Thursday, 1/19/23

Diving into Dynamical Systems

Dynamical system: a system whose states change with time.

Deterministic dynamical system: a dynamical system where the changes are governed by specific rules.

Stochastic dynamical system: a dynamical system where the changes are governed by random events or inputs instead of by specific rules.

In this class, we are primarily concerned with deterministic dynamical systems. However, random events (such as thermal noise) may set an otherwise deterministic dynamical system in motion.

Parameters of a system do not change with time (at least in simple models), and are often represented by letters near the beginning of the alphabet (A, c, k, m, etc.) or by Greek letters.

Variables are usually represented by letters near the end of the alphabet (t, u, v, x, y, z,). For our purposes, time (t) is the only independent variable. All other variables are dependent variables, functions of time.

Inside the state space, a phase space or phase plot is a plot of two of more states where the states are proportional to each other by derivatives. A 2-D phase plot could be position vs. velocity. A 3D phase plot could be position (x-axis), velocity (y-axis) and acceleration (z-axis).

Undamped second order dynamical system

Given $\ddot{x} + ax = 0$ where $x|_{t=0}$=1

Assume a solution of $x = x_o\cos(\omega_n t)$

Therefore: $\dot{x} = -x_o\omega_n\sin(\omega_n t)$

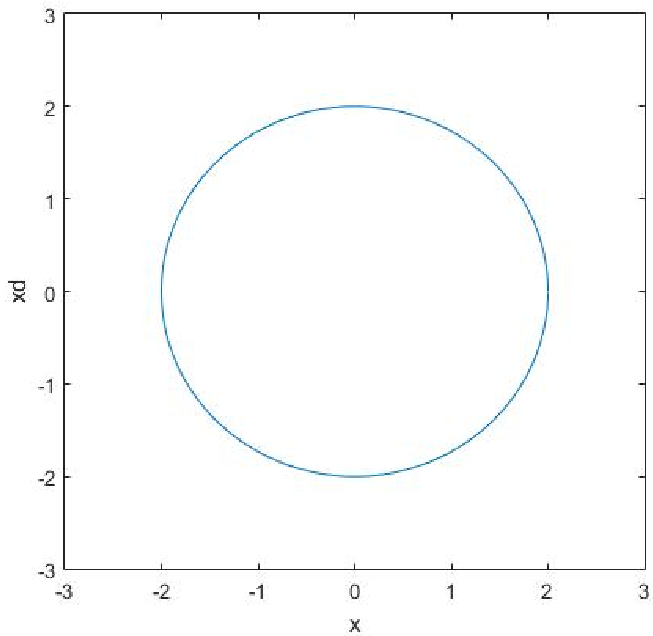
And $\ddot{x} = -x_o\omega_n^2\cos(\omega_n t)$

Equating terms: $-x_o\omega_n^2\cos(\omega_n t) + ax_o\cos(\omega_n t) = 0$

Resulting in $\omega_n = \sqrt{a}$

Let's let a = 1 and x_o = 2 so that $x = 2\cos(t)$ and $\dot{x} = -2\sin(t)$

Then the phase plot for x and $\dot{x}$ is:



It is a circle. This system is undamped, with no energy added to or removed from the system. Positive damping would result in the trajectory spiraling to the origin (0,0) as energy is removed from the system (likely as heat). Negative damping (an unstable system) would result in the trajectory spiraling out as the response "blows up".

Given that $\mathrm{x} = \mathrm{x}_o\cos(\omega_n \mathrm{t})$, the dynamical system of $\ddot{x} + ax = 0$ could represent a simplistic sinusoidal oscillator where "a" determines the frequency of oscillation.

This type of system is a "Conservative System," because energy is conserved, i.e. no energy is lost or dissipated from the system.

In a simple conservative mechanical system, like an ideal mass-spring system, the energy in the system moves between potential energy stored in the spring, $E_p = \frac{1}{2}kx^2$, and kinetic energy stored in the moving mass, $E_k = \frac{1}{2}mv^2$.

In an equivalent lossless electrical circuit, consisting of an ideal inductor and an ideal capacitor, energy oscillates from being stored in the inductor, $E_L = \frac{1}{2}LI^2$, to energy being stored in the capacitor, $E_c = \frac{1}{2}CV^2$. So, an ideal inductor in parallel with an ideal capacitor, with an initial voltage across the capacitor, would realize a simple but unrealizable electrical oscillator.

