Thursday 3/2/23

Chaos Theory

Definition: a branch of mathematics that deals with the behavior of nonlinear dynamical systems that:

(1) Are deterministic.
(2) Have a long term aperiodic response.
(3) Have an extreme sensitivity to initial conditions.
(4) Have topological mixing.
(5) Have a spread spectrum frequency response.

(1) Deterministic – The performance is completely described by the governing nonlinear differential equations, and not stochastic events.

(2) Long term aperiodic response – no periodicity.

(3) Extreme sensitivity to initial conditions – very small changes in initial conditions quickly result in an extreme divergence in the two responses (trajectories). Observing this characteristic led to the study of chaos theory.

(4) Topological mixing – in the phase space plot, some finite areas of the plot will eventually get filled as the system runs for a long time.

(5) Spread spectrum frequency response – the aperiodic characteristic results in a wide bandwidth frequency spectrum.

Historical Perspective

Edward Lorenz was a meteorology professor at MIT. In 1961, he was developing a simplified computer model of the weather using 12 variables. One day he ran a simulation, and the results interested him. So, he reran it and got completely different results. He discovered that the first time he ran it, he had used an initial condition of 0.506127. The second time he ran it, he had used an initial condition of 0.506. The two initial conditions differed by just 0.0251%! This would have had little effect on the performance of a linear system. His result was shocking and led to the development of chaos theory. This extreme sensitivity to initial condition led to this statement in 1972: "*Does the flap of a butterfly's wings in Brazil set off a tornado in Texas*" – Philip Merilees.

Eventually, Lorenz simplified the model down to these three equations:

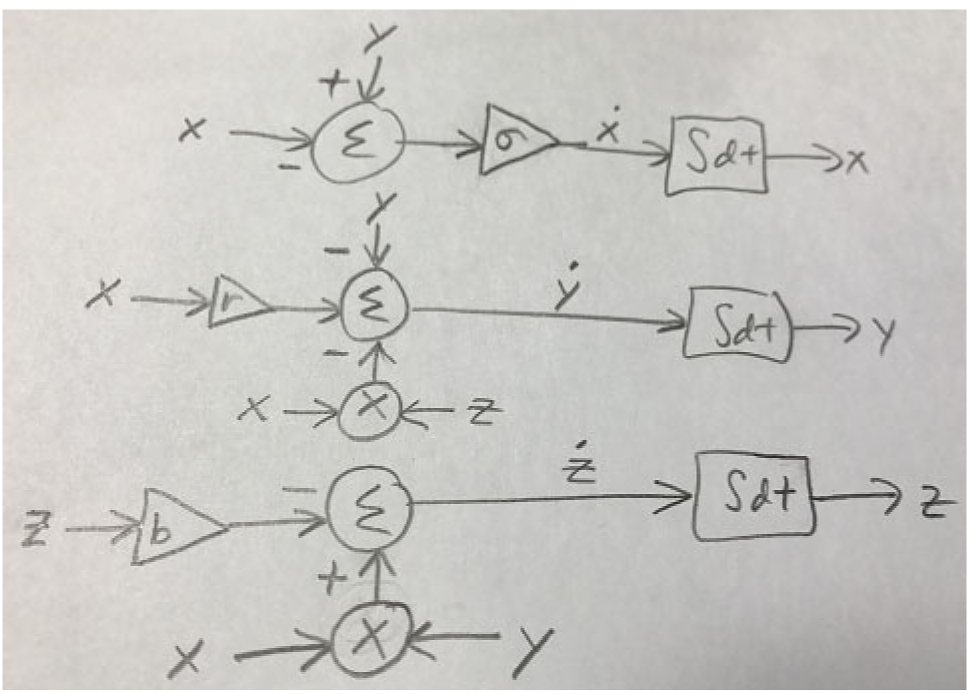
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - bz$$

σ, r and b are constants.

