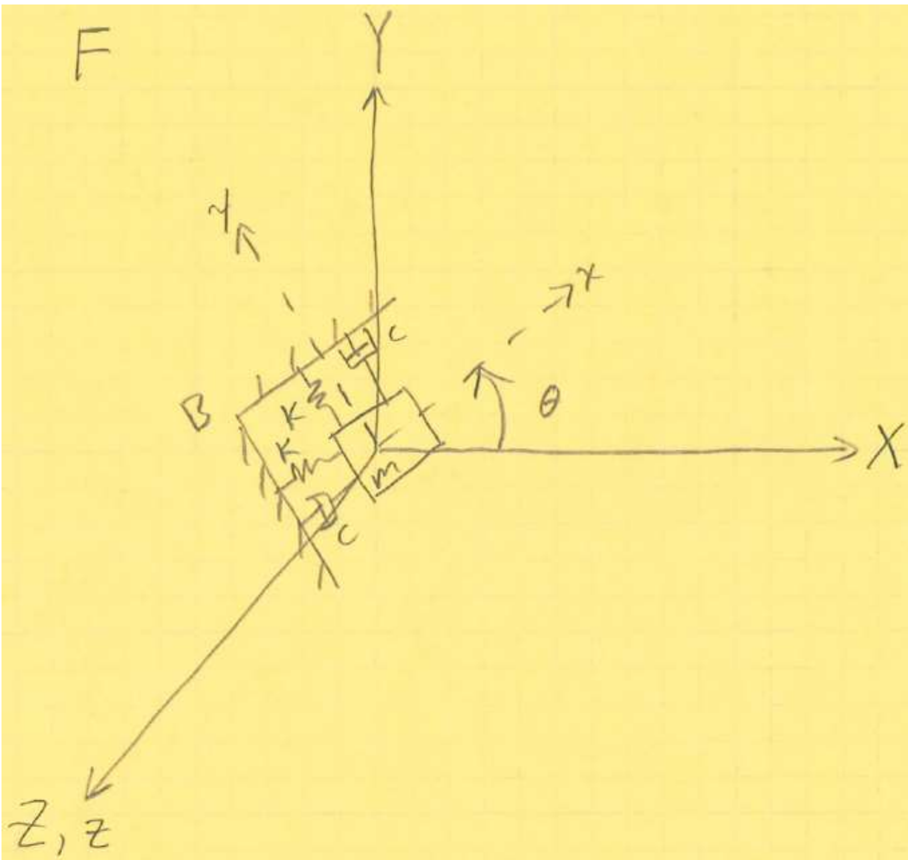
Lecture 11/7/22

Inertial Sensors (MEMS Gyroscopes)

1) Review from the previous lecture

Let reference frame B be in a fixed reference frame F, where B can rotate with respect to F about Z:



X, Y, Z → $\hat{I}$, $\hat{J}$, $\hat{K}$: unit vectors in F.
x, y, z → $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$: unit vectors in B.

Note: z and Z always point in the same direction.

Angular rate: $\dot{\theta} = \frac{d\theta}{dt} = \Omega$
Angular acceleration: $\ddot{\theta} = \frac{d\Omega}{dt} = \alpha$

<u>Identities</u>

$\hat{\imath} = \hat{I}cos(\theta) + \hat{J}sin(\theta)$
$\hat{\jmath} = -\hat{I}sin(\theta) + \hat{J}cos(\theta)$
$\dot{\hat{\imath}} = \Omega\hat{\jmath}$
$\dot{\hat{\jmath}} = -\Omega\hat{\imath}$

Corolis acceleration: $\vec{a}_c$

where $\vec{a}_c = 2\Omega\dot{x}\hat{\jmath} - 2\Omega\dot{y}\hat{\imath}$

→ motion in one axis ($\dot{x}$ or $\dot{y}$) plus rotation about z (Ω) results in motion in the opposite x or y axis:

$\dot{x}\hat{\imath}$ and $\Omega\hat{k} \rightarrow 2\Omega\dot{x}\hat{\jmath}$

$\dot{y}\hat{\jmath}$ and $\Omega\hat{k} \rightarrow -2\Omega\dot{y}\hat{\imath}$

However, higher order terms exist.

