## Tuesday 3/28/23

### Periodically Forced Chaotic Systems

Consider the Van der Pol Oscillator we previously discussed:

$$\ddot{x} + b(x^2 - 1)\dot{x} + x = 0$$

We previously discussed the case where b=1, for a linearly oscillating system with nonlinear AGC: $c = (x^2 - 1)$, and showed that it results in a limit cycle.

Now let's add a periodic forcing function:

$$\ddot{x} + b(x^2 - 1)\dot{x} + x = Asin(\Omega t)$$

First, rewrite this equation in state space notation:

Let $z = \Omega t$

Therefore $\dot{z} = \Omega$

Also let $y = \dot{x}$

That yields three state variables: x, y, z, and 3 state equations:

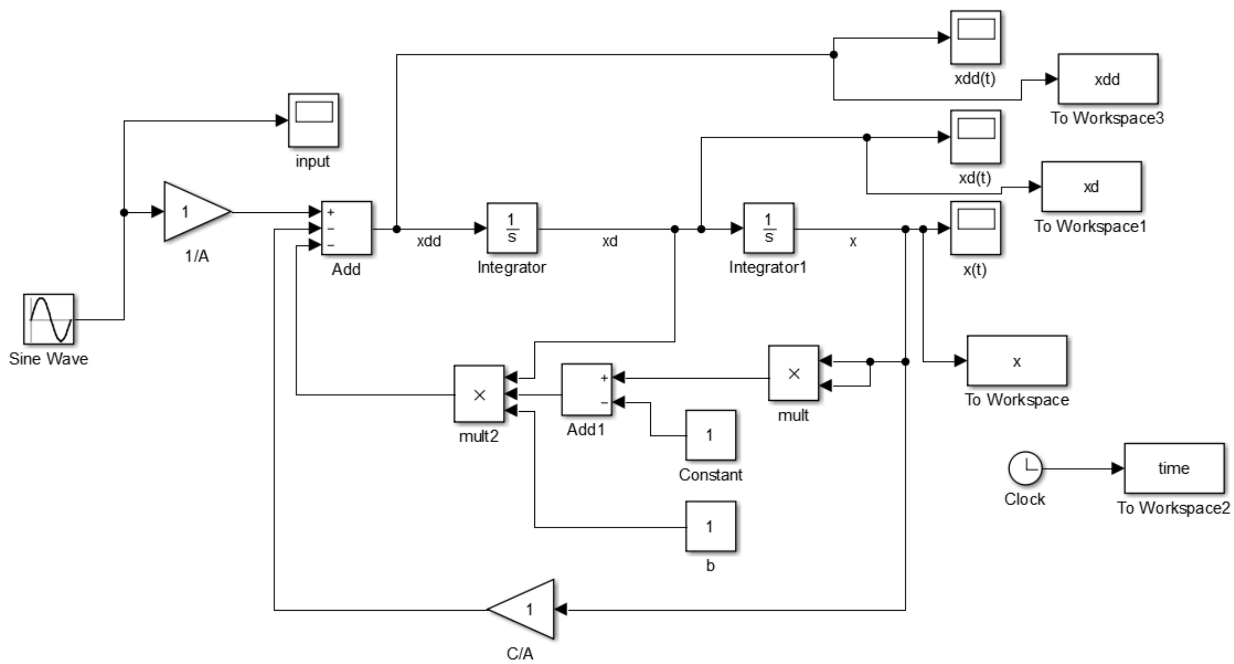
$$\dot{x} = y$$

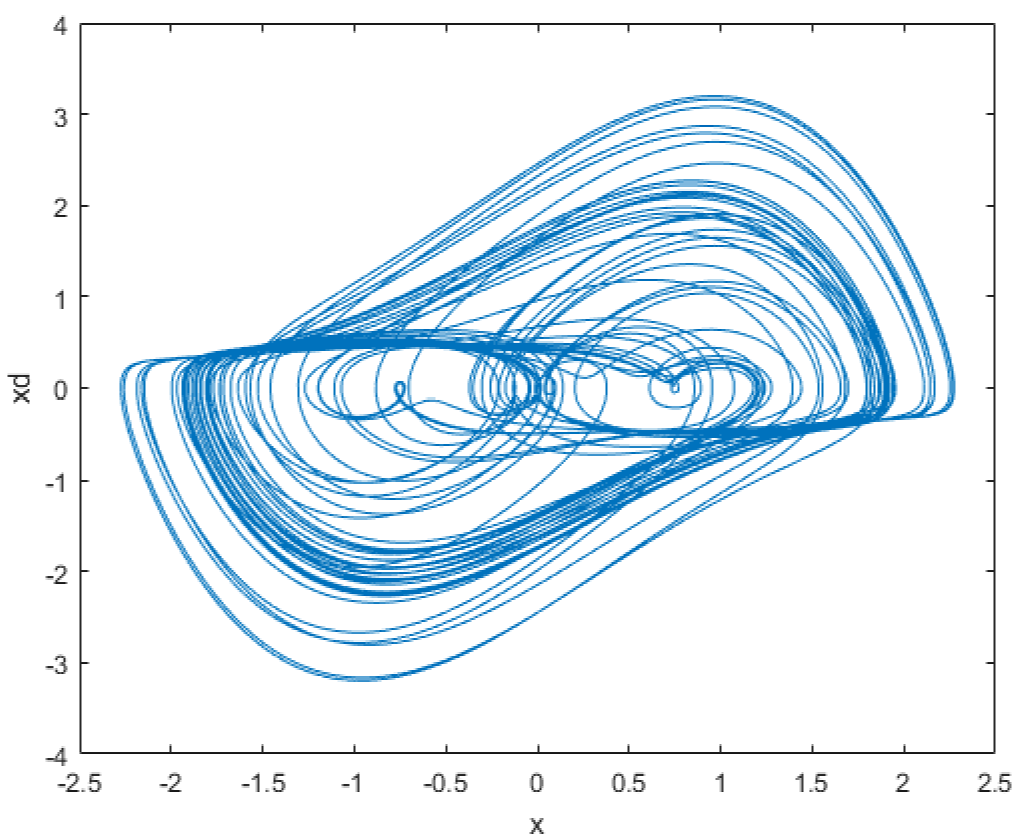
$$\dot{y} = Asin(z) - x - b(x^2 - 1)y$$

$$\dot{z} = \Omega$$

Observe that this is now a 3D system and is therefore capable of being chaotic.

This system was simulated in Simulink with A = b = 1, $\Omega$ = 0.45, and run for 500 s:


input
1
1/A
Sine Wave
Add
xdd
1/s
Integrator
xd
1/s
Integrator1
x
xdd(t)
xdd
To Workspace3
xd(t)
xd
To Workspace1
x(t)
x
To Workspace
mult2
Add1
mult
1
Constant
1
b
Clock
time
To Workspace2
1
C/A


xd
x
4
3
2
1
0
-1
-2
-3
-4
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

Two other similar periodically forced chaotic systems are the Raleigh Oscillator:

$$\ddot{x} = (\dot{x}^2 - 4)\dot{x} + x = 5\sin(4t)$$

And the Raleigh oscillator variant:

$$\ddot{x} + x\dot{x} = \sin(4t)$$

