Tuesday, 8/19/25

Introduction

There are three types of common analog subsystems:

(1) The rectifier – generally a nonlinear system
AC in, DC out

(2) The amplifier – can often be considered a linear system
Output Signal = Input Signal X Gain

(3) The oscillator:
  a. Linear Oscillator: DC in, stable AC sinusoid out
  b. Nonlinear Oscillator: DC in, complex periodic waveform out
  c. Chaotic Oscillator: DC in, deterministic non-periodic complex waveform out

Definition of a Linear System:

1) Mathematical definition of a linear differential equation:

A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in or can be expressed in the form:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

There are two types of linear differential equations: those with constant coefficients $\{a_o\}$ and those with variable coefficients $\{a_o(x)\}$.

In ECE and ME, we are most used to differential equations with constant coefficients. Example:

$$2\frac{d^2 x}{dt^2} + 5\frac{dx}{dt} + 3x = \sin(5t)$$

But this differential equation with variable coefficients is also linear:

$$2\frac{d^2x}{dt^2} + 5t\frac{dx}{dt} + 3x = \sin(5t)$$

Examples of ordinary differential equations that are nonlinear:

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x^2 = 0$$

$$\frac{d^2x}{dt^2} + 5\left(\frac{dx}{dt}\right)^2 + 6x = 0$$

$$\frac{d^2x}{dt^2} + 5x\frac{dx}{dt} + 3x = 0$$

2) Engineering definition of a linear system, H, where $x_1(t)$ is an input and $x_2(t)$ is an input, while $y_1(t) = H\{x_1(t)\}$ and $y_2(t) = H\{x_2(t)\}$:

   a. Superposition applies

      $ay_1(t) + by_2(t) = H\{ax_1(t) + bx_2(t)\}$, where a and b are scalars

   b. In steady state, the frequency components of the output are the same as the frequency components of the input

      $Y(f_1) = H\{X(f_1)\}$

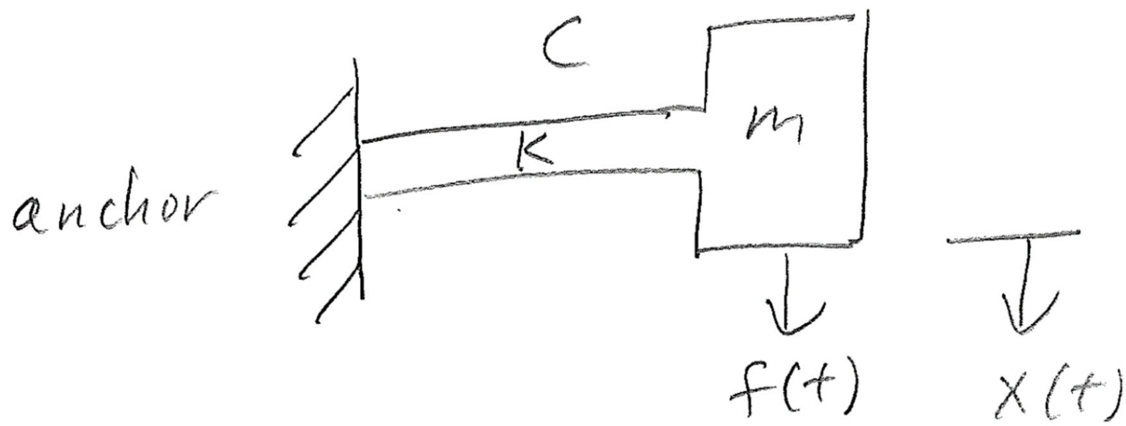
      (1) One exception: $H\{X_1(f_1) + X_2(f_1)\} = Y_1(f_2) + Y_2(f_2)$. This also falls under superposition.

   c. The system is not sensitive to small changes in initial conditions.

3) Most, if not all, “linear” engineering systems are modelled by linear ordinary differential equations with constant coefficients.

Consider some engineering examples of systems modeled by linear differential equations:

a. Simple cantilevered beam with attached mass



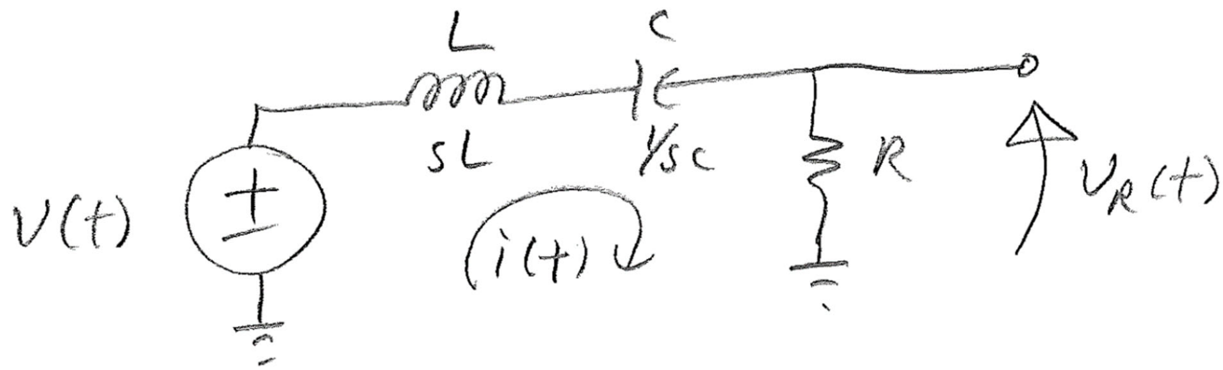
$f_{inertial} + f_{damping} + f_{spring} = f(t)$