Consider the motion of the proof mass:

$\vec{r} = x\hat{\imath} + y\hat{\jmath}$ displacement of m

$$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + x\dot{\hat{\imath}} + y\dot{\hat{\jmath}}$$
$$= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \Omega(x\hat{\jmath} - y\hat{\imath}) \quad \text{velocity of m}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \dot{x}\dot{\hat{\imath}} + \dot{y}\dot{\hat{\jmath}} + \dot{\Omega}(x\hat{\jmath} - y\hat{\imath}) + \Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \Omega(x\dot{\hat{\jmath}} - y\dot{\hat{\imath}})$$
$$= \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \alpha(x\hat{\jmath} - y\hat{\imath}) + \Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \Omega^2(-x\hat{\imath} - y\hat{\jmath})$$
$$= \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + 2\Omega(\dot{x}\hat{\jmath} - \dot{y}\hat{\imath}) + \alpha(x\hat{\jmath} - y\hat{\imath}) - \Omega^2(x\hat{\imath} + y\hat{\jmath})$$ acceleration of m

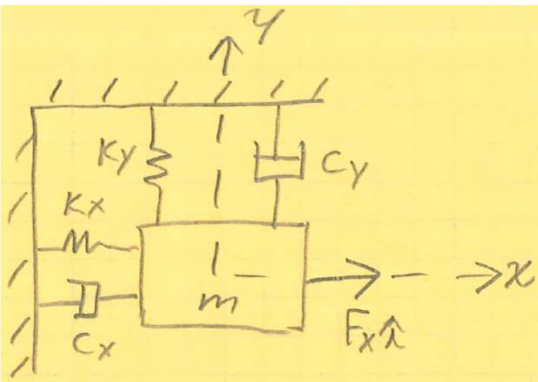
From this expression for $\vec{a}$:

$a_x = \ddot{x} - \alpha y - 2\Omega\dot{y} - \Omega^2 x$ Acceleration component along x

$a_y = \ddot{y} + \alpha x + 2\Omega\dot{x} - \Omega^2 y$ Acceleration component along y

2) System dynamics

Consider this model for the MEMS SMD mechanical system:



$F_x = A_x \sin(\omega_d t)$ → to force m to oscillate along x–axis (using an actuator)

$F_y = 0$ → no force applied to m along y-axis (with an actuator)

There exists a coupling of the equations of motion:

$$m a_x + c_x \dot{x} + k_x x = F_x \quad (1)$$

$$m a_y + c_y \dot{y} + k_y y = F_y = 0 \quad (2)$$

Expanding these equations:

$$m(\ddot{x} - \alpha y - 2\Omega\dot{y} - \Omega^2 x) + c_x\dot{x} + k_x x = A_x \sin(\omega_d t) \qquad (1)$$

$$m(\ddot{y} + \alpha x + 2\Omega\dot{x} - \Omega^2 y) + c_y\dot{y} + k_y y = 0 \qquad (2)$$

We want to solve this set of equations to obtain an expression for y(t). Thankfully, we can make some reasonable simplifying assumptions:

(1) Let $k_x = k_y = k$

(2) Let $c_x = c_y = c$

Note: with (1) and (2): $\omega_{nx} = \omega_{ny} = \omega_n$. Real MEMS gyroscopes usually have $\omega_s > \omega_d$: defined as $\omega_{ny} > \omega_{nx}$, where ω_s is in regard to the sense side and ω_d is in regard to the drive side. Having $\omega_s > \omega_d$ yields better stability and a measurable rotation rate bandwidth.

(3) Assume that the angular acceleration, α, is very slow and can be approximated as $\alpha = 0$ rad/s^2.

(4) Assume that the system natural frequency, ω_n, is much greater than **Ω**, the angular rate being measured. Therefore $\Omega^2 x$ and $\Omega^2 y$ can be approximated by 0.

Example: if $f_n = 10$ kHz: $\omega_n = 2\pi f_n = 62{,}831.8$ rad/s
If $\Omega = 300$ °/s = $300(2\pi/360) = 5.24$ rad/s
And 62,831.8 >> 5.24

Also from EQ 1: $m(\ddot{x} - \alpha y - 2\Omega\dot{y} - \Omega^2 x) + c_x\dot{x} + k_x x = A_x \sin(\omega_d t)$

Examine the "x" terms: $-m\Omega^2 x + k_x x \rightarrow m\left(\frac{k_x}{m} - \Omega^2\right) = m(\omega_n^2 - \Omega^2)$

From the Ω and f_n terms above → $\omega_n^2 = 3.9 \times 10^9$ rad/s and $\Omega^2 = 27.5$ rad/s. So, $\omega_n^2 - \Omega^2 \approx \omega_n^2$

(5) The amplitude of the motion of m along the x-axis will be tightly controlled as a closed loop resonator.