In these systems, a 2nd order nonlinear dynamical system is made 3rd order by periodically forcing it, adding the 3rd state.

## The Duffing Oscillator

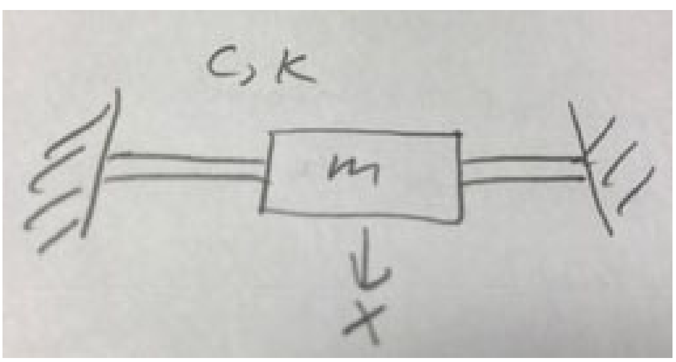
Consider this system:

$$\ddot{x} + b\dot{x} + (k_1 \pm k_2 x^2)x = Asin(\Omega t)$$

This equation is the Duffing equation.

For this particular system, the effective spring constant changes with the magnitude of x.

Consider, for example, the mechanical spring-mass-damper system below:




The 2-beam suspension system is statically indeterminate. For small displacements, a linear spring constant can be assumed. For large displacements, however, the two beam elements are deflected AND stretched, and the effective system spring constant increases with the magnitude of the displacement. So, this system could potentially be modeled by the Duffing oscillator with

$$k = k_1 + k_2 x^2$$

for the system spring constant. The system has a nonlinear stiffness that does not follow Hooke's Law.

When $k_1>0$ and $k_2>0$, the system spring constant increases with displacement (positive or negative displacement). For this case, the spring is referred to as a "hardening spring."

When $k_1>0$ and $k_2<0$, the system spring constant decreases with displacement (positive or negative displacement). For this case, the spring is referred to as a "softening spring."

Even in non-mechanical systems, the terms "hardening" and "softening" are often used in referring to the Duffing equation with regard to the sign of $k_2$.

