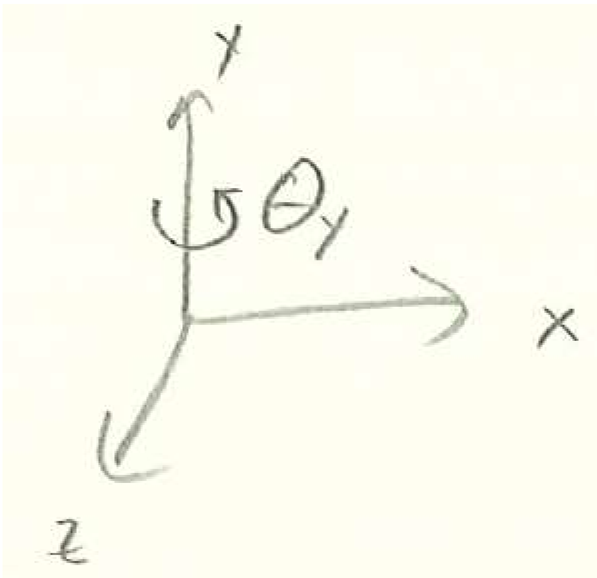
Lecture 11/4/22

Inertial Sensors (MEMS Gyroscopes)

1) Introduction

Gyroscopes measure angular motion, but not translational motion:



Newton's first law of motion:

An object in translational motion continues in that motion unless it is altered by an external force.

An object in rotational motion continues in that motion unless it is altered by an external torque.

∴ Conservation of momentum: both translational (linear) and rotational (angular).

(1) Macroscale gyroscopes

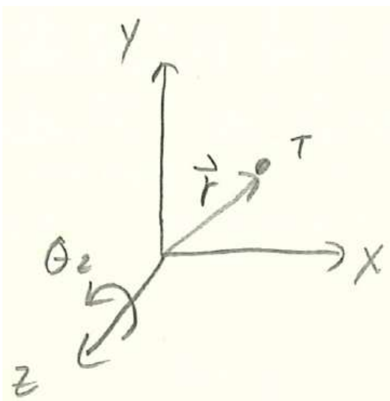
These gyroscopes use a large spinning mass and conservation of angular momentum to measure the angler rate of precession (off axis rotation rate of the spinning object due an applied torque).

(2) Microscale gyroscopes

The most common approach for these gyroscopes is to use a small mass and vibrate it in one direction and then detect orthogonal in-plane motion due to angular rotation about the 3rd axis → due to the Coriolis acceleration, a_c.

2) Let's examine the Coriolis force and acceleration

Consider:



Where the position vector is: $\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath}$, and

the velocity vector is: $\vec{V_r} = V_{rx}\hat{\imath} + V_{ry}\hat{\jmath} = \frac{d}{dt}\vec{r}$.

Given that the coordinate system rotates about the z-axis at: $\vec{\Omega}_z = \Omega_z\hat{k} = \dot{\theta}_z\hat{k}$, the object, T, experiences a "virtual force" in the x-y plane due the $\Omega_z\hat{k}$ rotation.

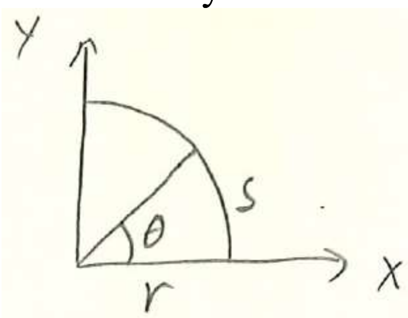
This "virtual force" is the Coriolis force, and it results in a Coriolis acceleration in the rotating x-y plane → $\vec{a}_c$.

$$\vec{a}_c = 2\vec{\Omega}_z \times \vec{V_r} = \Omega_z\hat{k} \times \left(V_{rx}\hat{\imath} + V_{ry}\hat{\jmath}\right) = 2\Omega_z V_{rx}\hat{\jmath} - 2\Omega_z V_{ry}\hat{\imath},$$

where: $a_{cy} = 2\Omega_z V_{rx}$ and $a_{cx} = -2\Omega_z V_{ry}$.

Observe that $a_{cy} \propto \Omega_z\ multiplied\ by\ V_{rx}$.