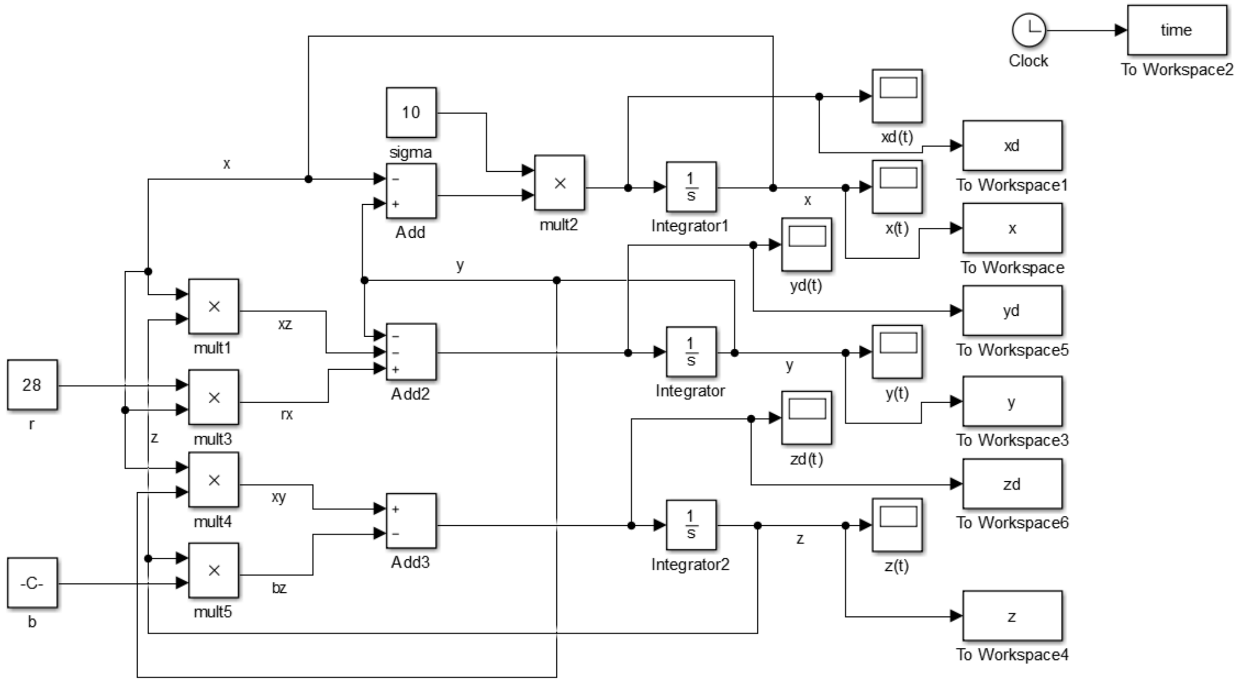
The only nonlinear terms are –xz and xy: two multiplications.

A solution is chaotic for (σ, r, b) = (10, 28, 8/3)

Building the system in hardware or simulation:



Simulink model below (run for 100s):



Clock
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To Workspace2
10
sigma
Add
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Integrator1
xd(t)
xd
To Workspace1
x(t)
x
To Workspace
yd(t)
yd
To Workspace5
y(t)
y
To Workspace3
zd(t)
zd
To Workspace6
z(t)
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To Workspace4
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Integrator
Integrator2
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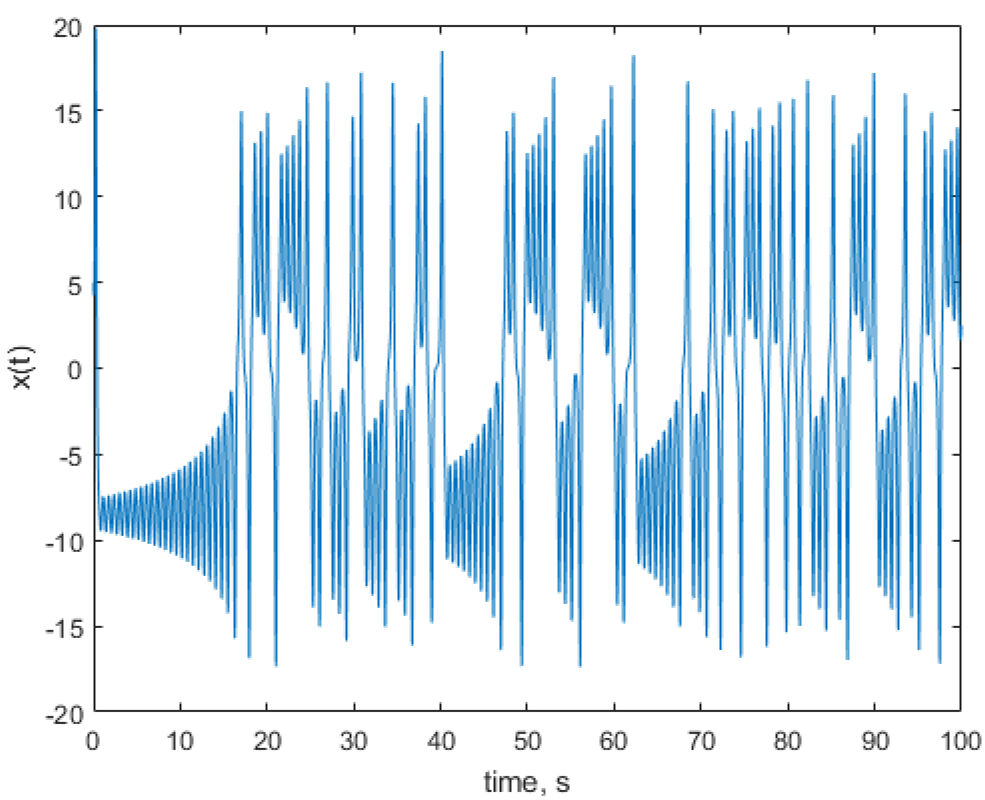