Consider the effect on the frequency response (transmissibility vs. normalized frequency) in the plot below. Note: the transmissibility is the magnitude of the output displacement of the proof mass divided by the input displacement to the mechanical system as a function of frequency (or normalized frequency):

$\mathrm{m}\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$ → for a linear system, y(t) is input displacement and x(t) is the output displacement.

This is a 2nd order mechanical system with a lowpass response, with a transmissibility of 1 at DC, a peak at the resonant frequency (approximately equal to Q for highly underdamped systems), and a roll off after the resonant frequency.

For the nonlinear system in the figure below, β is $K_2$.

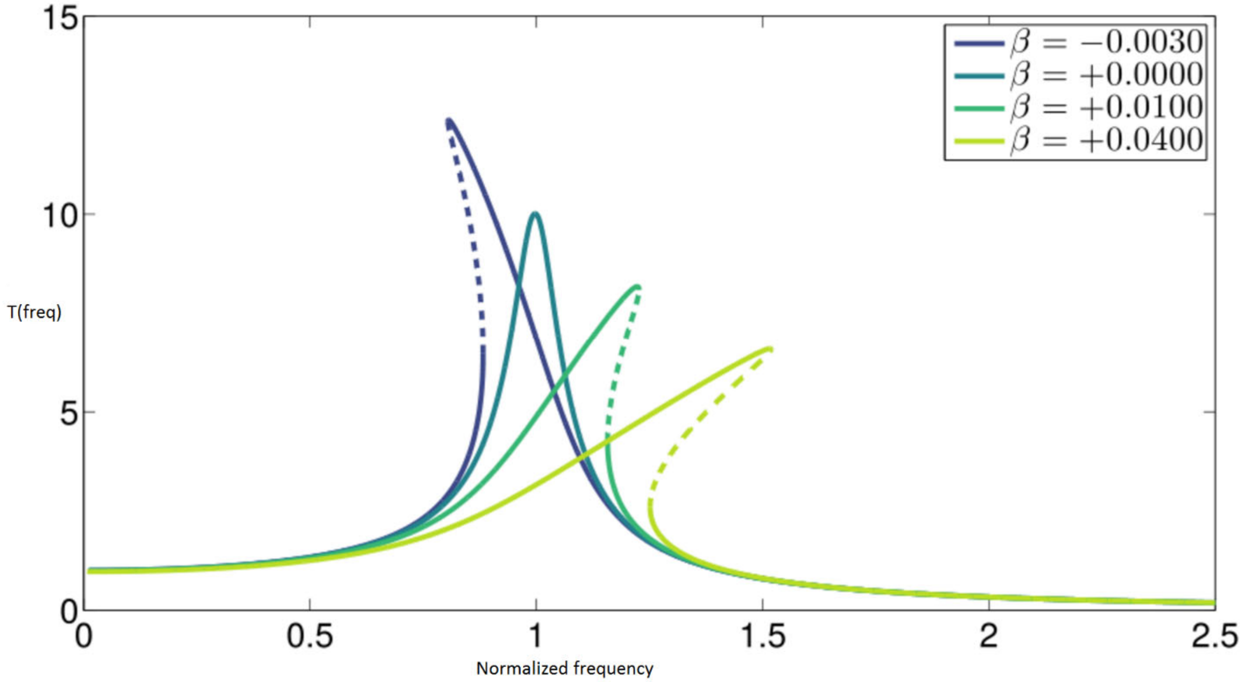
When β≠0, The dashed lines represent unstable regions of the trace <see close up>. When slowly increasing in frequency, once "A" is reached, the response abruptly jumps down to "B". However, when slowly decreasing in frequency, once "C" is reached, the response abruptly jumps up to "D". Observe that a hysteresis now exists in the transmissibility response: the jumps A-B and C-D do not coincide.

The case with $k_1<0$ and $k_2>0$ is called "Duffing's two-well oscillator" and models a ball rolling along a trough having two dips with a hump in between.

