

Thursday 10/16/25

The Logistic Map and Chaos

Chaos can be observed in a simple discrete model for the population dynamics within a species. For this:

r is the reproductive rate, which decreases as the population increases,

x is a measure of the population as a fraction of the carrying capacity,

x_0 is the starting population,

x_k is the population at the k^{th} interval,

x_{k+1} is the population at the $(k+1)^{\text{th}}$ interval,

In this simple system, the population completely dies out each interval, and the next interval's population is a function of this interval's reproductive rate and current population.

$$r(x) = r(1 - x)$$

Therefore, this equation models the population at successive generations (intervals), k :

$$x_{k+1} = rx_k(1 - x_k)$$

The plot of $x(k)$ vs. k is called the logistic map.

Consider the logistic maps for some cases for different reproductive rates, r , with an initial population, x_0 , of 0.1.

If $r < 1$, the population eventually dies out.

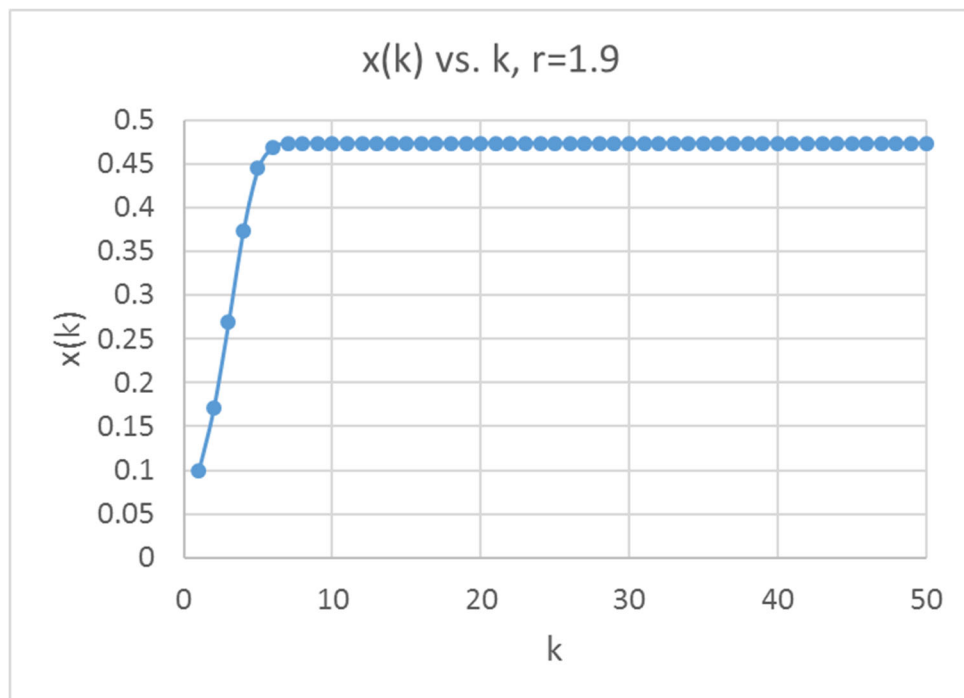
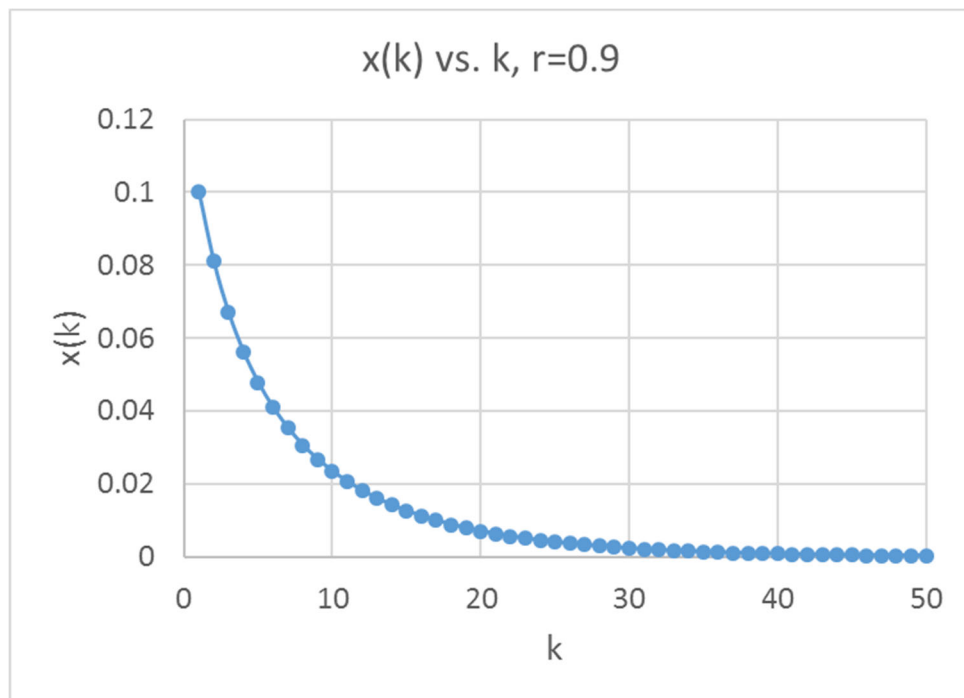
If $r=1.9$, the population settles out at 0.47 (47% of the carrying capacity).

