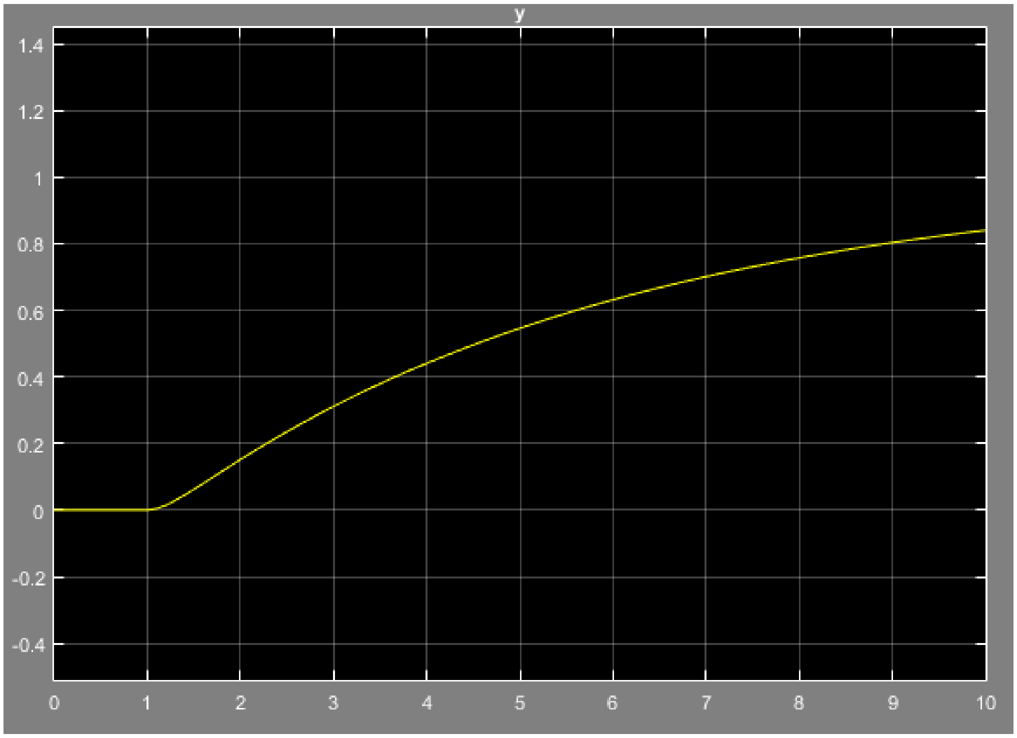
Output y(t) for Q = 1, underdamped. Observe the "ringing" at the resonant frequency.



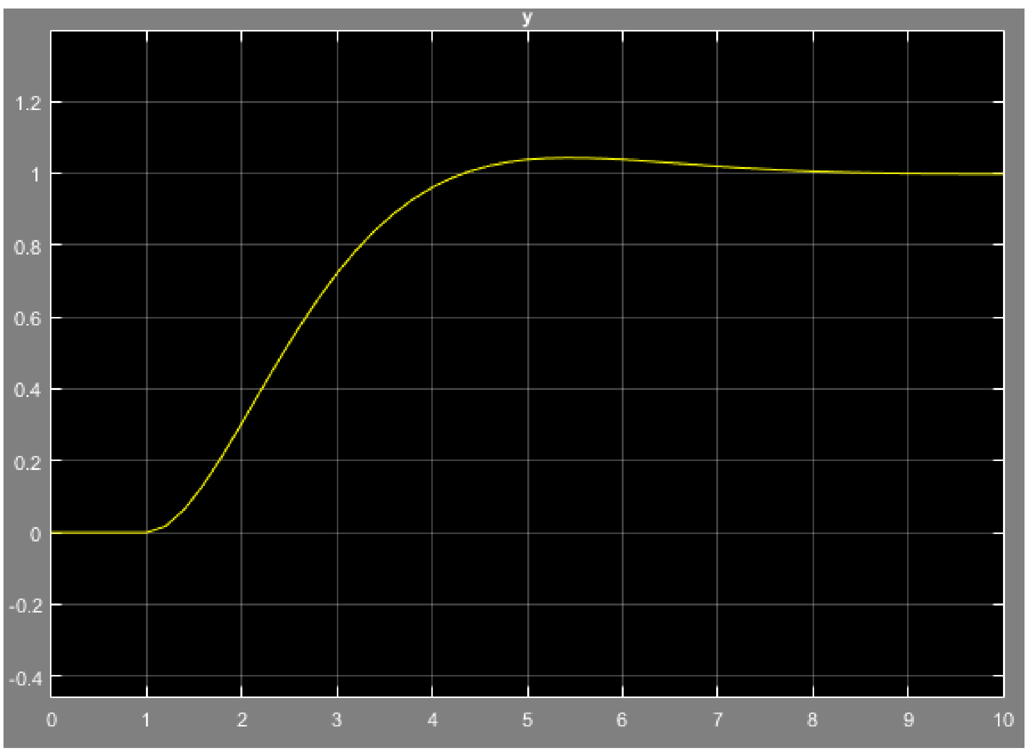
y(t) for Q = 1, but response run for 25s to observe decaying ringing.