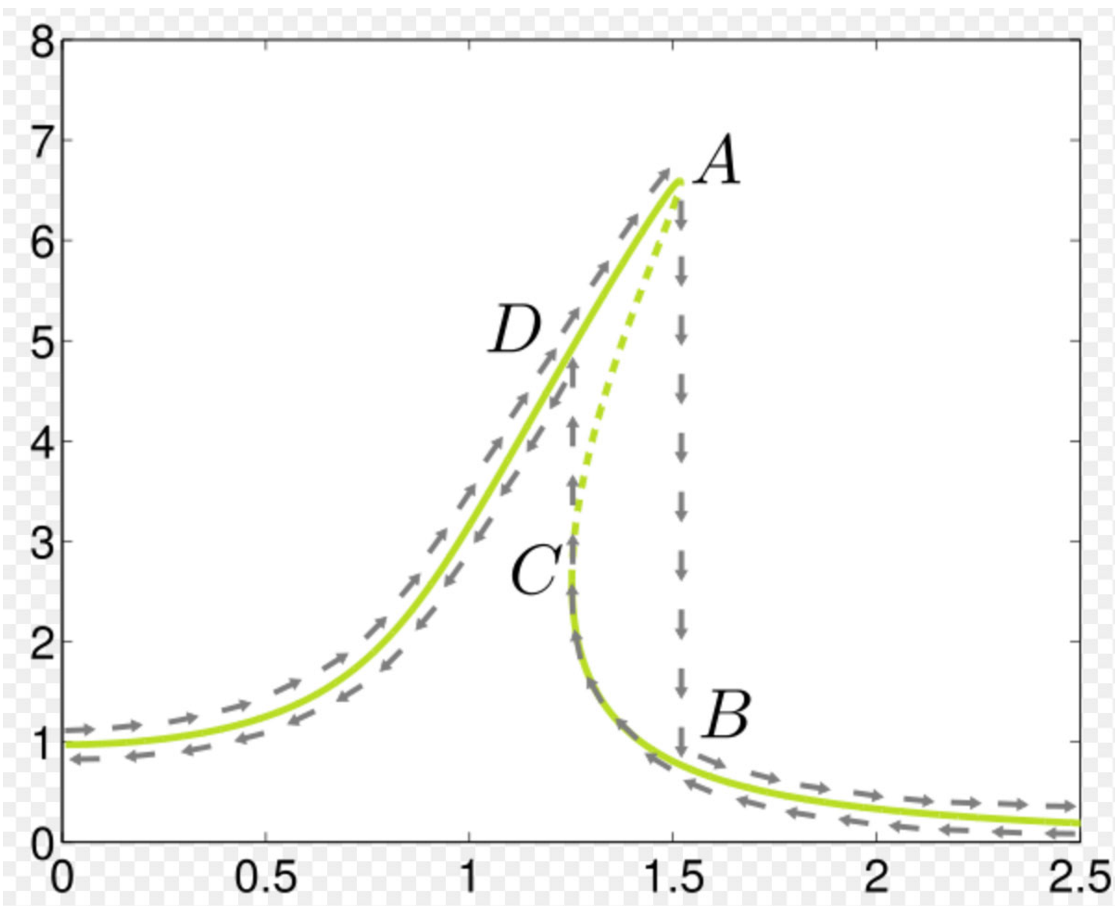
All of these cases can exhibit chaos with the right parameter values. For example:

$\ddot{x} + \dot{x} - x + x^3 = \sin{(0.8t)}$

Simulink model and phase plot shown below.