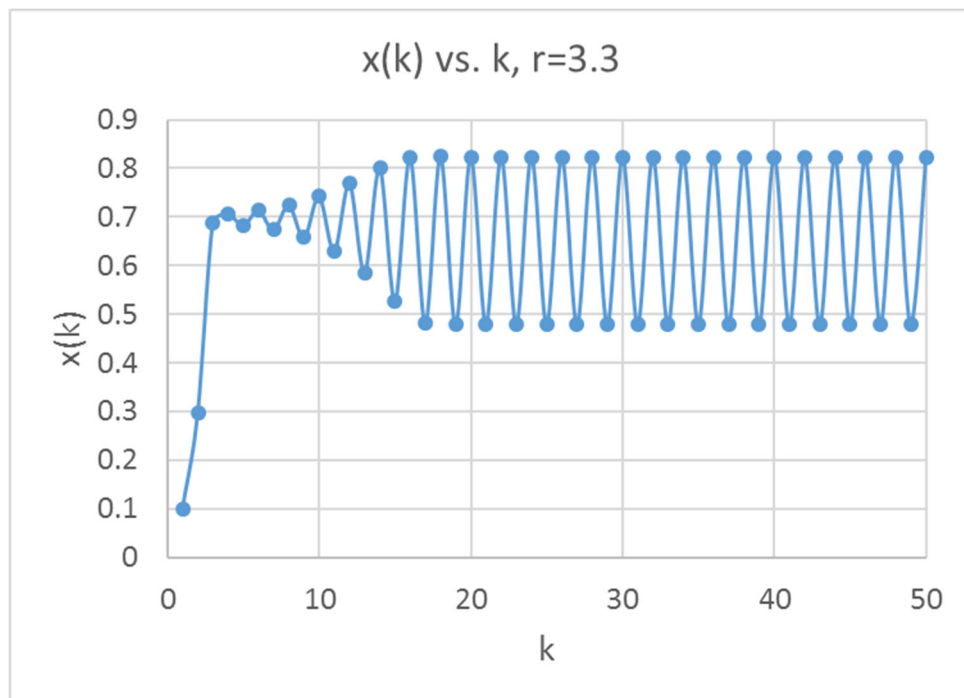
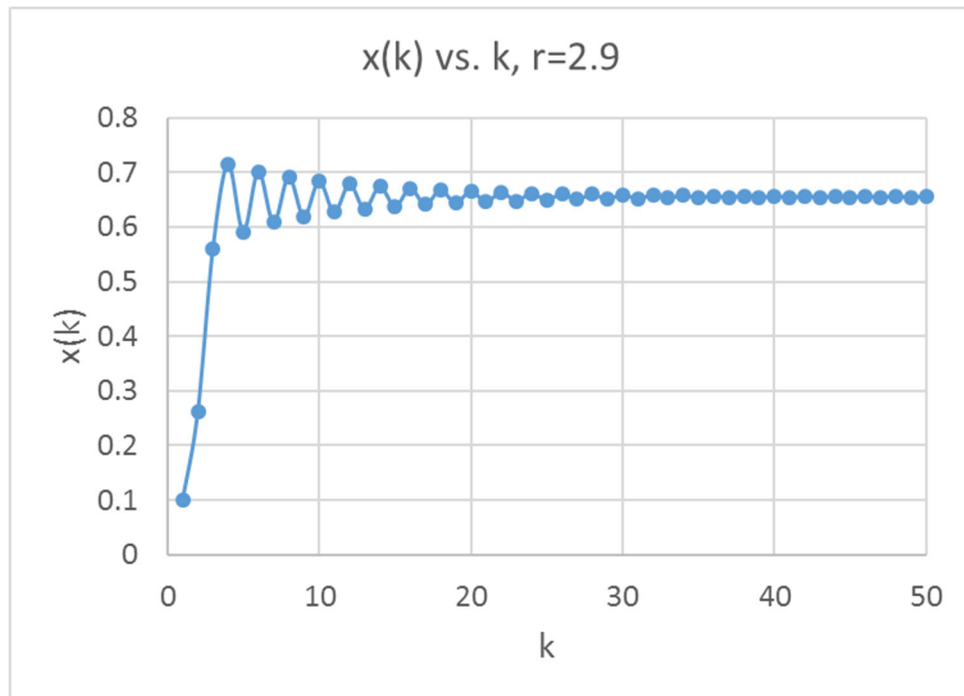
If $r=2.9$, the population stabilizes after several small boom and bust cycles with a population at 65.6% of the carrying capacity.