y(t) for Q = 0.5, critically damped response.



y(t) for Q = 0.2, overdamped response.



y(t) for Q = 0.707, maximally flat response. Observe a slight overshoot with a reasonably fast response time.

State Variable Modelling

Example: $G(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2+cs+k}$, a second order system

Obviously: $\ddot{y}(t) = \frac{1}{m}f(t) - \frac{k}{m}y(t) - \frac{c}{m}\dot{y}(t)$ (1)

Define the state variables:

Let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$

Then:

$\dot{x}_1(t) = x_2(t)$ and $\dot{x}_2(t) = \ddot{y}(t)$

Therefore (1) becomes: $\dot{x}_2(t) = \frac{1}{m}f(t) - \frac{k}{m}x_1(t) - \frac{c}{m}x_2(\mathrm{t})$

Now the dynamical system can be represented in matrix form:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\dfrac{k}{m} & -\dfrac{c}{m}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \dfrac{1}{m}\end{bmatrix}f(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

This is a very useful way for representing dynamical systems, and it is very applicable to numerical processing techniques. The general matrix form is:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

This is also applicable to higher order systems, systems described by multiple differential equations, and even nonlinear systems.

The state-space representation is a mathematic model of a physical system consisting of the input **u(t)**, output **y(t)** and state variables **x(t)** related by first order differential equations. The term "state space" refers to a dimensional space where the axes are the state variables. Therefore, the state of the modelled physical system can be represented as a vector within that space.

Diving into Dynamical Systems

Dynamical system: a system whose states change with time.

Deterministic dynamical system: a dynamical system where the changes are governed by specific rules.

Stochastic dynamical system: a dynamical system where the changes are governed by random events or inputs instead of by specific rules.

In this class, we are primarily concerned with deterministic dynamical systems. However, random events (such as thermal noise) may set an otherwise deterministic dynamical system in motion.

Parameters of a system do not change with time (at least in simple models), and are often represented by letters near the beginning of the alphabet (A, c, k, m, etc.) or by Greek letters.

Variables are usually represented by letters near the end of the alphabet (t, u, v, x, y, z,). For our purposes, time (t) is the only independent variable. All other variables are dependent variables, functions of time.

Inside the state space, a phase space or phase plot is a plot of two of more states where the states are proportional to each other by derivatives. A 2-D phase plot could be position vs. velocity. A 3D phase plot could be position (x-axis), velocity (y-axis) and acceleration (z-axis).