ma + cv + kx = f(t)

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$$

2nd order linear differential equation with constant coefficients

b. LCR circuit



$V_R(s)$ =RI(S)

$$I(s) = \frac{V(s)}{sL + {}^{1}\!/_{sC} + R}$$

$$V_R(s) = \frac{RV(s)}{sL + {}^{1}\!/_{sC} + R}$$

$V_R(s)(sL + {}^{1}\!/_{sC} + R) = RV(s)$ – Integro-differential equation.

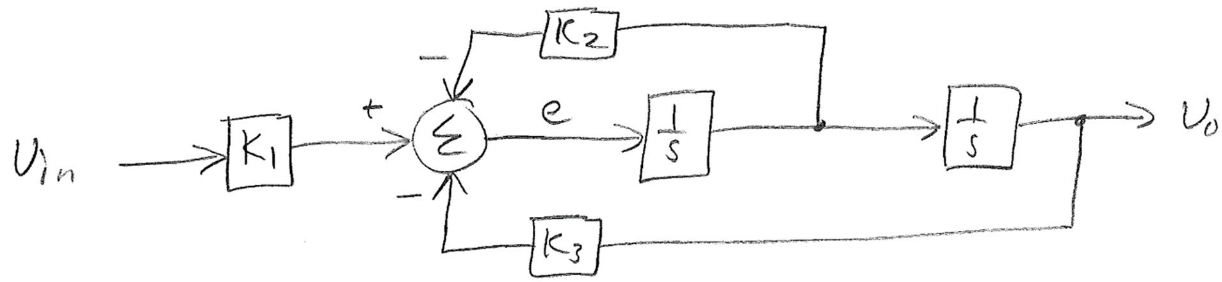
Therefore differentiate both sides and divide by R:

$$V_R(s)(s^2\,{}^{L}\!/_{R} + {}^{1}\!/_{RC} + s) = sV(s)$$

$$\frac{L}{R}\frac{d^2V_R}{dt^2} + \frac{dV_R}{dt} + \frac{V_R}{RC} = \frac{dV}{dt}$$

2nd order linear differential equation with constant coefficients

c. Op amp based voltage feedback system



$E(s) = k_1 V_{in} - E(s)\frac{k_2}{s} - E(s)\frac{k_3}{s^2}$ (1)

$V_o(s) = \frac{E(s)}{s^2}$ (2)

Rearranging (1):

$$E(s)(1 + \frac{k_2}{s} + \frac{k_3}{s^2}) = k_1 V_{in}$$

Therefore:

$E(s) = \frac{k_1 V_{in}}{1+\frac{k_2}{s}+\frac{k_3}{s^2}} = \frac{k_1 V_{in} s^2}{s^2+k_2 s+k_3}$ (3)

(3) into (2):

$V_o(s) = \frac{k_1 V_{in}}{s^2+k_2 s+k_3}$ (4)

Rearranging (4):

$$V_o(s)(s^2 + k_2 s + k_3) = k_1 V_{in}$$

In the time domain:

$$\frac{1}{k_1}\frac{d^2 v_o}{dt^2} + \frac{k_2}{k_1}\frac{dv_o}{dt} + \frac{k_3}{k_1} v_o = v_{in}$$

2nd order linear differential equation with constant coefficients

Conservative Systems

Conservative systems: have no energy dissipating elements (resistances in circuits, damping in mechanical systems)

Example: $a\frac{d^2x}{dt^2}$+bx = 0

Let an initial condition be x = x_o at t=0

Assume a solution of x(t) = $x_o\cos(\omega t)$

Therefore $\dot{x}(t) = -x_o\omega\sin(\omega t)$

And $\ddot{x}(t) = -x_o\omega^2\cos(\omega t)$

Then: $-ax_o\omega^2\cos(\omega t) + bx_o\cos(\omega t) = 0$

Therefore: $\omega = \sqrt{\frac{b}{a}}$

It will oscillate indefinitely with frequency ω. Not realizable, but the concept is useful later in explaining the operation of real oscillators.