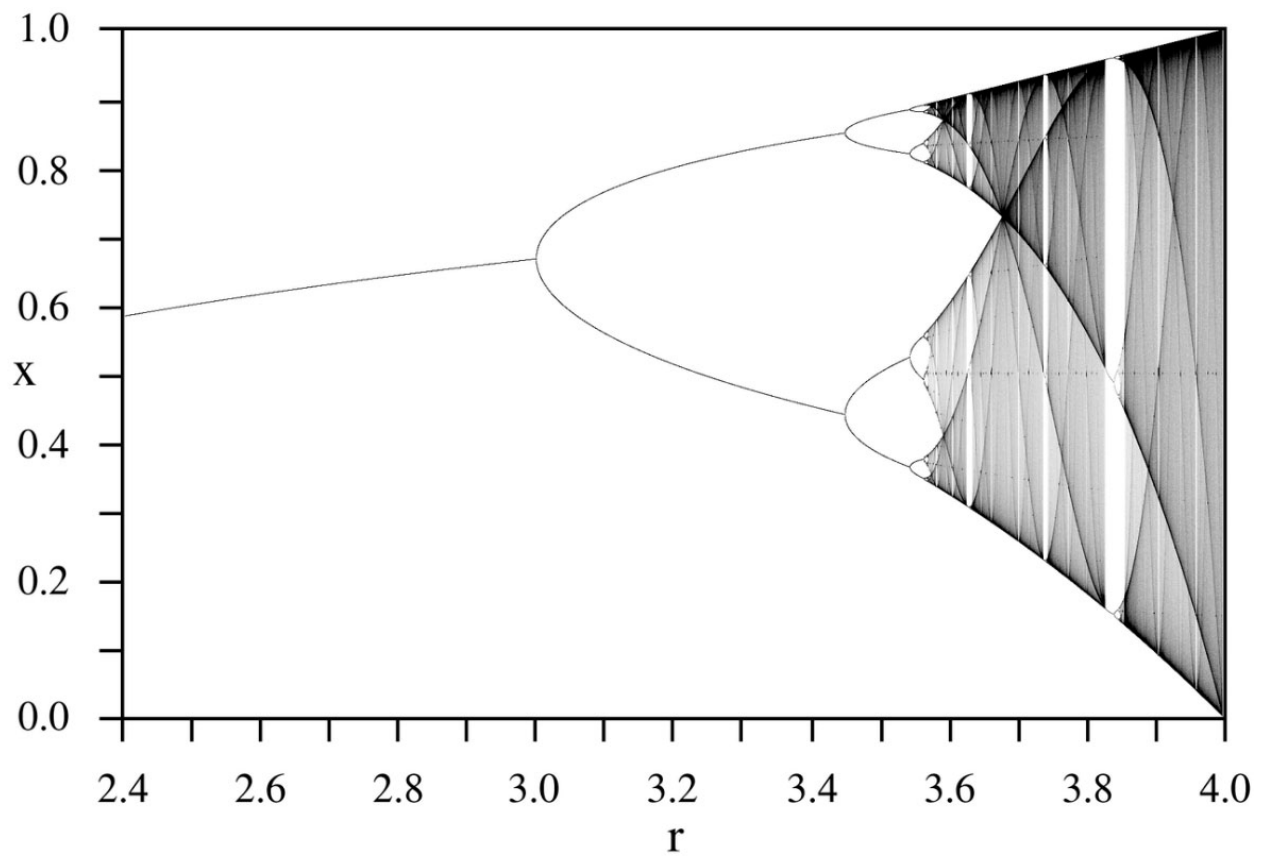
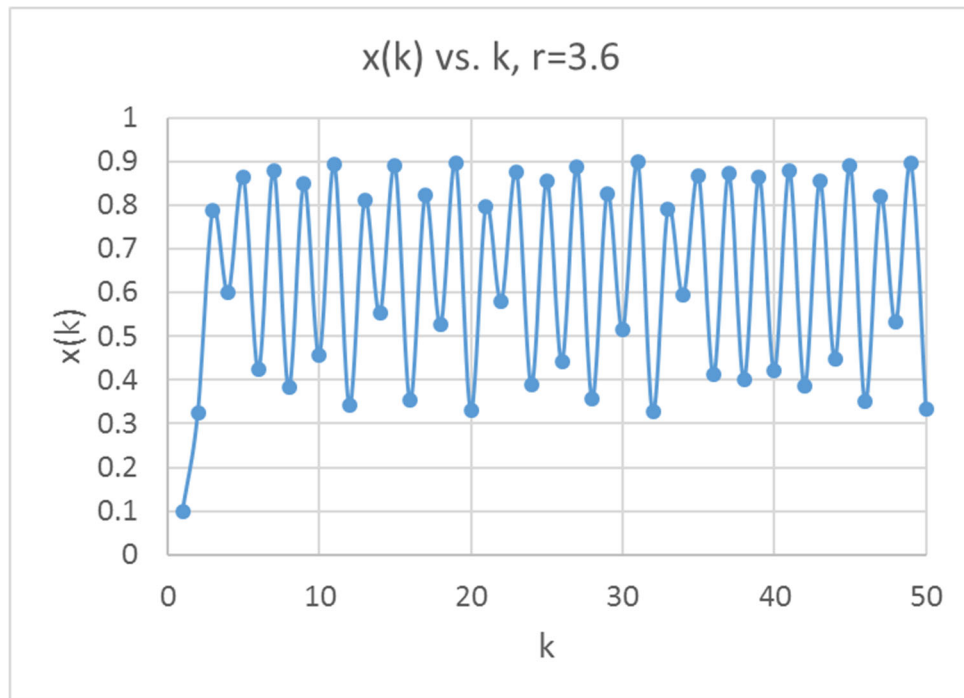
If $r=3.3$, the population settles to continually having cyclical boom and bust cycles.