Real Dynamical Systems

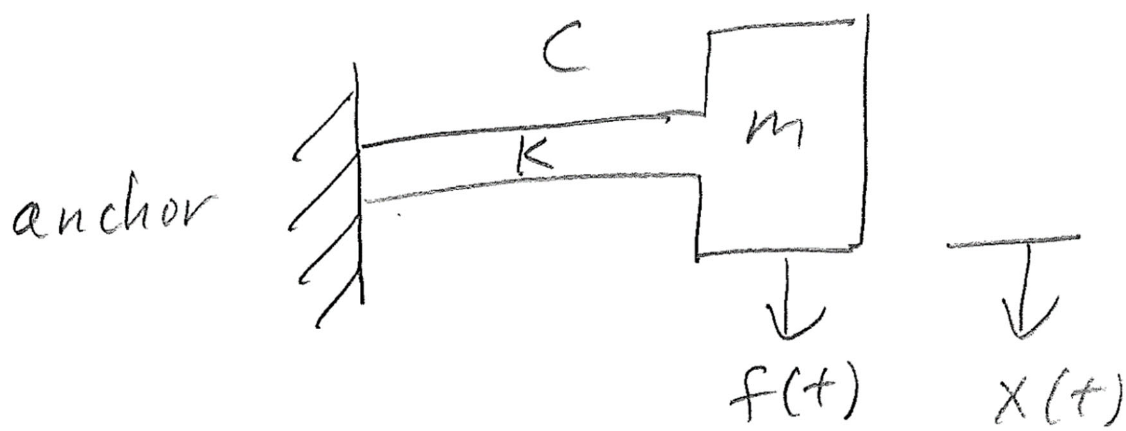
All real dynamical systems have inherent energy loss mechanisms, typically involving the conversion of mechanical or electrical energy into heat:

All real mechanical systems have losses due to friction.

All real inductors and capacitors have resistive losses in them and in their electrical connections. Note, the phenomenon of superconductivity will not be considered here.

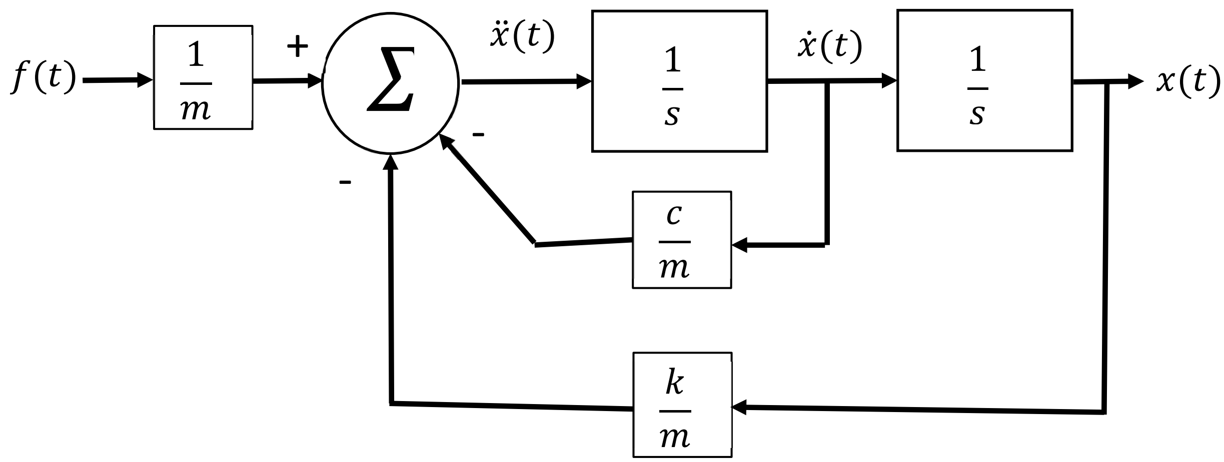
These types of systems are called "Dissipative Systems," since energy is dissipated over time.