Dissipative Systems

Dissipative systems: have energy dissipating elements (resistances in circuits, damping in mechanical systems)

Free oscillations will dampen out over time as energy is dissipated from the system, usually in the form of heat (resistive losses, frictional losses, etc.)

<u>All real systems are dissipative</u>.

Sustained oscillating systems (i.e. oscillators) use some form of feedback to cancel out the dissipative term(s) from the system so that the system behaves like a conservative system.

Evaluation of Oscillation Quality

Desired response: $y(t) = Acos(\omega t)$

Actual response: $y(t) = (A_o + a(t))cos(\omega t + \varphi(t))$

where a(t) and φ(t) are imperfections from the desired oscillatory response.

They could be due to the system (deterministic) or due to noise (stochastic).

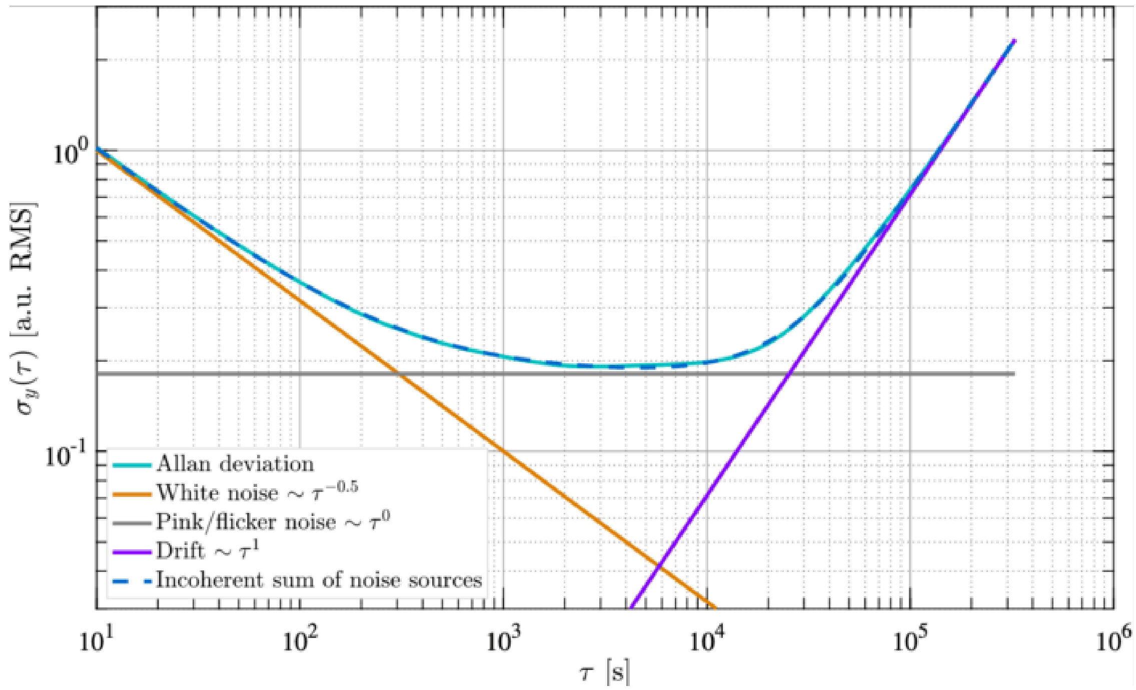
Analytical tools exist for analyzing oscillation quality:

1) Standard statistical tools: such as Standard Deviation.

2) The Allan Variance (invented by David W. Allan for measuring stability in clocks and oscillators). It allows different noise sources to be identified from the time series data from the oscillating system. The basic formula is:

$$\sigma_y^2(\tau) = \frac{1}{2}\langle(\bar{y}_{i+1} - \bar{y}_i)^2\rangle$$

Where y(t) is a continuous time series, $\sigma_y^2(\tau)$ is its Allan variance, τ is a segment length of data over which its average $\bar{y}$ is calculated. $\bar{y}_i$ is $\bar{y}$ calculated for the i^{th} τ segment length. The Allan deviation, $\sigma_y(\tau)$ is the $\sqrt{\sigma_y^2(\tau)}$. A plot of $\sigma_y(\tau)$ versus τ, called an Allan deviation plot, in a log-log scale can be made to determine various noise processes from the slopes. Example:



https://liquidinstruments.com/white-papers/understanding-and-performing-allan-variance-measurements/

Table I. Slopes of common noise sources for gyroscopes and clocks

Slope Value	Gyroscope Noise Source	Clock Noise Source
-1	Quantization noise	White or flicker PM
-1/2	Angle random walk	White FM
0	Bias instability	Flicker FM
+1/2	Rate random walk	Random walk FM
+1	Rate Ramp	Frequency drift

The x-axis τ(s) represents the averaging time. The minima of $\sigma_y(\tau)$ corresponds with the τ representing the optimal averaging time to achieve maximum oscillation measurement stability.