Another way to look at this:



$S = r\theta$

$\dot{S} = \dot{r}\theta + r\dot{\theta}$

$\ddot{S} = \dot{r}\dot{\theta} + \ddot{r}\theta + r\ddot{\theta} + \dot{r}\dot{\theta}$

$= 2\dot{r}\dot{\theta} + \ddot{r}\theta + r\ddot{\theta}$

$= 2\dot{r}\Omega_z + \ddot{r}\theta + r\alpha_z$

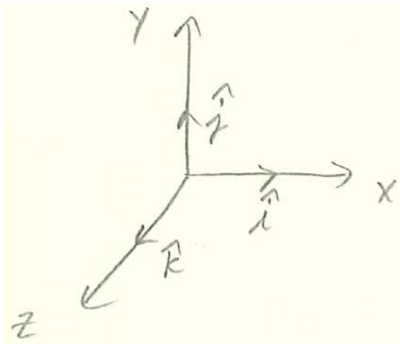
If $\theta \approx 0$ and $\ddot{\theta} \approx 0$, then $\ddot{S} \approx a_{cy} = 2V_{rx}\Omega_z$

If we were technicians, then knowing the $\vec{a}_c$ equation would be sufficient. But we are engineers and we should therefore know more.

3) Review of unit vectors

A unit vector is a normalized vector of length 1.

In Cartesian co-ordinates: 3 unit vectors in the x, y, and z directions:



For the cross product of unit vectors: use the right hand rule:

(1) $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$

(2) $\hat{\imath} \times \hat{\jmath} = \hat{k}$

$\hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{k} = \hat{\imath}$

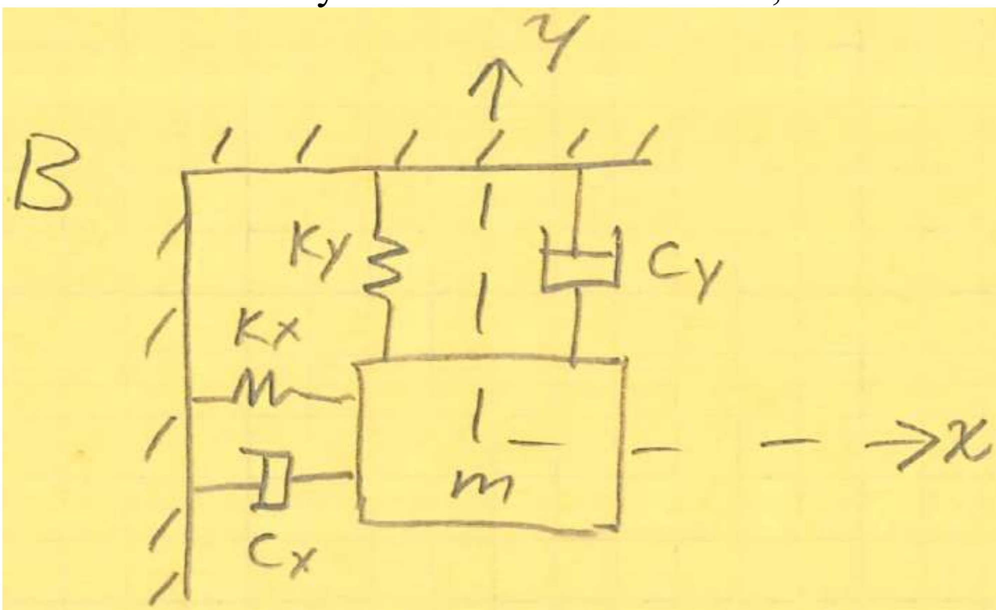
(3) $\hat{\jmath} \times \hat{\imath} = -\hat{k}$

$\hat{k} \times \hat{\jmath} = -\hat{\imath}$

$\hat{\imath} \times \hat{k} = -\hat{\jmath}$

4) Modeling an SMD in an inertial reference frame

Consider a SMD system in a reference frame, B:



Assume that the mass, m, can only move in the x and y directions with no rotation in B.