x(t)
time, s

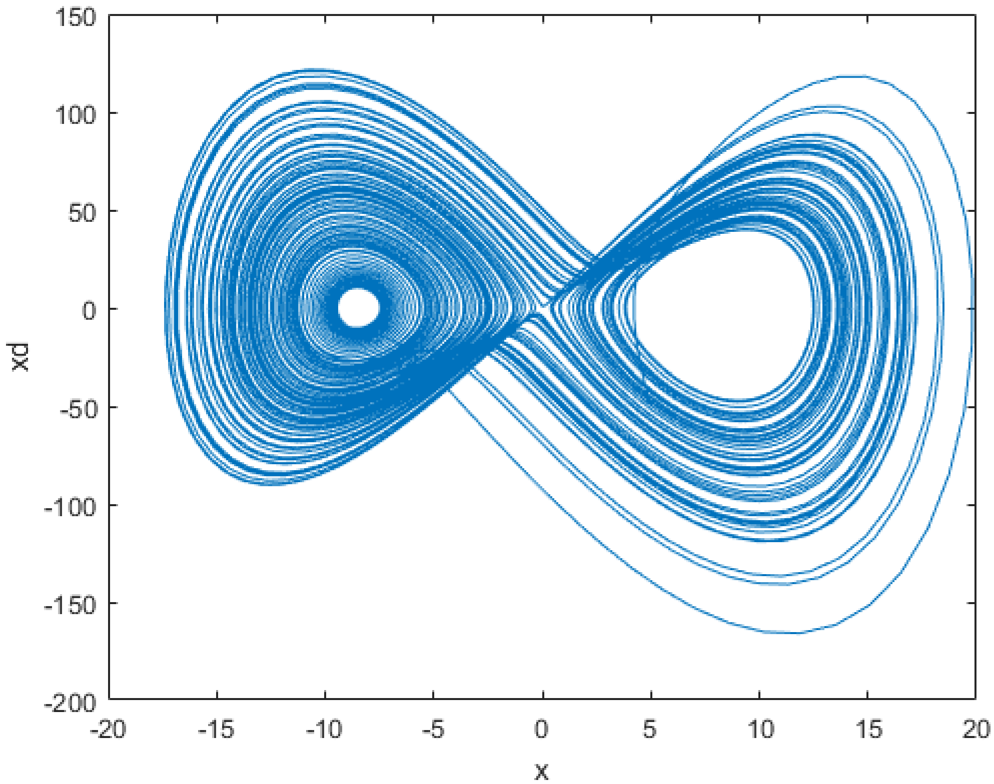
Observe that x(t) appears to oscillate about one point for some period of time, with the amplitude of the oscillation increasing over time. Then at some point in time, x(t) jumps to some new point about which it oscillates for a while before jumping back to the first point. This "jump" happens because some "guard condition" was met that allows the trajectory to then move from orbiting about some point in the phase space to orbiting about some other point in the phase space, and vice versa. So over time, x(t) jumps back and forth between these two points at which it oscillates for some amount of time. These two points by which x(t) oscillates are called "attractors", and specifically due to the chaotic nature of the system they are called "strange attractors." A phase plot of xd vs. x clearly shows this <below>.

Each complete loop around an attractor on the phase plot represents one oscillation of x(t) about the point of oscillation. If the radius of the orbit about an attractor grows, the amplitude of oscillation is increasing.

As shown in the x(t) vs. t plot, the xd vs. x phase plot also shows that after some number of oscillations about one attractor, the trajectory jumps to oscillating about a second attractor.

Observe that the trajectory is only allowed in a certain region of the phase plot, but in that region it can go anywhere. This boundedness of the trajectory shows that the system is globally stable.