If $r=3.6$, the population is now chaotic.

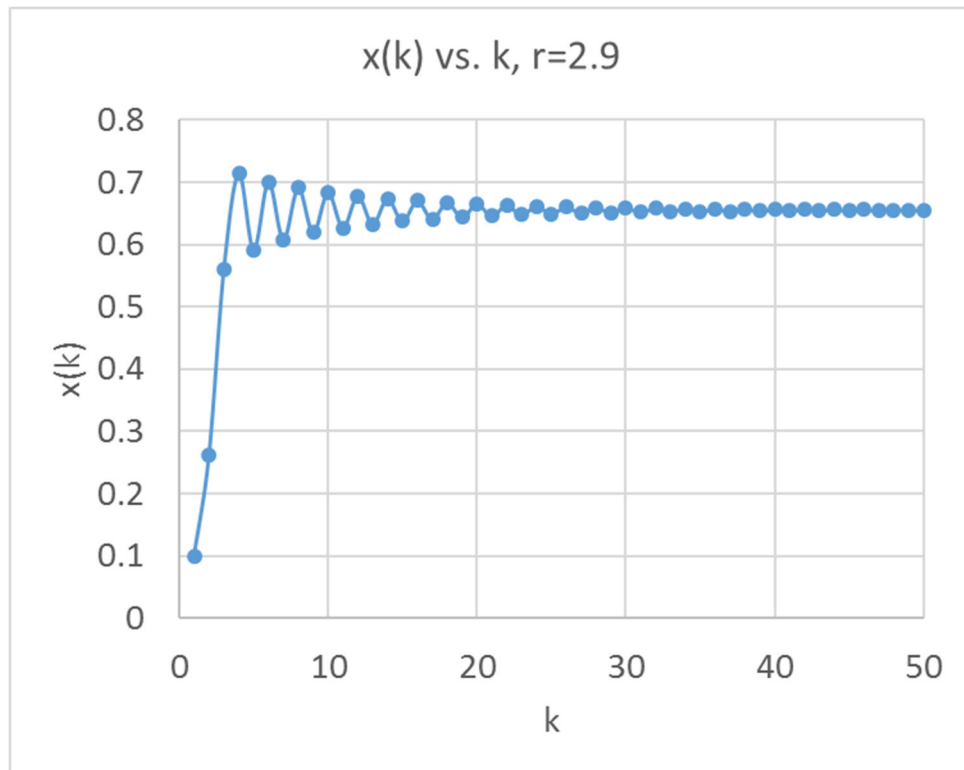
A bifurcation diagram is useful in understanding the dynamics of this and other systems that can have chaotic regions. A bifurcation diagram shows the values visited (or asymptotically approached) of a system as a function of a system bifurcation parameter.