A feedback control system will adjust $F_x = A_x \sin(\omega_d t)$ to precisely keep the motion along the x-axis exactly as desired. Therefore, we can drop the $a_{cx} = -2\Omega\dot{y}$ term in EQ 1, since the controller will null out its effect.

(6) The ω_d from $F_x = A_x \sin(\omega_d t)$ is usually selected so that:

$\omega_d = \omega_n = \sqrt{\frac{k}{m}}$.

This minimizes the amplitude of F_x required to achieve sufficient motion of m along the x-axis → due to high Q.

Therefore, the equations of motion simply to:

$$m\ddot{x} + c\dot{x} + kx = A_x \sin(\omega_n t) \qquad (1)$$

$$m\ddot{y} + c\dot{y} + ky + 2m\Omega\dot{x} = 0 \qquad (2)$$

Clearly, $\Omega\hat{k}$ and motion along the x-axis produces corresponding motion along the y-axis {useful if $\dot{x}$ is consistent (periodic and known)}.

3) Solve for x(t) in steady state

We will start by assuming a solution of the form:

$$x(t) = X_d \cos(\omega_n t)$$

Then: $\dot{x}(t) = -X_d \omega_n \sin(\omega_n t)$

And: $\ddot{x}(t) = -X_d\omega_n^2\cos(\omega_n t)$

Therefore $m\ddot{x} + c\dot{x} + kx = A_x\sin(\omega_n t)$ becomes:

$$-mX_d\omega_n^2\cos(\omega_n t) - cX_d\omega_n\sin(\omega_n t) + kX_d\cos(\omega_n t) = A_x\sin(\omega_n t)$$

Equate cos() and sin() terms:

(1) cos() terms:

$$-mX_d\omega_n^2\cos(\omega_n t) + kX_d\cos(\omega_n t) = 0$$

Reduces to: $kX_d = mX_d\omega_n^2$

And finally to: $\omega_n^2 = \frac{k}{m}$ → true but not helpful.

(2) sin() terms:

$$-cX_d\omega_n\sin(\omega_n t) = A_x\sin(\omega_n t)$$

Reduces to: $X_d = \frac{-A_x}{c\omega_n}$

Therefore: $x(t) = \frac{-A_x}{c\omega_n}\cos(\omega_n t)$

Then: $\dot{x}(t) = -X_d\omega_n\sin(\omega_n t) = \frac{A_x}{c}\sin(\omega_n t)$

F_x produces this motion of m along the x-axis: $x(t) = \frac{-A_x}{c\omega_n}\cos(\omega_n t)$.

Observe that as c increases: Q decreases and the amplitude of x(t) decreases.

4) Solve for the steady state motion of y(t)

From EQ (2): $m\ddot{y} + c\dot{y} + ky + 2m\Omega\dot{x} = 0$

Which can be rewritten as: $m\ddot{y} + c\dot{y} + ky = -2m\Omega\dot{x}$

Plugging in for $\dot{x}$:

$$m\ddot{y} + c\dot{y} + ky = -2m\Omega\frac{A_x}{c}\sin(\omega_n t) = A_y\sin(\omega_n t)$$

Where: $A_y = -2m\Omega\frac{A_x}{c}$

Let's assume a solution for y(t):

$y(t) = Y_d\cos(\omega_n t)$

Then: $\dot{y}(t) = -Y_d\omega_n\sin(\omega_n t)$

And: $\ddot{y}(t) = -Y_d\omega_n^2\cos(\omega_n t)$

Therefore: $m\ddot{y} + c\dot{y} + ky = A_y\sin(\omega_n t)$ becomes:

$$-mY_d\omega_n^2\cos(\omega_n t) - cY_d\omega_n\sin(\omega_n t) + kY_d\cos(\omega_n t) = A_y\sin(\omega_n t)$$

Equating the sin() terms:

$$-cY_d\omega_n\sin(\omega_n t) = A_y\sin(\omega_n t)$$

Therefore: $Y_d = -\frac{A_y}{c\omega_n} = \frac{2m\Omega A_x}{c^2\omega_n}$: NOTE: <u>use this for HW#9 probs 9&10</u>

So: $y(t) = \frac{2mA_x}{c^2\omega_n}\Omega\cos(\omega_n t) = G_1\,\Omega\cos(\omega_n t)$

Where: $G_1 = \frac{2mA_x}{c^2\omega_n}$

With the resulting motion along the y-axis: measure y(t), multiply that measurement by $A\cos(\omega_n t)$ and then LPF the product, which results in a DC signal proportional to **Ω**.