The "Basin of Attraction" is the region of the phase space where if a trajectory began at a point via an initial condition, the resulting trajectory would eventually be pulled into orbits oscillating about the attractor(s).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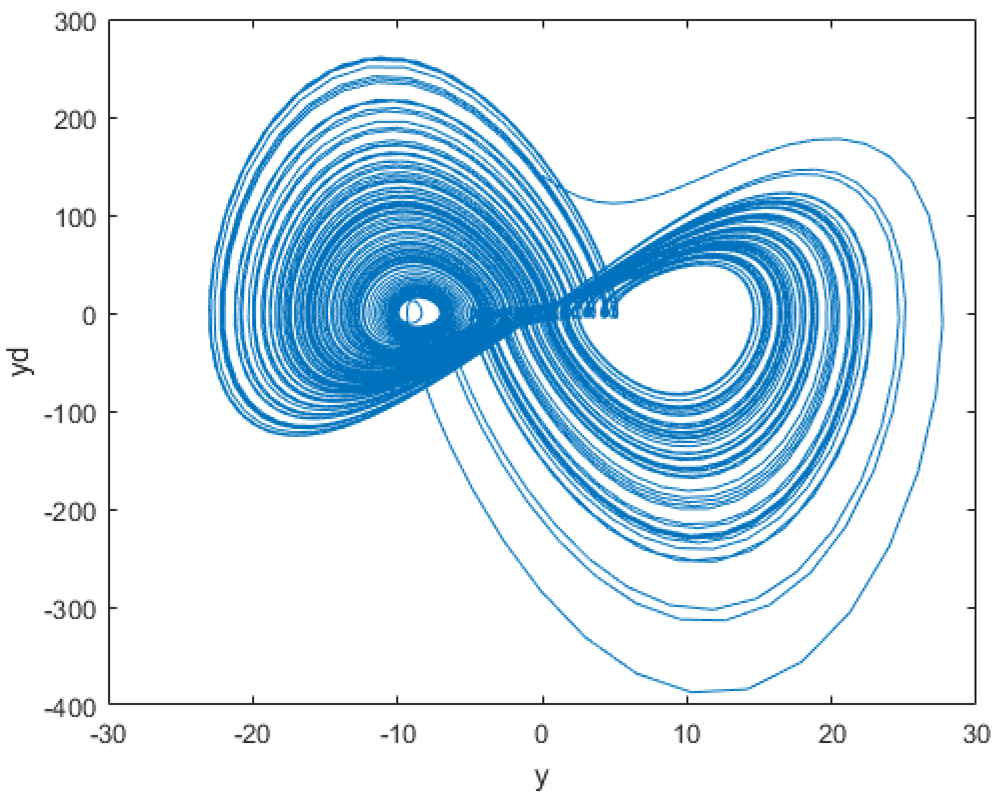
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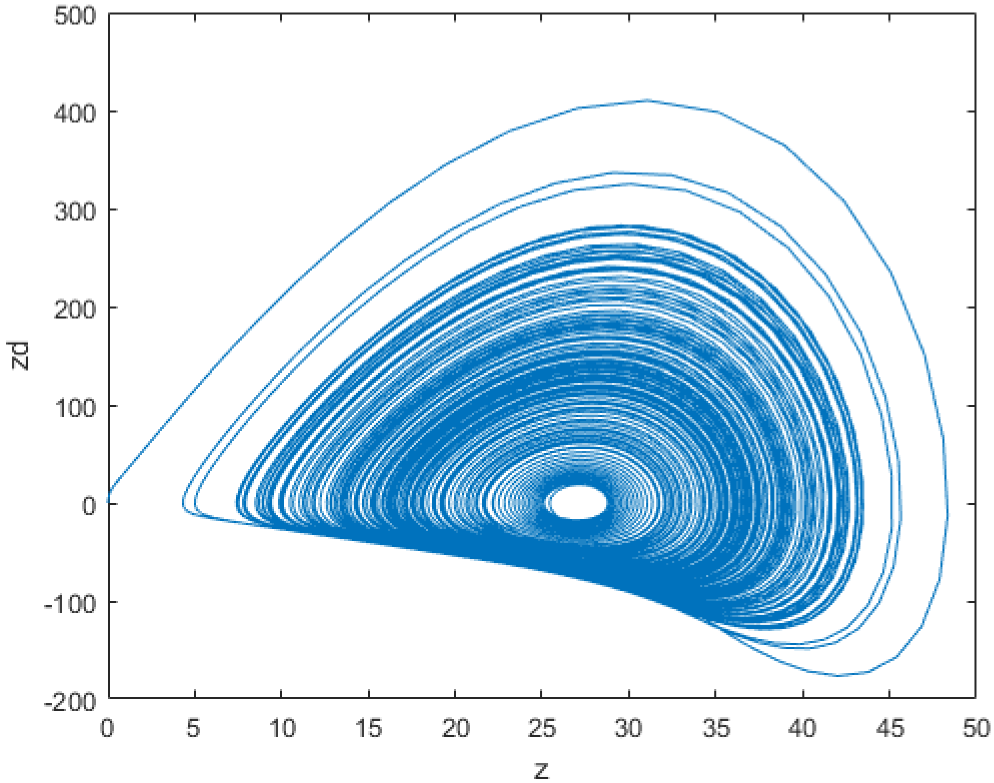
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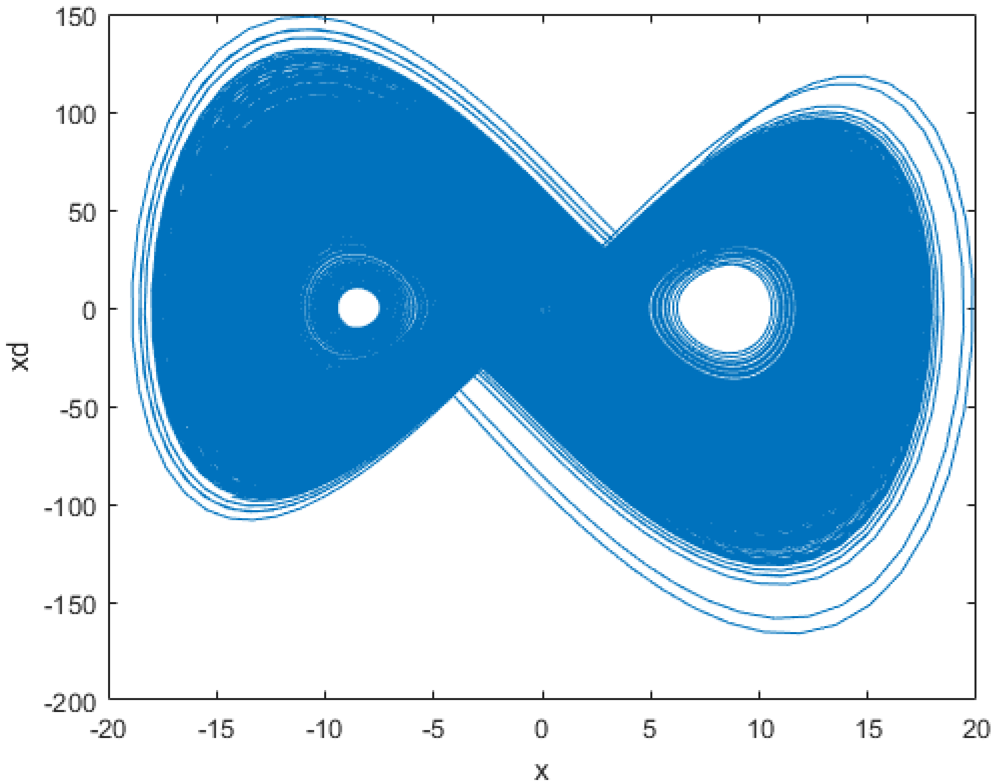
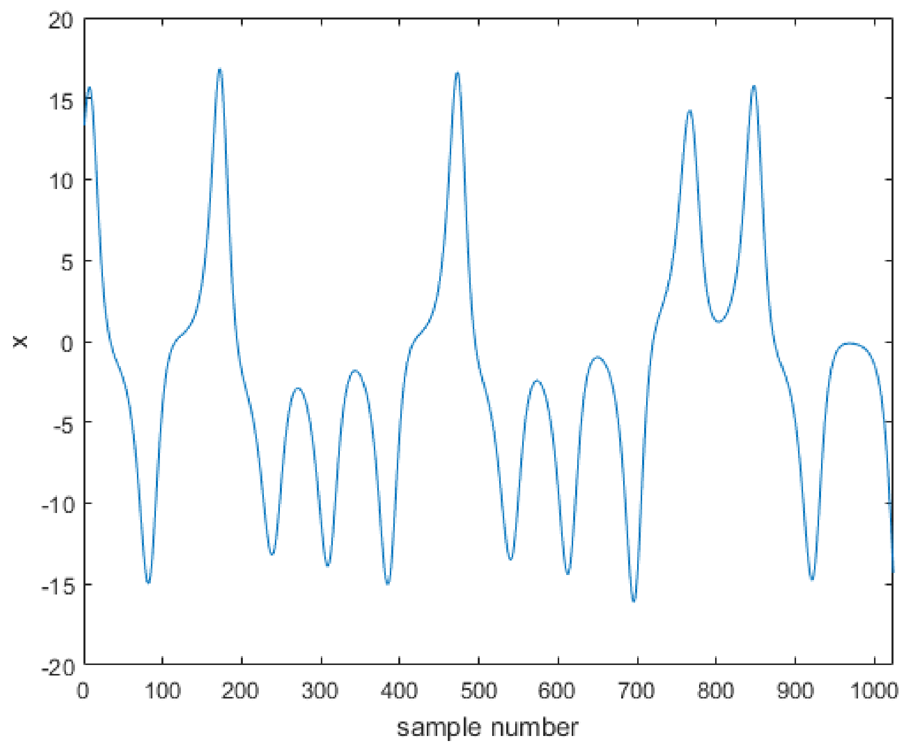
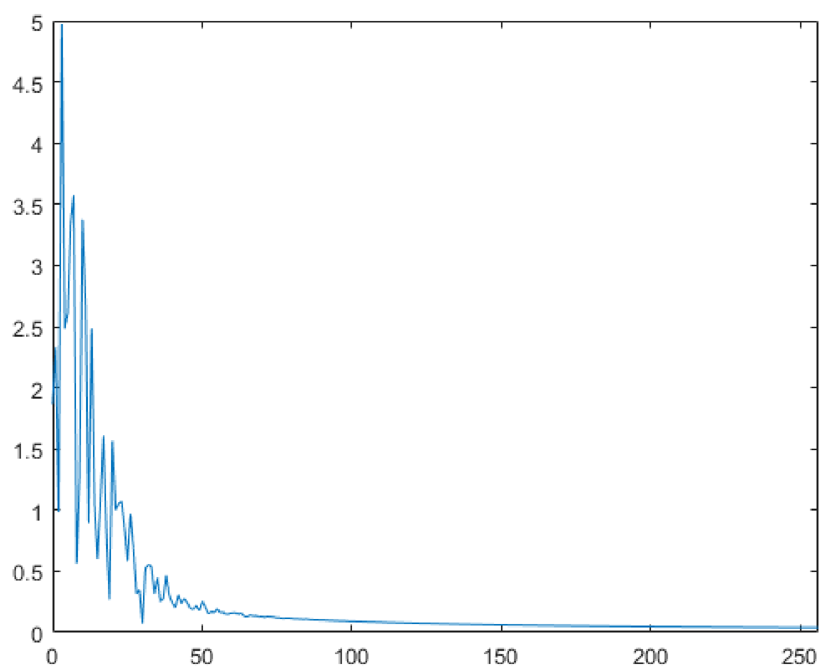