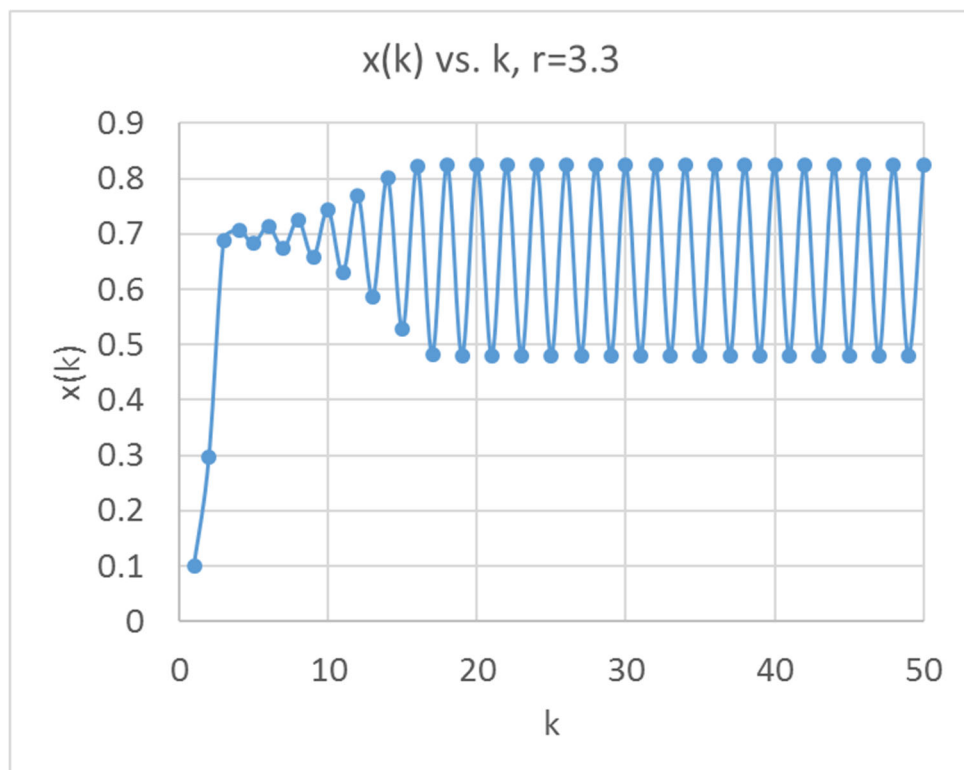




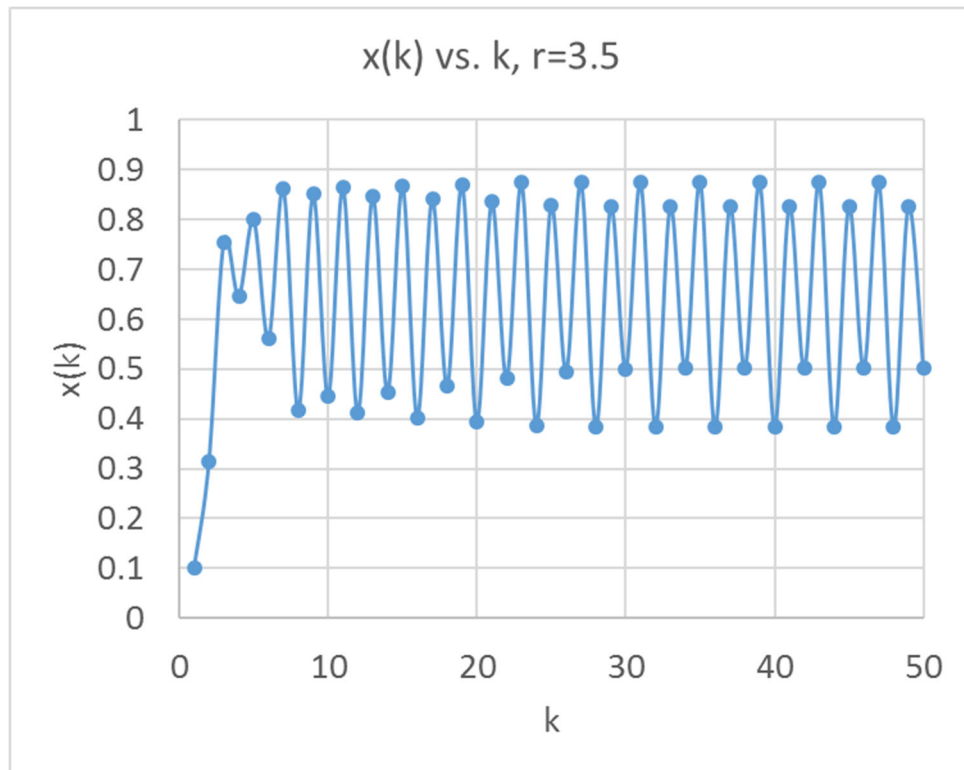
Bifurcation diagram for this system.



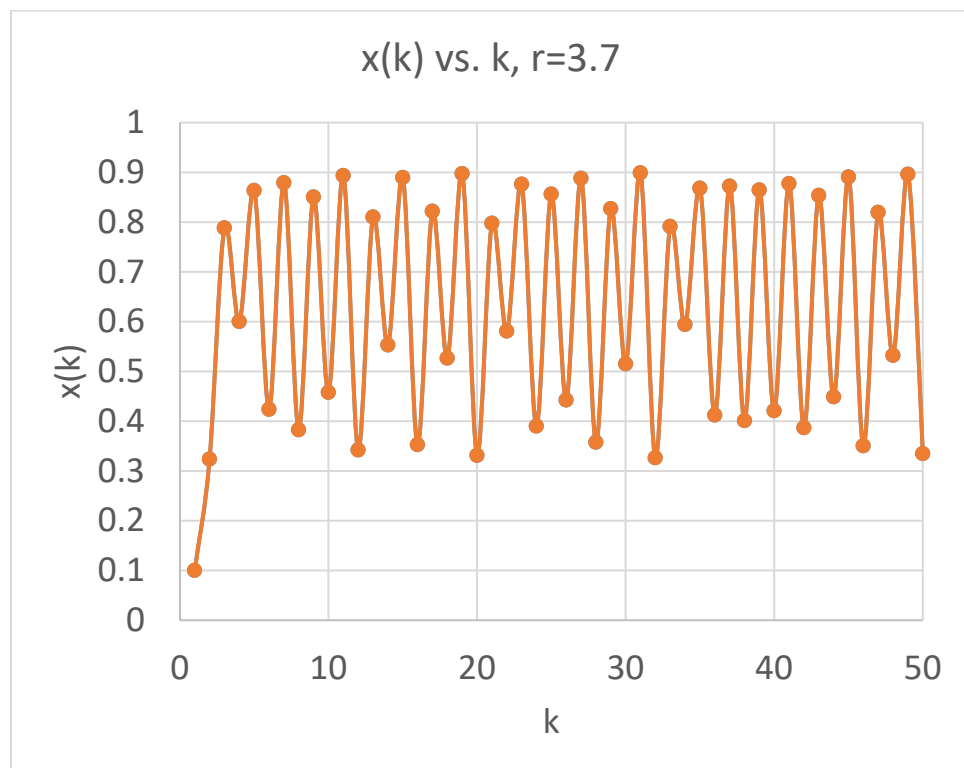
Relate to Bifurcation diagram (steady state $x(k) \sim .66$)