Undamped second order dynamical system

Given $\ddot{x} + ax = 0$ where $x|_{t=0}=1$

Assume a solution of $x = x_o\cos(\omega_n t)$

Therefore: $\dot{x} = -x_o\omega_n\sin(\omega_n t)$

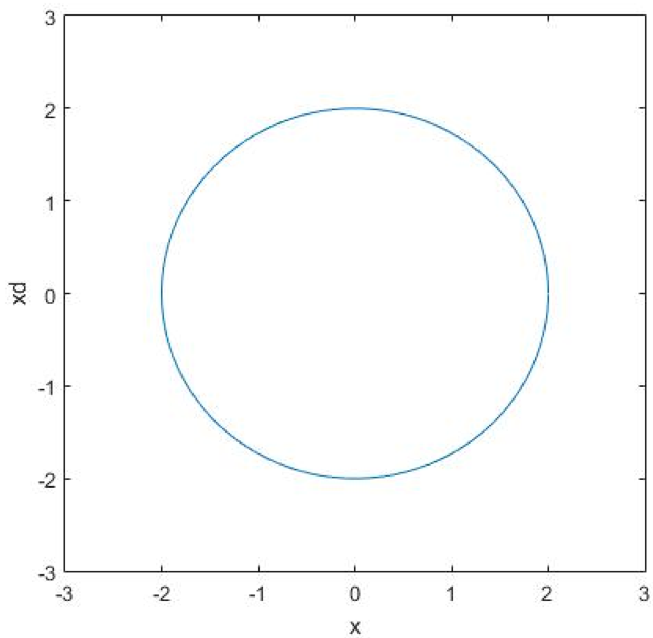
And $\ddot{x} = -x_o\omega_n^2\cos(\omega_n t)$

Equating terms: $-x_o\omega_n^2\cos(\omega_n t) + ax_o\cos(\omega_n t) = 0$

Resulting in $\omega_n = \sqrt{a}$

Let's let a = 1 and $x_o = 2$ so that $x = 2\cos(t)$ and $\dot{x} = -2\sin(t)$

Then the phase plot for x and $\dot{x}$ is:



It is a circle. This system is undamped, with no energy added to or removed from the system. Positive damping would result in the trajectory spiraling to the origin (0,0) as energy is removed from the system (likely as heat). Negative damping (an unstable system) would result in the trajectory spiraling out as the response "blows up".

Given that $\mathrm{x} = \mathrm{x}_o \cos(\omega_n \mathrm{t})$, the dynamical system of $\ddot{x} + ax = 0$ could represent a simplistic sinusoidal oscillator where "a" determines the frequency of oscillation.

This type of system is a "Conservative System," because energy is conserved, i.e. no energy is lost or dissipated from the system.

In a simple conservative mechanical system, like an ideal mass-spring system, the energy in the system moves between potential energy stored in the spring, $E_p = \frac{1}{2}kx^2$, and kinetic energy stored in the moving mass, $E_k = \frac{1}{2}mv^2$.

In an equivalent lossless electrical circuit, consisting of an ideal inductor and an ideal capacitor, energy oscillates from being stored in the inductor, $E_L = \frac{1}{2}LI^2$, to energy being stored in the capacitor, $E_c = \frac{1}{2}CV^2$. So, an ideal inductor in parallel with an ideal capacitor, with an initial voltage across the capacitor, would realize a simple but unrealizable electrical oscillator.

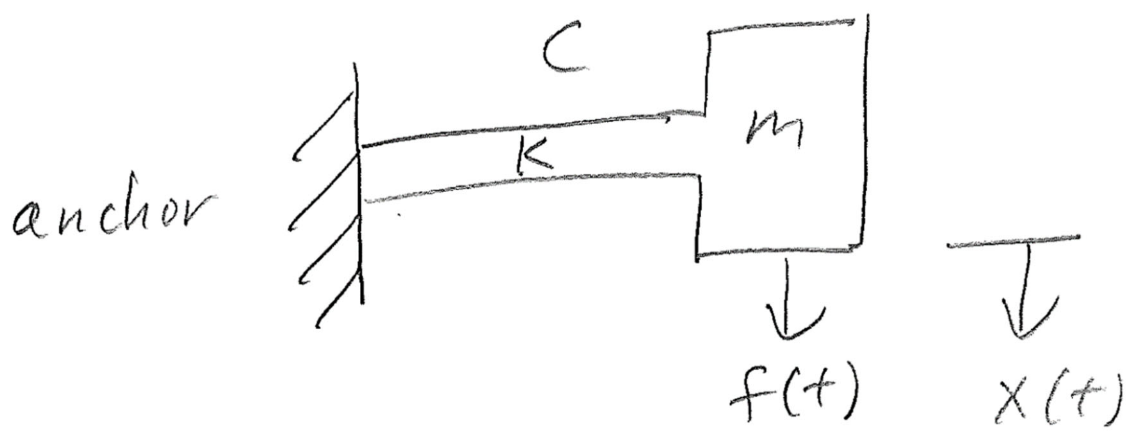
Real Dynamical Systems

All real dynamical systems have inherent energy loss mechanisms, typically involving the conversion of mechanical or electrical energy into heat:

> All real mechanical systems have losses due to friction.
>
> All real inductors and capacitors have resistive losses in them and in their electrical connections. Note, the phenomenon of superconductivity will not be considered here.