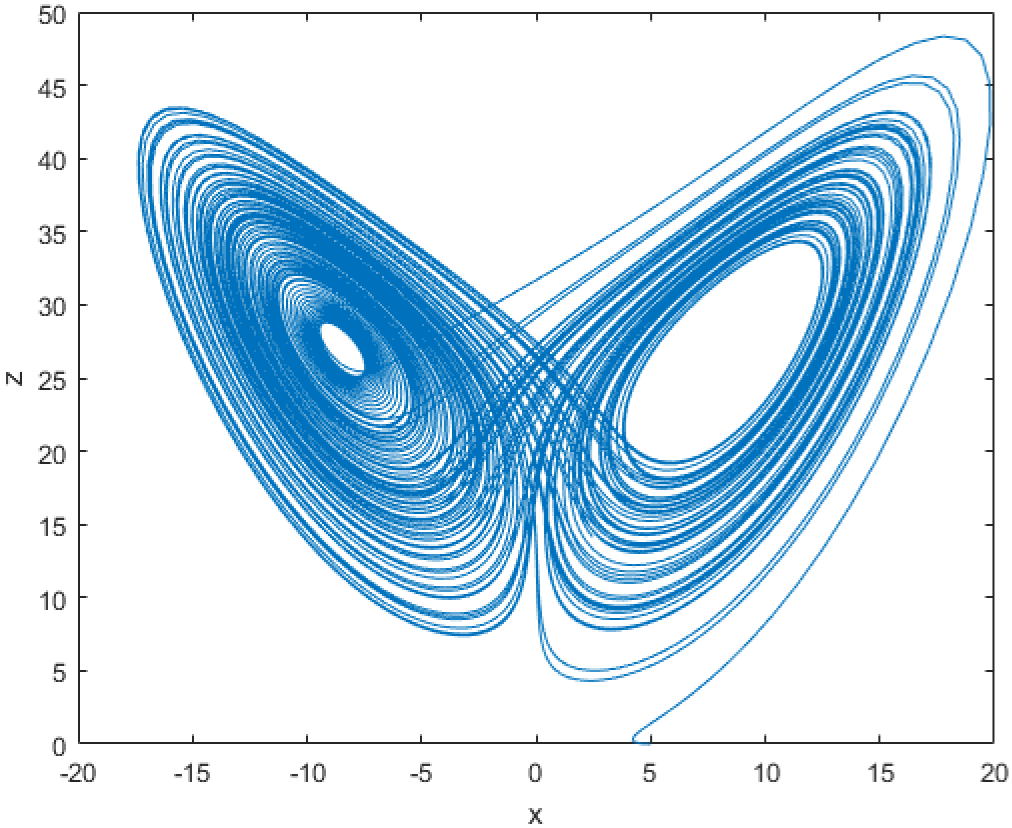
A plot of xd vs. x after running 1000s, illustrating topological mixing.