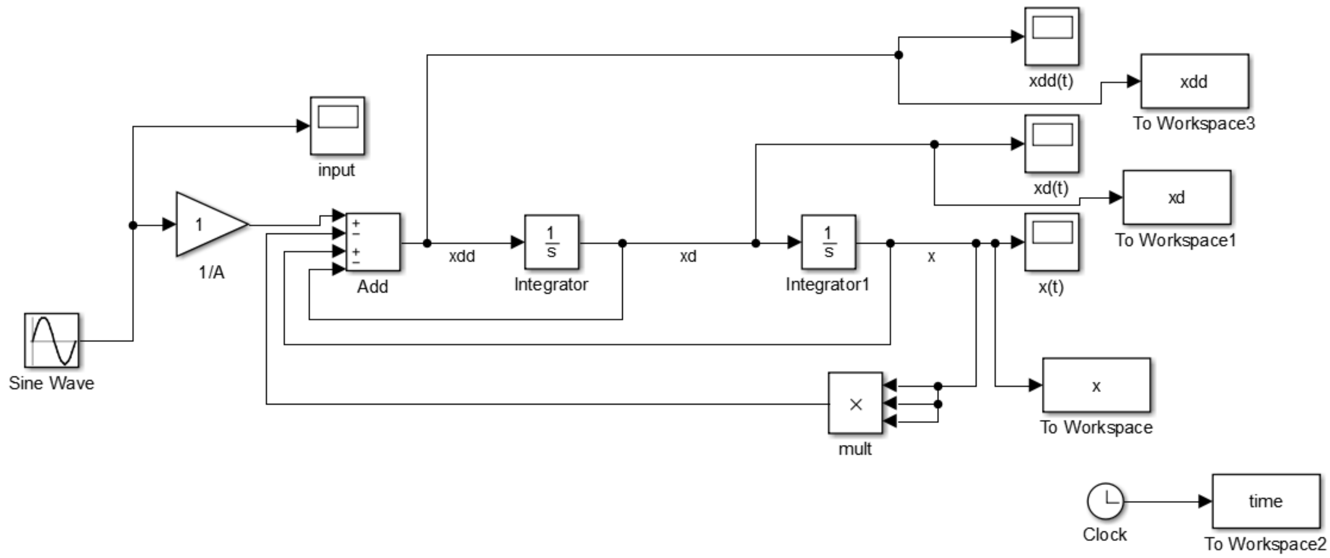


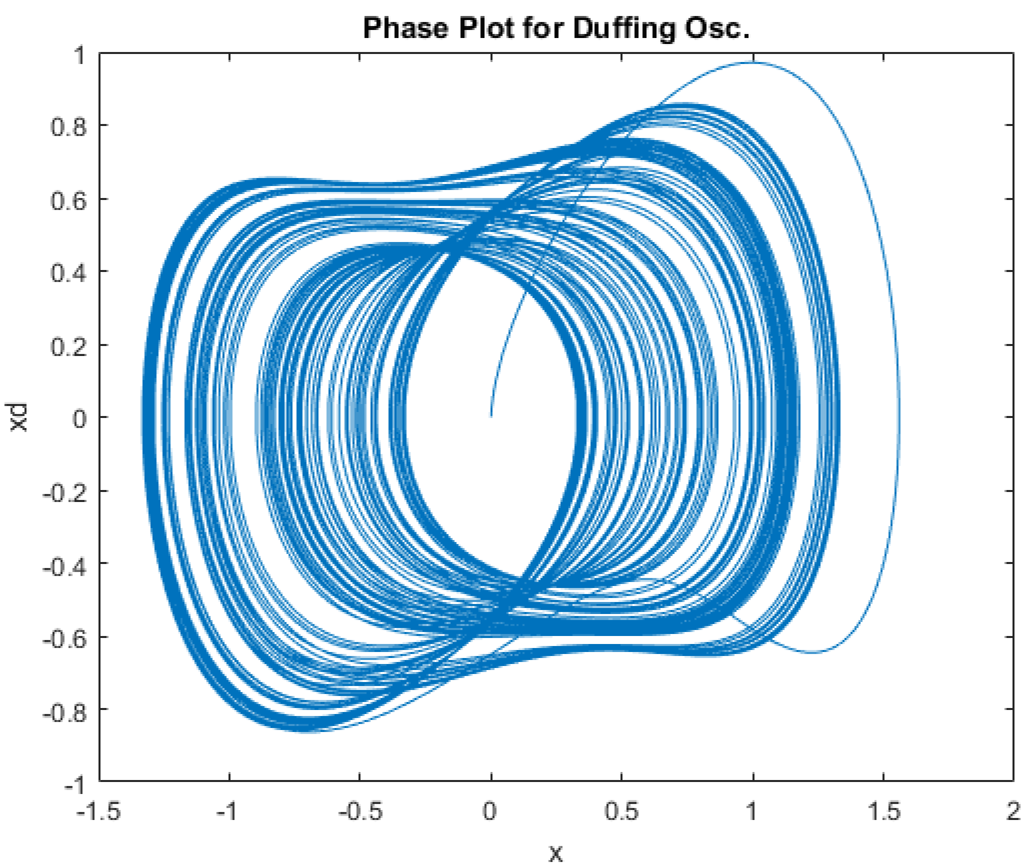
$\beta = -0.0030$
$\beta = +0.0000$
$\beta = +0.0100$
$\beta = +0.0400$
T(freq)
Normalized frequency

A
B
C
D

https://en.wikipedia.org/wiki/Duffing_equation



## Autonomous Chaotic Systems

These continuous-time chaotic systems are chaotic without any external forcing function (i.e. without any explicit time dependence).

Therefore, they must be at least 3rd order systems (3 derivatives).

The simplest examples are systems with 3 first order ordinary differential equations.

We already discussed the Lorenz system:

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz
\end{aligned}$$

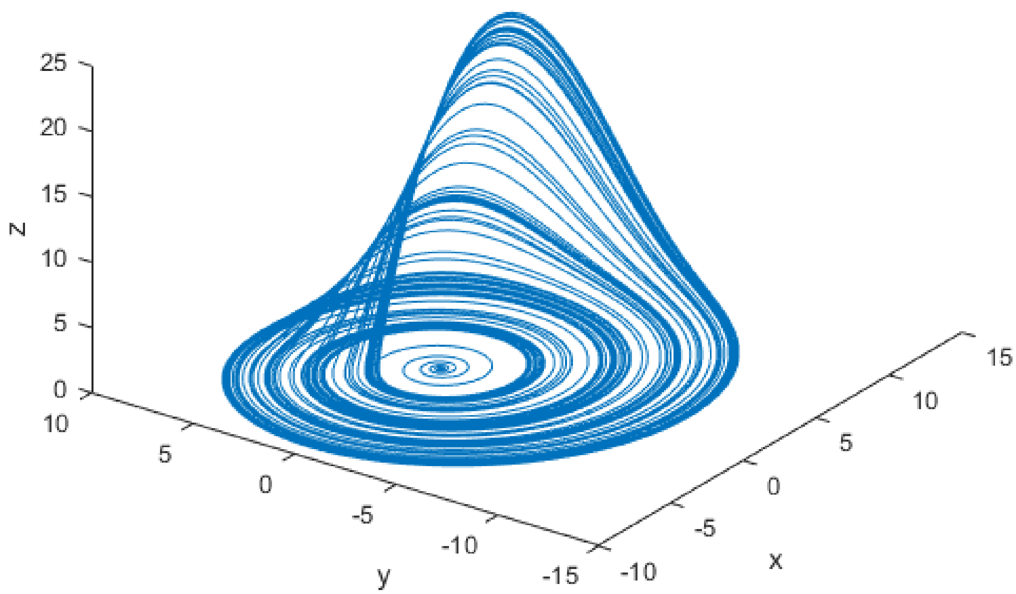
A similar (even simpler) chaotic system is the Rössler System (Dr. Otto Rössler, 1976, a non-practicing medical doctor):

$$\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}$$

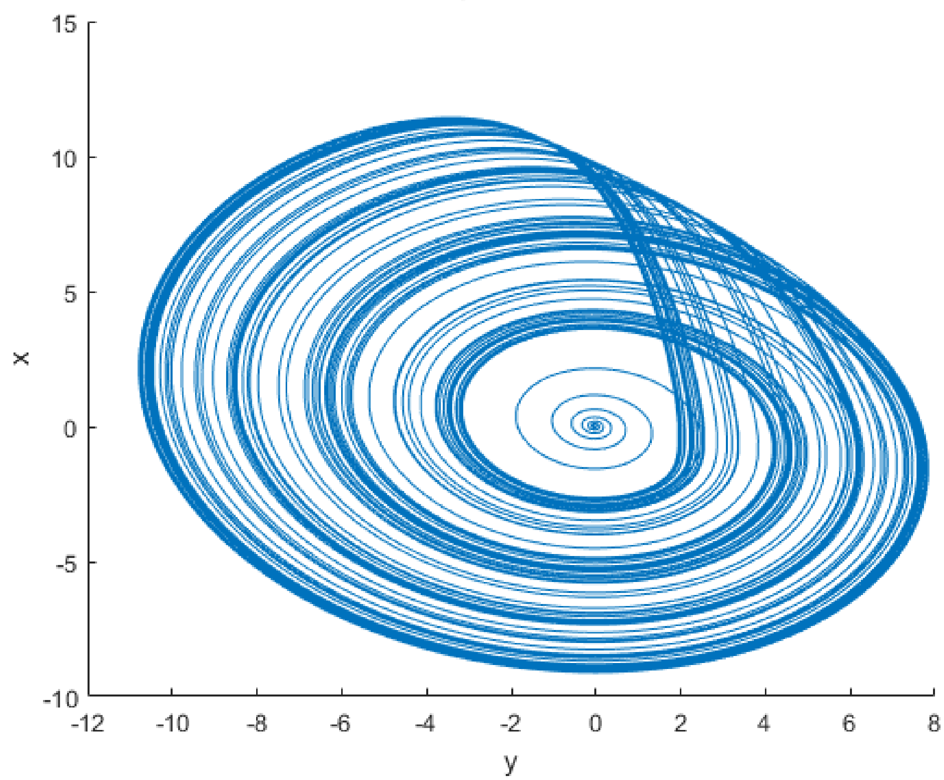
a, b and c are constants. Observe that the first two equations are linear and the only nonlinear term is the multiplication term, zx, in the 3rd equation.

The system is chaotic for a=0.2, b=0.2 and c=5.7 (see phase plots below)

**Rossler System Phase Plot**




**Rossler System Phase Plot**



**Chua's Circuit (also called Chua circuit)**

Dr. Leon Shua (U.C. Berkeley) invented the circuit bearing his name in 1983. Its purpose was to illustrate that the Lorenz system was not a numerical artifact, but rather a robust physical phenomenon. Many variants of this circuit have been realized. The Chua circuit is not a simple chaotic system compared to others we have discussed, but the circuit implementation is relatively simple.

