**The Duffing Oscillator**

Consider this system:

\[ \ddot{x} + b \dot{x} + (k_1 \pm k_2 x^2) x = A \sin(\Omega t) \]

This equation is the Duffing equation.

For this particular system, the damping changes with the magnitude of \( x \).

Consider, for example, the MEMS resonator below:

![Diagram of MEMS resonator](image)

The 2-beam suspension system is statically indeterminate. For small displacements, a linear spring constant can be assumed. For large displacements, however, the two beam elements are deflected AND stretched, and the effective system spring constant increases with the magnitude of the displacement. So this system could potentially be modeled by the Duffing oscillator with

\[ k = k_1 + k_2 x^2 \]

for the system spring constant. The system has a nonlinear stiffness that does not follow Hooke’s Law.

When \( k_1 > 0 \) and \( k_2 > 0 \), the system spring constant increases with displacement (positive or negative displacement). For this case, the spring is referred to as a “hardening spring.”

When \( k_1 > 0 \) and \( k_2 < 0 \), the system spring constant decreases with displacement (positive or negative displacement). For this case, the spring is referred to as a “softening spring.”

Even in non-mechanical systems, the terms “hardening” and “softening” are often used in referring to the Duffing equation with regard to the sign of \( k_2 \).
Consider the effect on the frequency response (transmissibility vs. normalized frequency) in the plot below. Note: the transmissibility is the magnitude of the output displacement of the proof mass divided by the input displacement to the mechanical system as a function of frequency (or normalized frequency):

\[ m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \rightarrow \text{for a linear system, } y(t) \text{ is input displacement.} \]

This is a 2\textsuperscript{nd} order mechanical system with a lowpass response, with a transmissibility of 1 at DC, a peak at the resonant frequency (approximately equal to Q for highly underdamped systems), and a roll off after the resonant frequency.

For the nonlinear system in the figure below, \( \beta \) is \( K_2 \).

When \( \beta \neq 0 \), the dashed lines represent unstable regions of the trace <see close up>. When slowly increasing in frequency, once “A” is reached, the response abruptly jumps down to “B”. However, when slowly decreasing in frequency, once “C” is reached, the response abruptly jumps up to “D”. Observe that a hysteresis now exists in the transmissibility response: the jumps A-B and C-D do not coincide.

The case with \( k_1<0 \) and \( k_2>0 \) is called “Duffing’s two-well oscillator” and models a ball rolling along a trough having two dips with a hump in between.

All of these cases can exhibit chaos with the right parameter values. For example:

\[ \ddot{x} + \dot{x} - x + x^3 = \sin(0.8t) \]

Simulink model and phase plot shown below.
https://en.wikipedia.org/wiki/Duffing_equation
Autonomous Chaotic Systems

These continuous-time chaotic systems are chaotic without any external forcing function (i.e. without any explicit time dependence).

Therefore they must be at least 3rd order systems (3 derivatives).

The simplest examples are systems with 3 first order ordinary differential equations.

We already discussed the Lorenz system:

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz
\end{align*}
\]

A similar (even simpler) chaotic system is the Rössler System (Dr. Otto Rössler, 1976, a non-practicing medical doctor):

\[
\begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{align*}
\]

a, b and c are constants. Observe that the first two equations are linear and the only nonlinear term is the multiplication term, zx, in the 3rd equation.

The system is chaotic for a=0.2, b=0.2 and c=5.7 (see phase plots below)
Rossler System Phase Plot

Rossler System Phase Plot