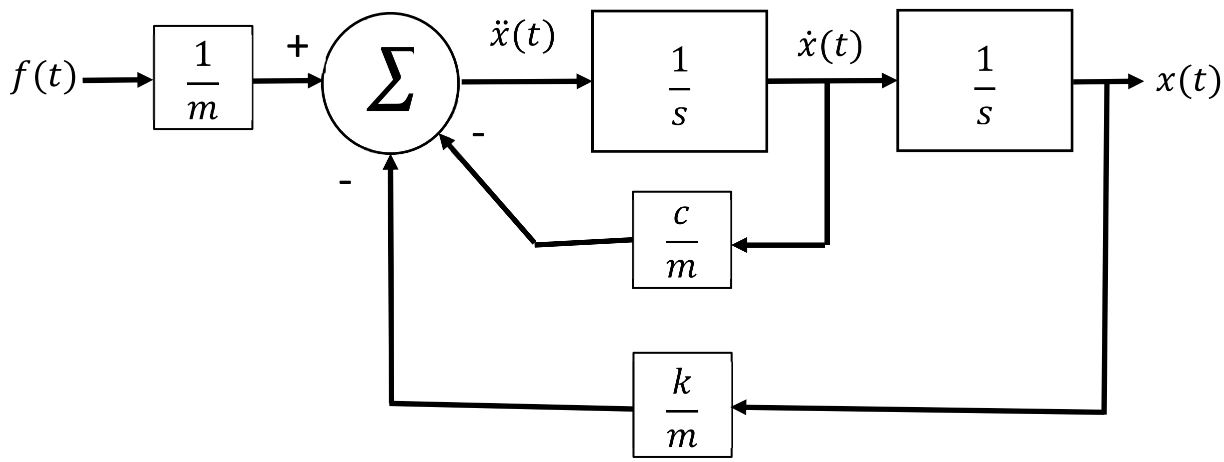
These types of systems are called "Dissipative Systems," since energy is dissipated over time.

Consider the spring-mass-damper system previously discussed:

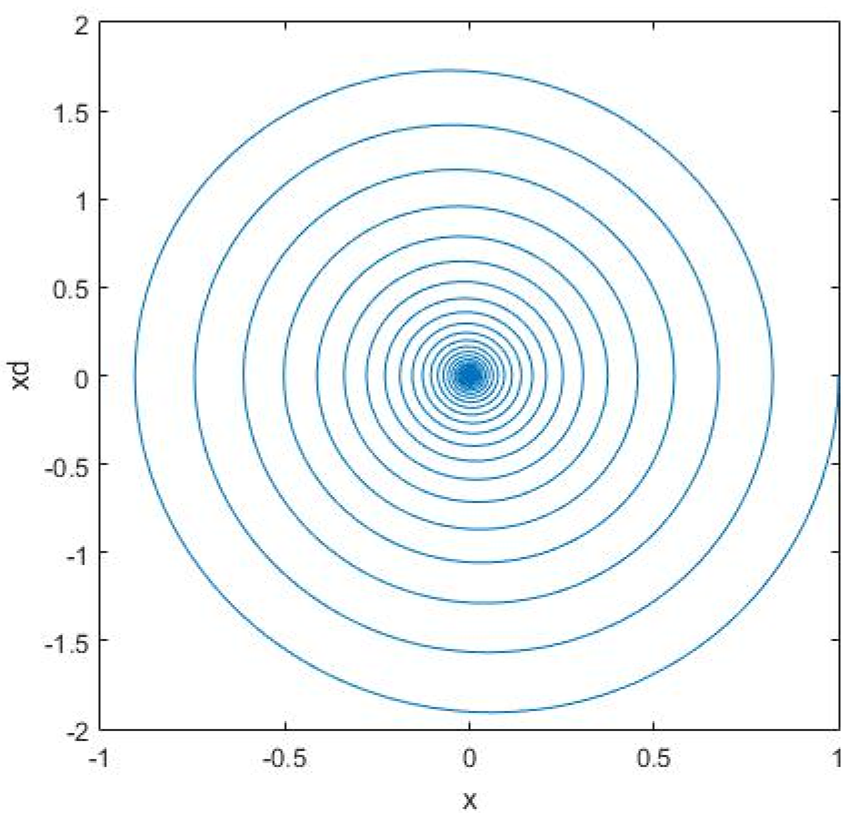


Where : $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$

We can model the system using a simulation diagram from $\ddot{x} = \frac{1}{m}f(t) - \frac{c}{m}\dot{x} - \frac{k}{m}x$:

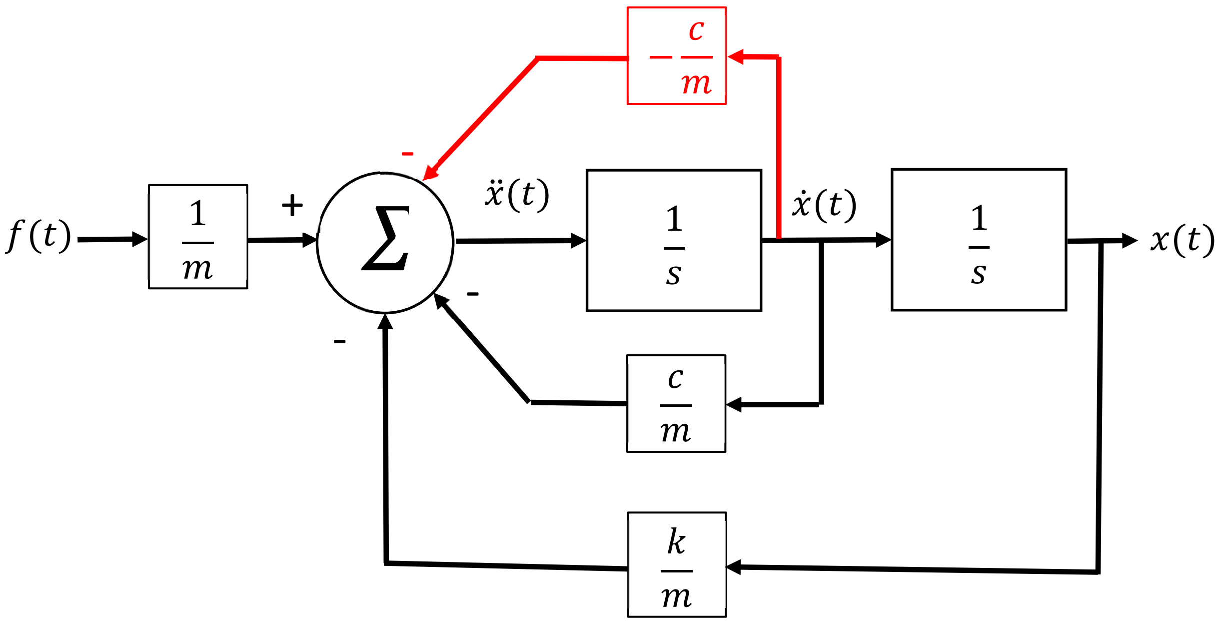


Picking values for (1/m) = 0.5, (c/m) = 0.125, and (k/m) = 4, with no input and an initial condition of x =1, the phase space portrait is:



Which starts at x = 1, xd = 0, and decays to (0,0) as energy is dissipated out of the system. Note, for the sake of simplicity here, parameter units were not used.

How do you make a dissipative system oscillate? Consider this:



A parallel feedback term or network with a gain equal to (-c/m) is added to the system to cancel out the loss due to $\frac{c}{m}\dot{x}$. Now the composite system is "lossless." Actually, realizing the $-\frac{c}{m}\dot{x}$ term requires energy, so the overall system is still dissipative, however it still oscillates.

Notice that from our state variable model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t)$$

reduces to:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t)$$

where the **A** "\" diagonal is all zeros.