Relate to Bifurcation diagram (steady state $x(k) \sim .40$ & $.82$)



Relate to Bifurcation diagram (steady state $x(k)$: period doubling)



Relate to Bifurcation diagram (steady state $x(k)$: chaotic response)

Implementing the logistics map in circuitry

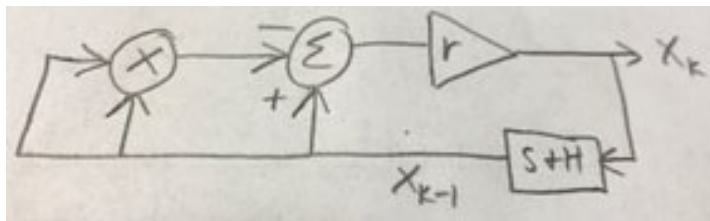
Recall: $x_{k+1} = rx_k(1 - x_k)$

Rewrite as: $x_k = r(x_{k-1} - x_{k-1}^2)$

This is a nonlinear difference equation.

It could be implemented in a simulation tool such as MATLAB, in software on a microcontroller, in digital logic circuitry, or in analog circuitry.

Consider this analog circuit implementation block diagram:



S&H is a Sample-and-Hold circuit.

Consider the Analog Devices AD783 Sample-and-Hold Amplifier chip,

We have already discussed multiplier and summer circuits.

The amplifier gain has a value of r .

In a circuit implementation, noise will be present in the circuit, which will change the output compared to that of a digital simulation. Show simulation results.



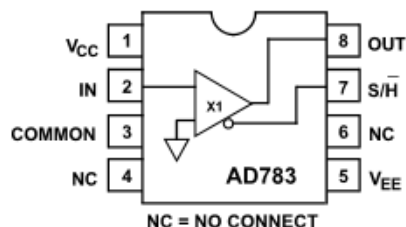
Complete Very High Speed Sample-and-Hold Amplifier

AD783

FEATURES

- Acquisition Time to 0.01%: 250 ns Typical
- Low Power Dissipation: 95 mW
- Low Droop Rate: 0.02 $\mu\text{V}/\mu\text{s}$
- Fully Specified and Tested Hold Mode Distortion
- Total Harmonic Distortion: -85 dB
- Aperture Jitter: 50 ps Maximum
- Internal Hold Capacitor
- Self-Correcting Architecture
- 8-Pin Mini Cerdip and SOIC Packages