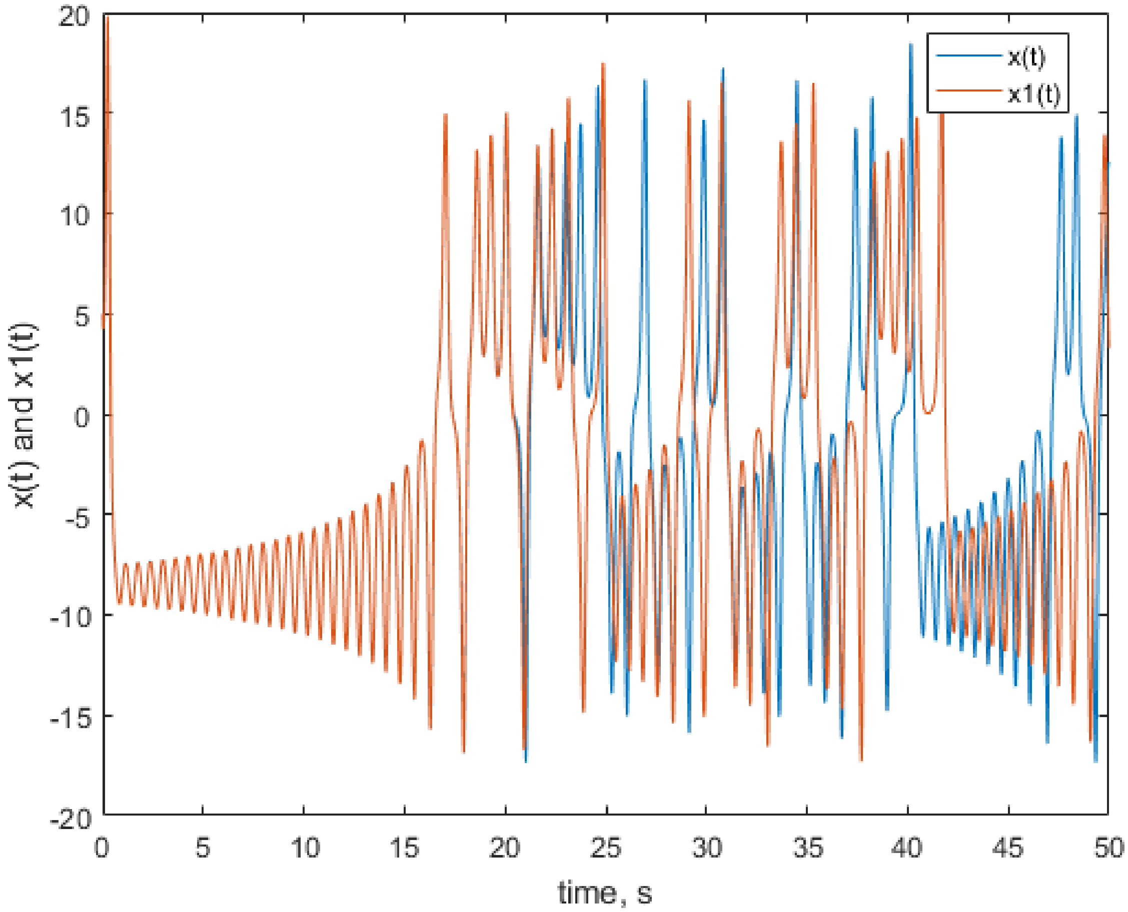
Last 1024 data points of 1000s of x(t)



Illustrative 1024 point FFT of last 1024 data points for 1000s of x(t). Note: only first 256 bins shown. Illustrates wide bandwidth characteristic.