An equation set for a Chua circuit:

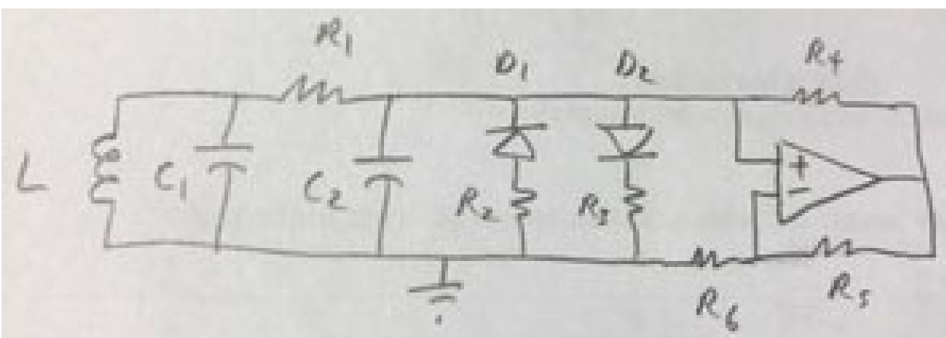
$$\dot{x} = \alpha[y - x - f(x)]$$

$$\dot{y} = \sigma(x - y) + \rho z$$

$$\dot{z} = -\beta y$$

f(x) is the electrical response of a nonlinear resistor. α, β, σ, and ρ are constants.

Consider the circuit below:




$R_4$, $R_5$, $R_6$ and the op amp form a negative resistance where:

$$R_{eq} = -R_4\frac{R_6}{R_5} = -R_{neg}$$

Assume that the diodes are ideal diodes with a 0.5V turn on voltage. Let $V_2$ be the voltage across $C_2$. Then this relationship holds for the resistance, looking to the right of $C_2$, $R_{in}$:

For $-0.5V < V_2 < 0.5V$: $R_{in} = -R_{neg}$

For $V_2 < -0.5V$: $R_{in} = R_2 // -R_{neg}$

For $V_2 > 0.5V$: $R_{in} = R_3 // -R_{neg}$

Let $R_2 = R_3 = R_{pos}$.

Therefore: $R_{in} = \frac{-R_{pos}R_{neg}}{R_{pos}-R_{neg}}$ for $|V_2| > 0.5V$

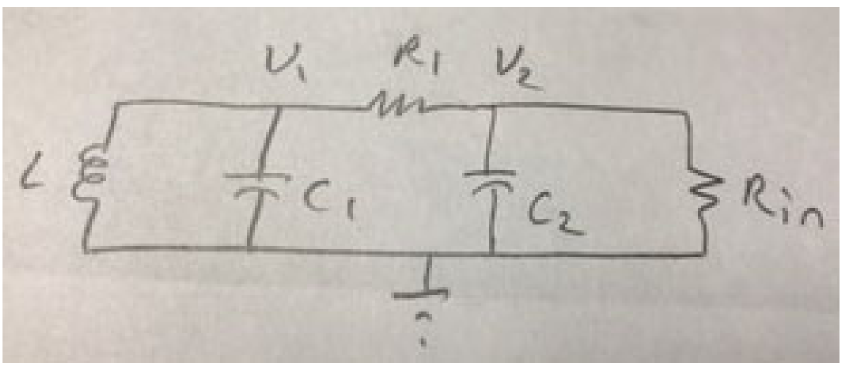
If $|-R_{neg}| > R_{pos}$, then $R_{in}$ is positive when either diode is on:

For $-0.5V < V_2 < 0.5V$: $R_{in} = -R_{neg}$: a negative resistance

For $V_2 < -0.5V$: $R_{in} = R_2//-R_{neg}$: a positive resistance

For $V_2 > 0.5V$: $R_{in} = R_3//-R_{neg}$: a positive resistance

Consider the circuit model below for analysis purposes:




$$V_1\left(\frac{1}{sL} + sC_1 + \frac{1}{R_1}\right) - V_2\left(\frac{1}{R_1}\right) = 0 \qquad (1)$$

$$V_2\left(sC_2 + \frac{1}{R_{in}} + \frac{1}{R_1}\right) - V_1\left(\frac{1}{R_1}\right) = 0 \qquad (2)$$

Therefore (2) can be rewritten as:

$$V_1 = V_2\left(sR_1C_2 + \frac{R_1}{R_{in}} + 1\right) \qquad (3)$$

(3)→(1):

$$V_2\left(sR_1C_2 + \frac{R_1}{R_{in}} + 1\right)\left(\frac{1}{sL} + sC_1 + \frac{1}{R_1}\right) - V_2\left(\frac{1}{R_1}\right) = 0$$

$$V_2\left(\frac{R_1C_2}{L} + \frac{R_1}{sLR_{in}} + \frac{1}{sL} + s^2R_1C_1C_2 + \frac{sR_1C_1}{R_{in}} + sC_1 + sC_2 + \frac{1}{R_{in}} + \frac{1}{R_1} - \frac{1}{R_1}\right) = 0$$