FUNCTIONAL BLOCK DIAGRAM



PRODUCT DESCRIPTION

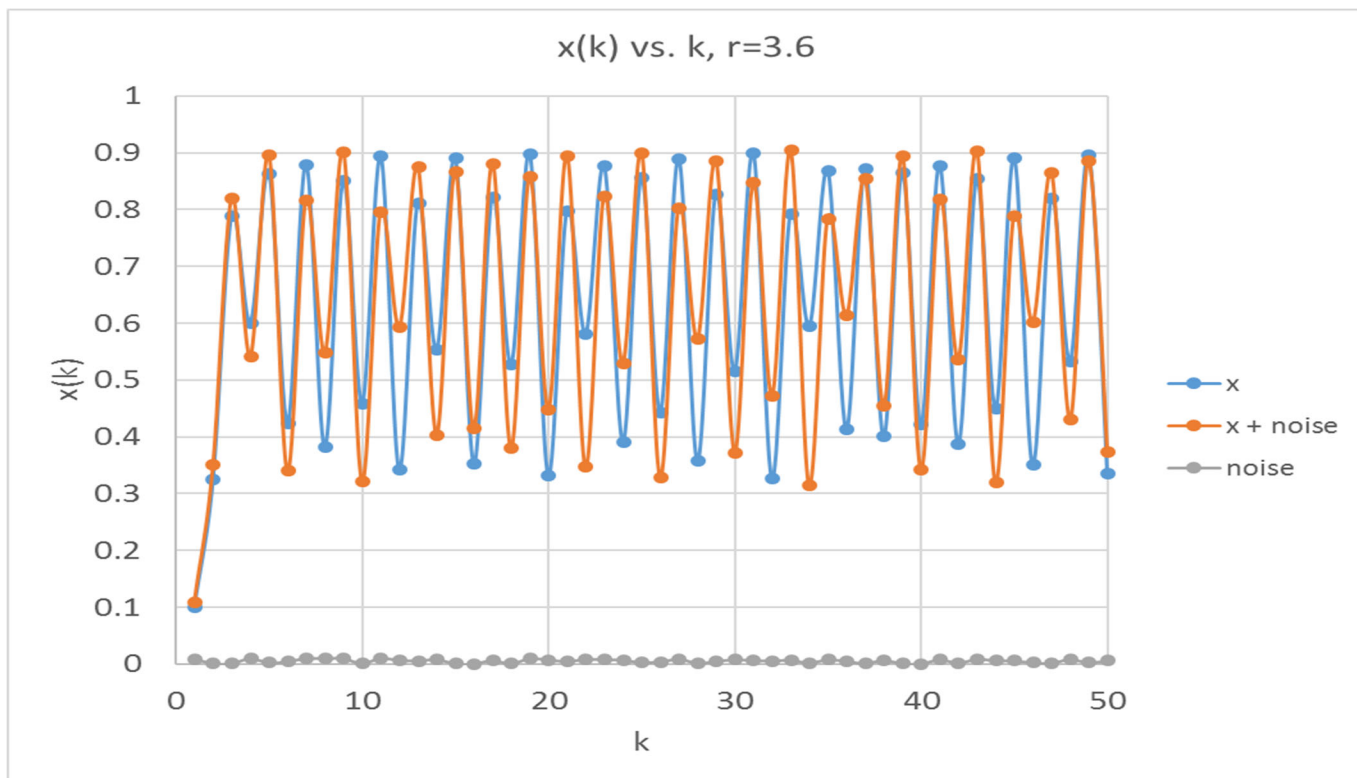
The AD783 is a high speed, monolithic sample-and-hold amplifier (SHA). The AD783 offers a typical acquisition time of 250 ns to 0.01%. The AD783 is specified and tested for hold mode total harmonic distortion with input frequencies up to 100 kHz. The AD783 is configured as a unity gain amplifier and uses a patented self-correcting architecture that minimizes hold mode errors and ensures accuracy over temperature. The AD783 is self-contained and requires no external components or adjustments.

The AD783 retains the held value with a droop rate of 0.02 $\mu\text{V}/\mu\text{s}$. Excellent linearity and hold mode dc and dynamic performance make the AD783 ideal for high speed 12- and 14-bit analog-to-digital converters.

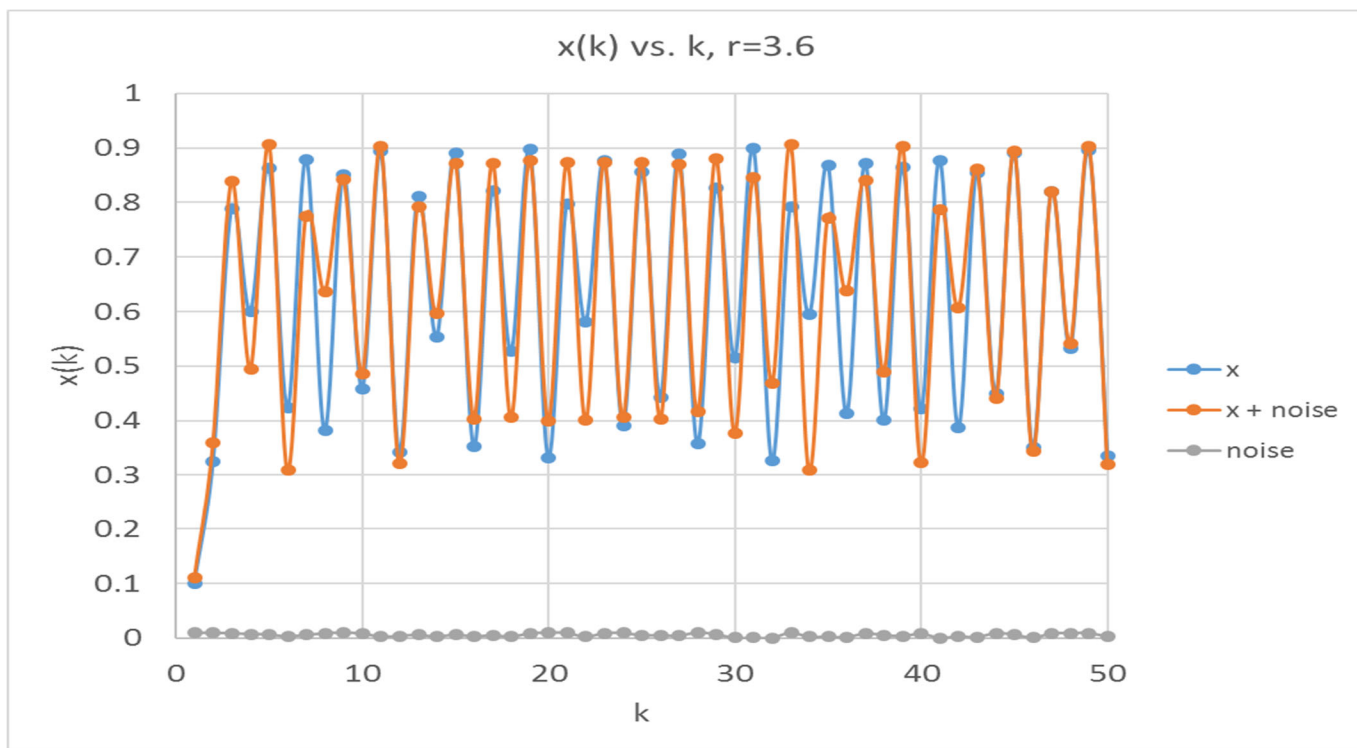
The AD783 is manufactured on Analog Devices' ABCMOS process which merges high performance, low noise bipolar circuitry with low power CMOS to provide an accurate, high speed, low power SHA.