Consider the spring-mass-damper system previously discussed:

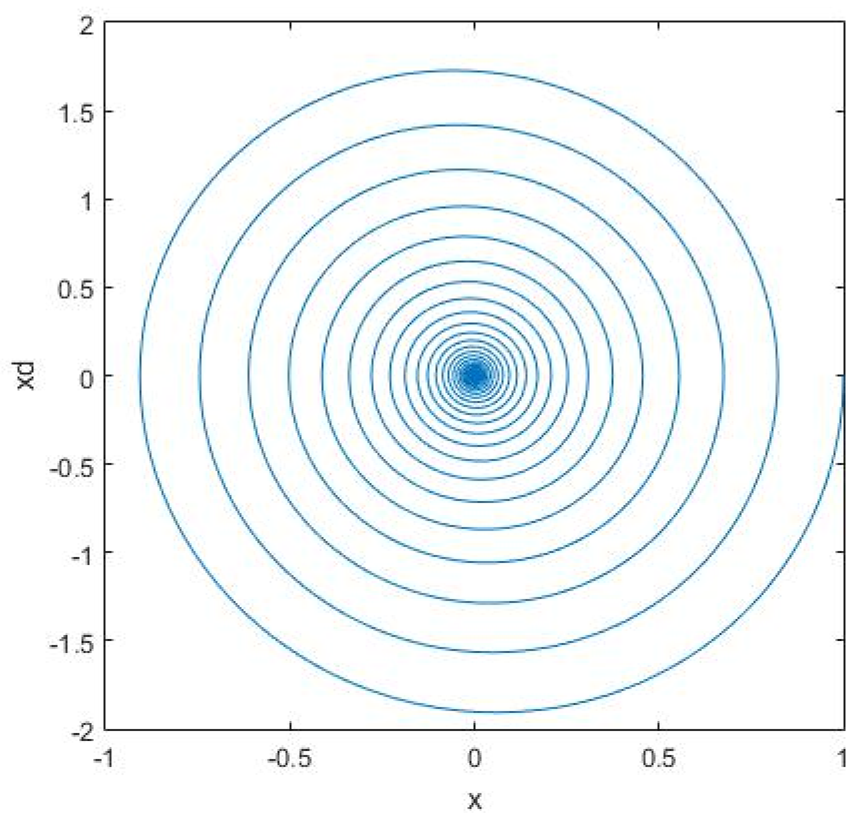


Where : $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$

We can model the system using a simulation diagram from $\ddot{x} = \frac{1}{m}f(t) - \frac{c}{m}\dot{x} - \frac{k}{m}x$:

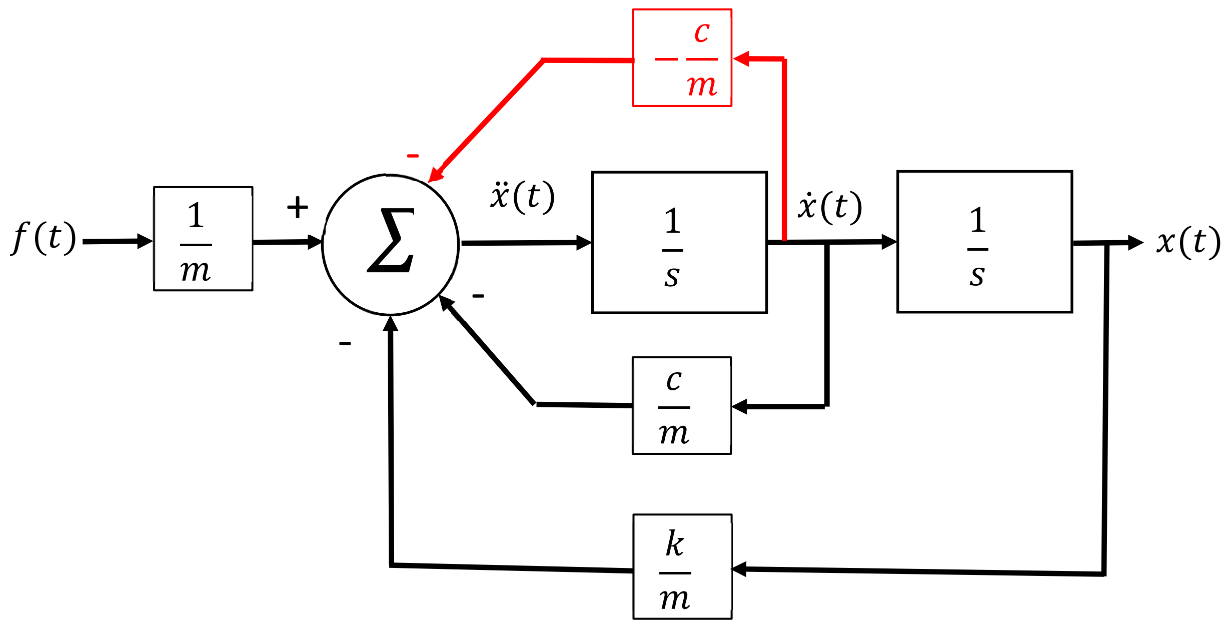


Picking values for (1/m) = 0.5, (c/m) = 0.125, and (k/m) = 4, with no input and an initial condition of x =1, the phase space portrait is:



Which starts at x = 1, xd = 0, and decays to (0,0) as energy is dissipated out of the system. Note, for the sake of simplicity here, parameter units were not used.

How do you make a dissipative system oscillate? Consider this:



A parallel feedback term or network with a gain equal to (-c/m) is added to the system to cancel out the loss due to $\frac{c}{m}\dot{x}$. Now the composite system is "lossless." Actually, though, realizing the $-\frac{c}{m}\dot{x}$ term requires energy, so the overall system is still dissipative, but it still oscillates.

Notice that from our state variable model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t)$$

reduces to:

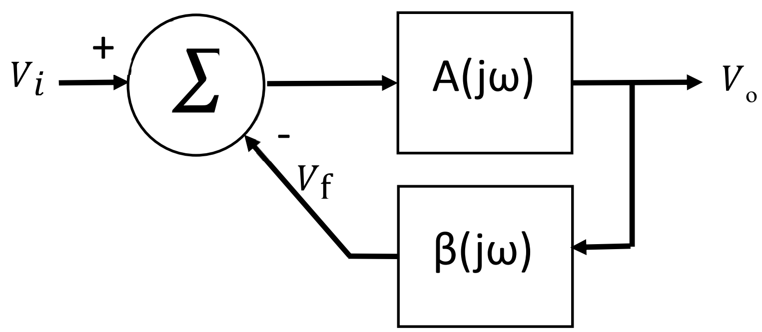
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f(t)$$

where the diagonal is all zeros.

The Barkhausen Stability Criterion

Historically: the idea came from a German physicist, Dr. Heinrich Georg Barkhausen, in 1921.

Consider a linear system modelled by:



This is a <u>negative feedback system</u>, where V_i is an input signal (such as a voltage), V_o is an output signal, A(jω) is the plant, β(jω) is a feedback network and V_f is a feedback signal. The transfer function, G(jω) is

$$G(j\omega) = \frac{V_o}{V_i}(j\omega) = \frac{A(j\omega)}{1 + A(j\omega)\beta(j\omega)}$$

For an oscillator, the input, V_i, is zero.

The loop gain is then defined as A(jω)β(jω).

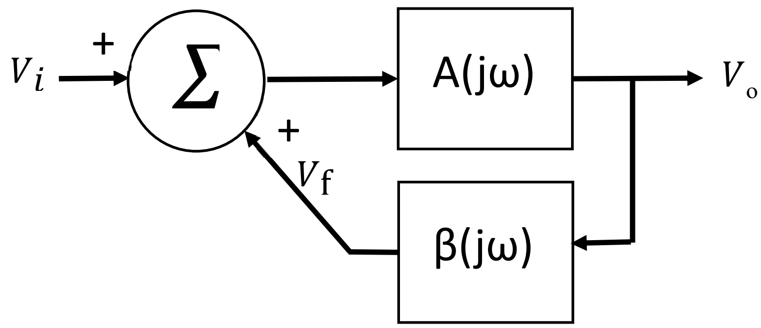
Observe that when $1 + A(j\omega)\beta(j\omega) = 0$, G(jω) → ∞. This is the condition for which the system will oscillate, when the input is zero,

Furthermore,

$$A(j\omega)\beta(j\omega) = -1 = 1\underline{|180°} = 1\underline{|-180°}$$

This typically occurs at one frequency, the frequency of oscillation, although it could occur at more than one frequency for more complex A(jω) and β(jω) networks.