$$V_2\left(\frac{sR_1C_2}{L}+\frac{R_1}{LR_{in}}+\frac{1}{L}+s^3R_1C_1C_2+\frac{s^2R_1C_1}{R_{in}}+s^2C_1+s^2C_2+\frac{s}{R_{in}}\right)=0$$

$$V_2\left[s^3R_1C_1C_2+s^2\left(\frac{R_1C_1}{R_{in}}+C_1+C_2\right)+s\left(\frac{R_1C_2}{L}+\frac{1}{R_{in}}\right)+\frac{R_1}{LR_{in}}+\frac{1}{L}\right]=0$$

$$\dddot{V}_2R_1C_1C_2+\ddot{V}_2\left(\frac{R_1C_1}{R_{in}}+C_1+C_2\right)+\dot{V}_2\left(\frac{R_1C_2}{L}+\frac{1}{R_{in}}\right)+V_2\left(\frac{R_1}{LR_{in}}+\frac{1}{L}\right)=0$$

Observe that this is a 3rd order system. So, it has the potential to go chaotic.

When $R_{in}$ is negative:

$\ddot{V}_2\left(\frac{R_1C_1}{R_{in}}+C_1+C_2\right)$ could be a negative term

$\dot{V}_2\left(\frac{R_1C_2}{L}+\frac{1}{R_{in}}\right)$ could be a negative term

$V_2\left(\frac{R_1}{LR_{in}}+\frac{1}{L}\right)$ could be a negative term

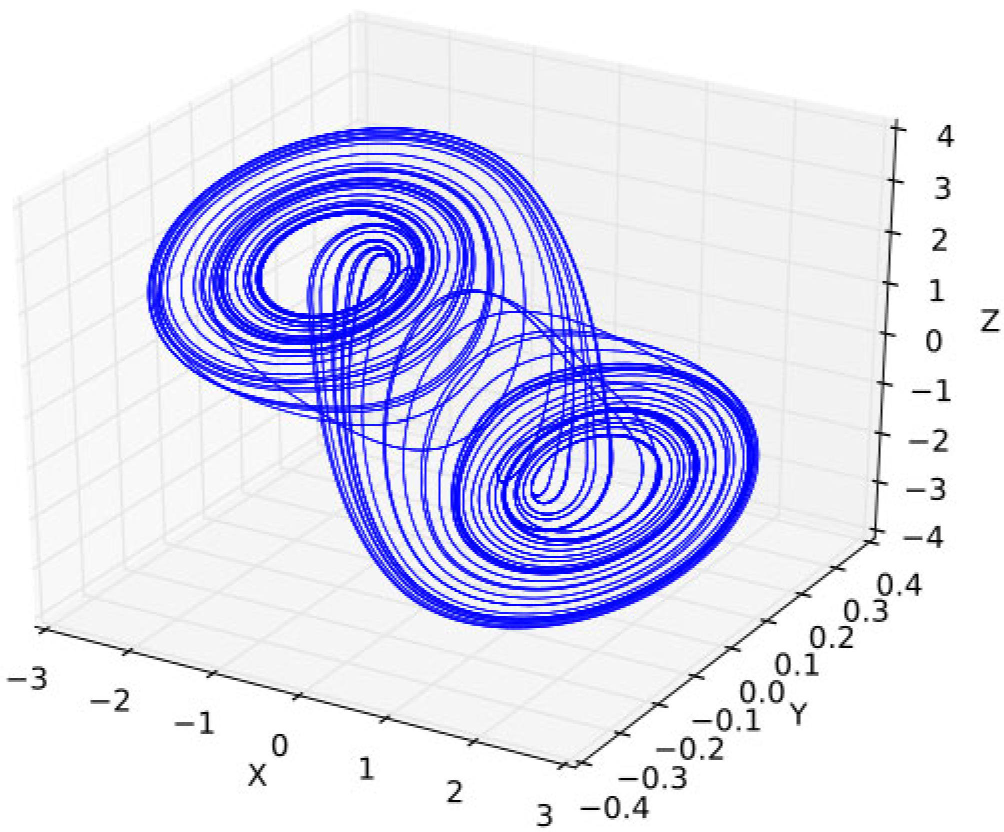
If any one of these terms is negative, then the system is unstable and the size of $V_2$ will increase over time (i.e. the system will oscillate with growing amplitude).

However, when $V_2 > 0.5V$ or $V_2 < -0.5V$, $R_{in}$ becomes positive and the system becomes dissipative.

Due to this nonlinear resistance, the system can oscillate chaotically <see below>.

The strange attractor pattern is called a “double scroll attractor”

Chua Circuit Phase Plot




https://en.wikipedia.org/wiki/Chua%27s_circuit