Phase plot of z vs. x, the classic Lorenz attractor “butterfly.”



Exact same simulations except σ=10 for the x(t) system and σ=10.001 for the x1(t) system. The two trajectories quickly diverge, illustrating the extreme sensitivity to initial conditions, a hallmark of chaotic systems.

Building this system in electronic circuitry:

1) Integrators: one op amp, one R and one C per negative integrator
2) Summers: one op amp, R's per negative summer
3) Fixed gain amplifiers: one op amp, R's per amplifier (inv or non-inv)
4) Multipliers: low frequency four quadrant multiplier IC's available

For low frequency operation, it is relatively easy to implement this chaotic system as an electronic circuit.

For high frequency operation, electronic circuit implementation issues arise:

1) Four quadrant multiplier ICs are generally low frequency devices.
2) Required bandwidth increases with operating speed. Chaotic systems are often very sensitive to nonlinear phase responses of op amps. Many high speed op amps are generally not designed for extremely wide bandwidth operation where a linear phase response over the entire bandwidth is required.
3) Op amp gain roll off and phase delay change the realized system governing differential equations.
4) EMI issues such as electromagnetic coupling can adversely affect circuit performance (extreme sensitivity to initial conditions)
5) Noise issues can also adversely affect circuit performance (extreme sensitivity to initial conditions). The thermal noise floor is proportional to the square root of the bandwidth, and therefore increases as the speed of the chaotic system increases.

Philosophical Implications

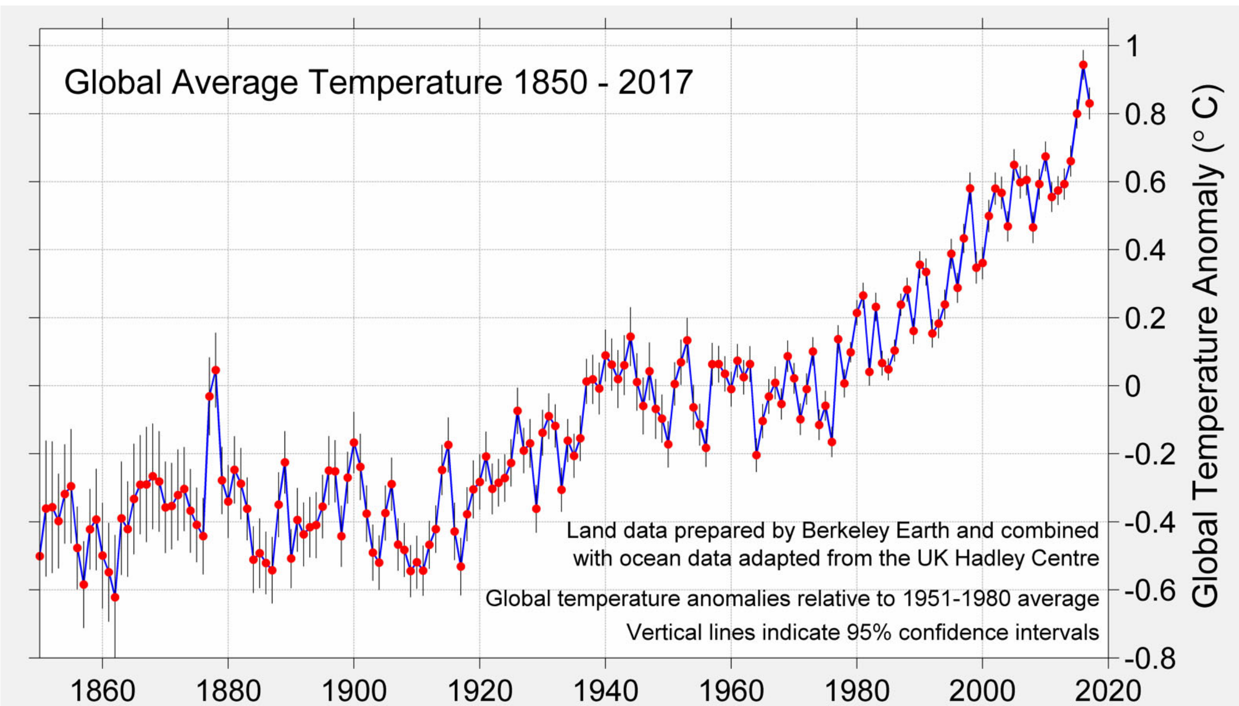
Some people ascribe nearly magical powers to chaotic systems, that they produce order out of disorder, and are responsible for biological evolution, and does this so quickly that it is missed in the fossil record. Why this is a fallacy:

1) Chaotic systems are no more ordered than linear oscillating systems. In fact, some dynamical systems can be linearly oscillating for some parameters and chaotic for others.