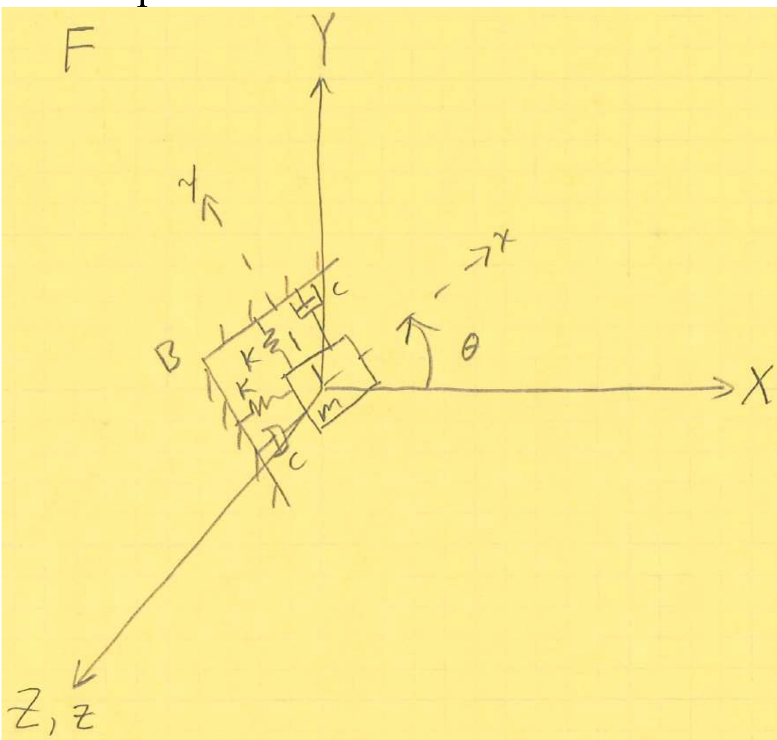
Although k_x is not necessarily the same as k_y, let's let $k_x = k_y = k$ for simplicity.

Therefore $\omega_{nx} = \omega_{ny}$ here.

Similarly, c_x is not necessarily the same as c_y, but let's let $c_x = c_y = c$ for simplicity.

5) Modeling an SMD in a rotating inertial reference frame

Let reference frame B be in a fixed reference frame F where B can rotate with respect to F:



X, Y, Z are in F → $\hat{I}$, $\hat{J}$, $\hat{K}$: unit vectors in F.

x, y, z are in B → $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$: unit vectors in B.

Note: z and Z always point in the same direction.

Let's explore the relationship between B and F:

$\hat{\imath} = f\left(\hat{I},\hat{J}\right)$ and $\hat{\jmath} = f\left(\hat{I},\hat{J}\right)$

$\therefore \hat{\imath} = \hat{I}cos(\theta) + \hat{J}sin(\theta)$

Note: if $\theta = 0^o \rightarrow \hat{\imath} = \hat{I}$

if $\theta = 90^o \rightarrow \hat{\imath} = \hat{J}$

$\therefore \hat{\jmath} = -\hat{I}sin(\theta) + \hat{J}cos(\theta)$

Note: if $\theta = 0^o \rightarrow \hat{\jmath} = \hat{J}$

if $\theta = 90^o \rightarrow \hat{\jmath} = -\hat{I}$

Angular rate: $\dot{\theta} = \frac{d\theta}{dt} = \Omega$

Angular acceleration: $\ddot{\theta} = \frac{d\Omega}{dt} = \alpha$

6) Derivatives of unit vectors

Note: $\frac{d}{dt}(\cos(\theta)) = -\dot{\theta}\sin(\theta) = -\Omega\sin(\theta)$

And: $\frac{d}{dt}(\sin(\theta)) = \dot{\theta}\cos(\theta) = \Omega\cos(\theta)$

With that in mind:

$$\therefore \frac{d}{dt}(\hat{\imath}) = \frac{d}{dt}\left(\hat{I}cos(\theta) + \hat{J}sin(\theta)\right)$$

$$= -\hat{I}\Omega sin(\theta) + \hat{J}\Omega cos(\theta)$$

$$= \Omega\left(-\hat{I}sin(\theta) + \hat{J}cos(\theta)\right)$$

But: $\hat{\jmath} = -\hat{I}sin(\theta) + \hat{J}cos(\theta)$

$$\therefore \frac{d}{dt}(\hat{\imath}) = \Omega\hat{\jmath}$$

$$\therefore \; \frac{d}{dt}(\hat{j}) = \frac{d}{dt}\left(-\hat{I}sin(\theta) + \hat{J}cos(\theta)\right)$$

$$= -\hat{I}\Omega cos(\theta) - \hat{J}\Omega sin(\theta)$$

$$= -\Omega\left(\hat{I}cos(\theta) + \hat{J}sin(\theta)\right)$$

$$= -\Omega\hat{\imath}$$

Therefore: $\dot{\hat{\boldsymbol{\imath}}} = \Omega\hat{\boldsymbol{\jmath}}$ **and** $\dot{\hat{\boldsymbol{\jmath}}} = -\Omega\hat{\boldsymbol{\imath}}$