Consider a positive feedback system:



Here:

$$G(j\omega) = \frac{V_o}{V_i}(j\omega) = \frac{A(j\omega)}{1 - A(j\omega)\beta(j\omega)}$$

and the criterion for oscillation is

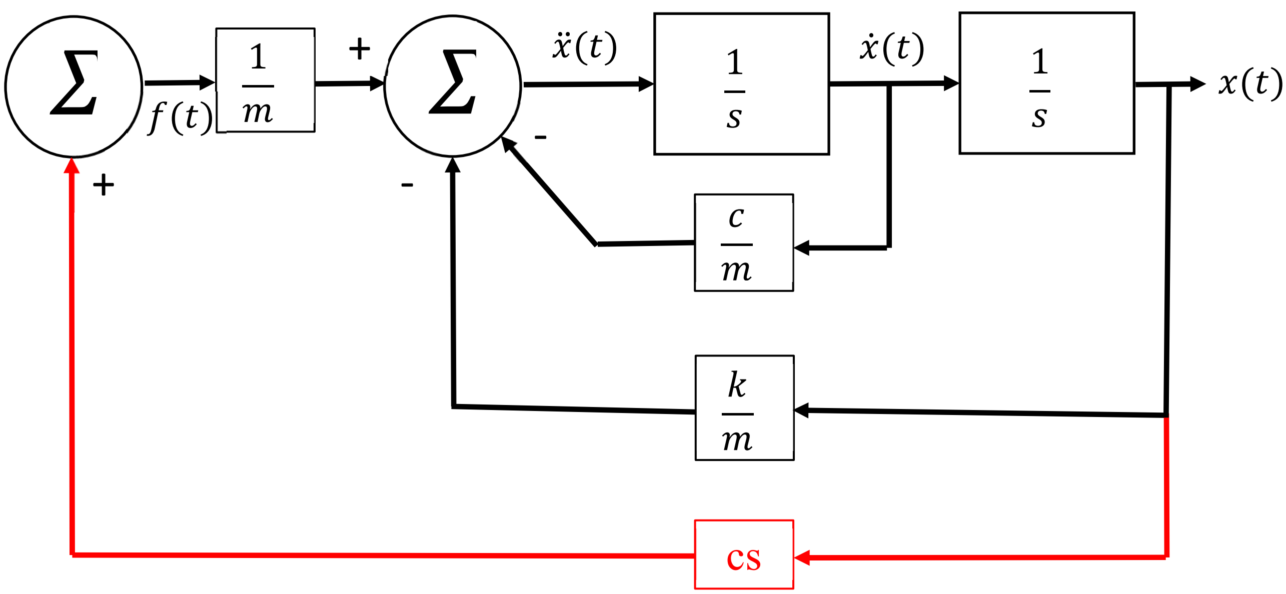
$$A(j\omega)\beta(j\omega) = 1 = 1\underline{|360°} = 1\underline{|n360°}$$

Example: mechanical spring-mass-damper system:

Plant differential equation: $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t)$

Plant transfer equation: $A(j\omega) = \frac{X}{F}(j\omega) = \frac{\frac{1}{m}}{s^2+\frac{c}{m}s+\frac{k}{m}}$

We know that in order to achieve constant oscillation, the damping term must be nulled out. Velocity feedback can be used to achieve this, or to increase or just decrease damping. But for the sake of sticking with the block diagram above, we will feed back the position, x(t), and differentiate it to get a term proportional to velocity:



$$\beta(j\omega) = cs$$

Since this is a positive feedback system, for oscillation according to the Barkhausen stability criterion:

$$A(j\omega)\beta(j\omega) = 1 = 1\underline{|360^\circ} = 1\underline{|0^\circ}$$

Therefore

$$A(j\omega)\beta(j\omega) = \frac{\frac{c}{m}s}{s^2+\frac{c}{m}s+\frac{k}{m}}$$

And

$$|A(j\omega)\beta(j\omega)| = \frac{\frac{c}{m}\omega}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{c}{m}\omega\right)^2}}$$

Therefore

$$|A(j\omega)\beta(j\omega)| = 1 \; at \; \omega = \sqrt{\frac{k}{m}}$$

And

$$phase(A(j\omega)\beta(j\omega)) = \tan^{-1}\left(\frac{\frac{c}{m}\omega}{0}\right) - \tan^{-1}\left(\frac{\frac{c}{m}\omega}{\frac{k}{m} - \omega^2}\right) = 0 \; at \; \omega = \sqrt{\frac{k}{m}}$$

Same result as $\ddot{x} + ax = 0$ where $a = \frac{k}{m}$.