2) Whether a dynamical system is linear, nonlinear or chaotic, the observation that it is now functioning in no way explains how that system came into existence.
3) Whether a dynamical system is linear, nonlinear or chaotic, it is still bound by the 2nd law of thermal dynamics: the total entropy (energy given up in a system and not available to do any work) of an isolated system can only remain constant or increase over time. In other words, in an isolated system where work is being done, less energy is available to do additional work as time increases. Systems become less ordered over time and eventually cease functioning (examples: iron rusting, appliances breaking, death of a biological system).
4) Complexity cannot be used as an excuse for missing data: for example, "Because of its high complexity, an event happened so fast there is not any data of its occurrence." If an event cannot be observed, i.e. it cannot be tested, the Scientific Method cannot be applied.

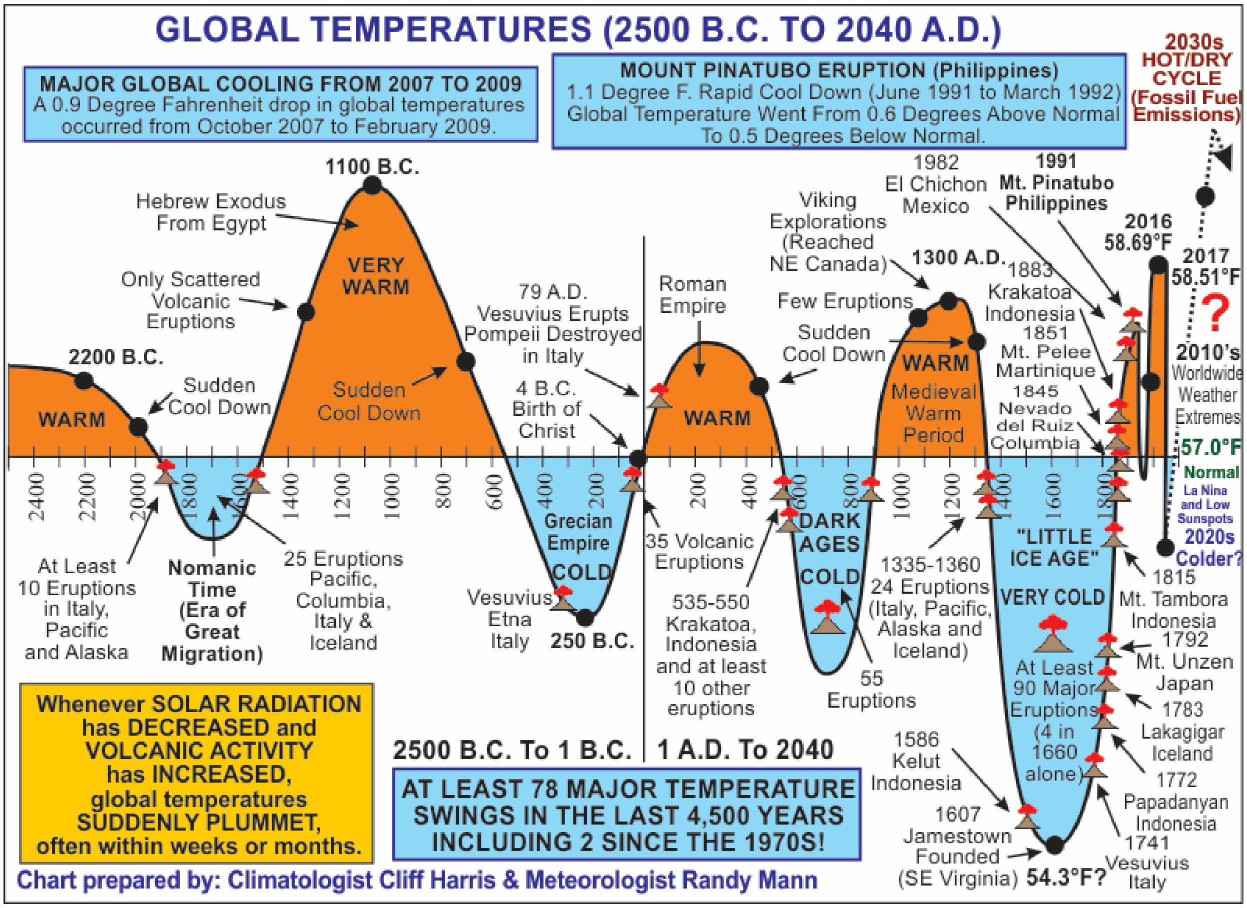
Applications to Weather and Climate: Dr. Lorenz (MIT meteorology professor) demonstrated that weather is a nonlinear system fully capable of chaotic behavior. By inference, climate would also possess these characteristics (nonlinear dynamics). If the climate orbits about an attractor (i.e. if the weather fluctuates about some bias point for a period of time), and then switches to orbiting about a different attractor, this should be expected as normal.

Typical global temperature chart showing global warming since the mid or late 1800s, used to demonstrate anthropogenic global warming:



Curtesy U.C. Berkeley

Global temperatures over the past ~4500 years:



Curtesy www.longrangeweather.com

Climate appears to orbit about warm and cold attractors over time-periods of centuries.