PRODUCT HIGHLIGHTS

1. Fast acquisition time (250 ns), low aperture jitter (20 ps) and fully specified hold mode distortion make the AD783 an ideal SHA for sampling systems.
2. Low droop (0.02 $\mu\text{V}/\mu\text{s}$) and internally compensated hold mode error result in superior system accuracy.
3. Low power (95 mW typical), complete functionality and small size make the AD783 an ideal choice for a variety of high performance applications.
4. The AD783 requires no external components or adjustments.
5. The AD783 is an excellent choice as a front-end SHA for high speed analog-to-digital converters such as the AD671, AD7586, AD674B, AD774B, AD7572 and AD7672.
6. Fully specified and tested hold mode distortion guarantees the performance of the SHA in sampled data systems.



Run first time

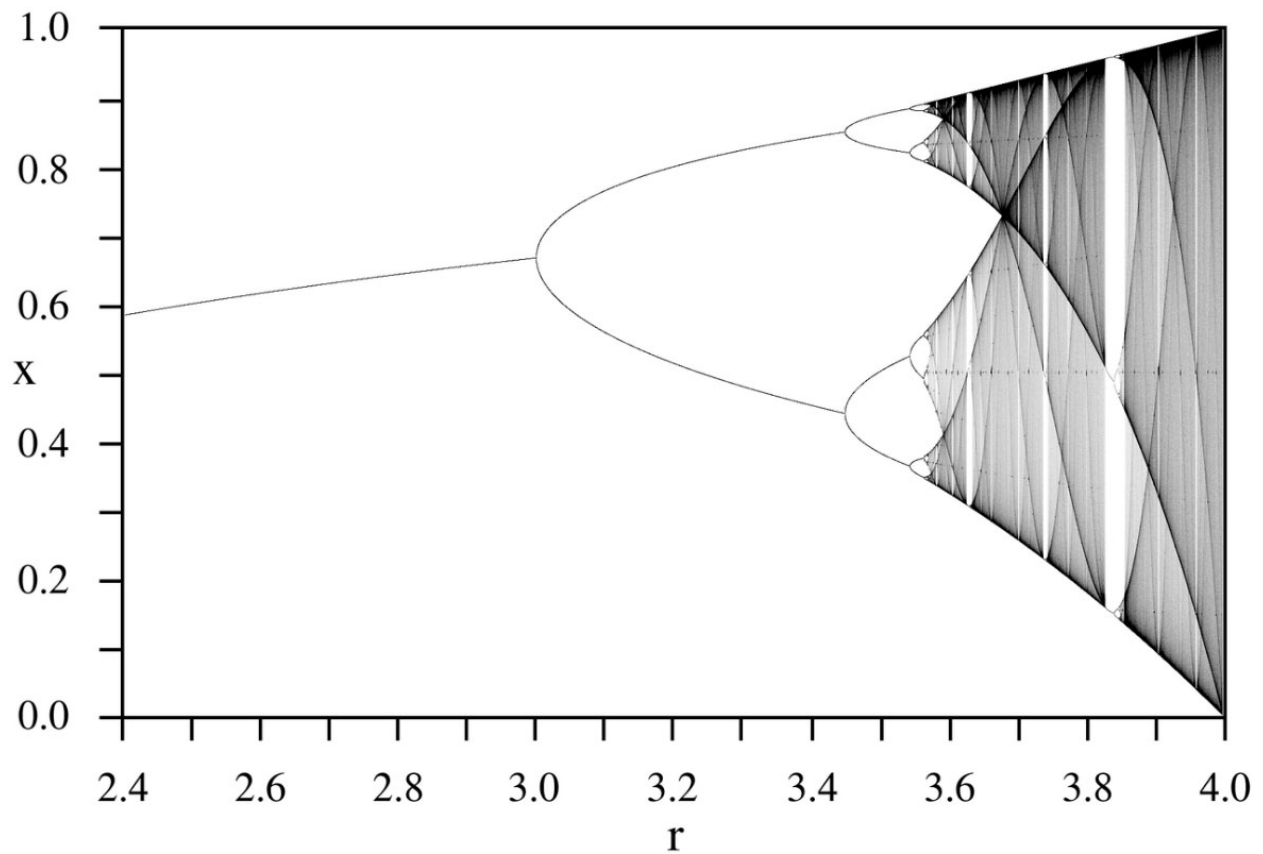


Run second time (only random noise values changed)

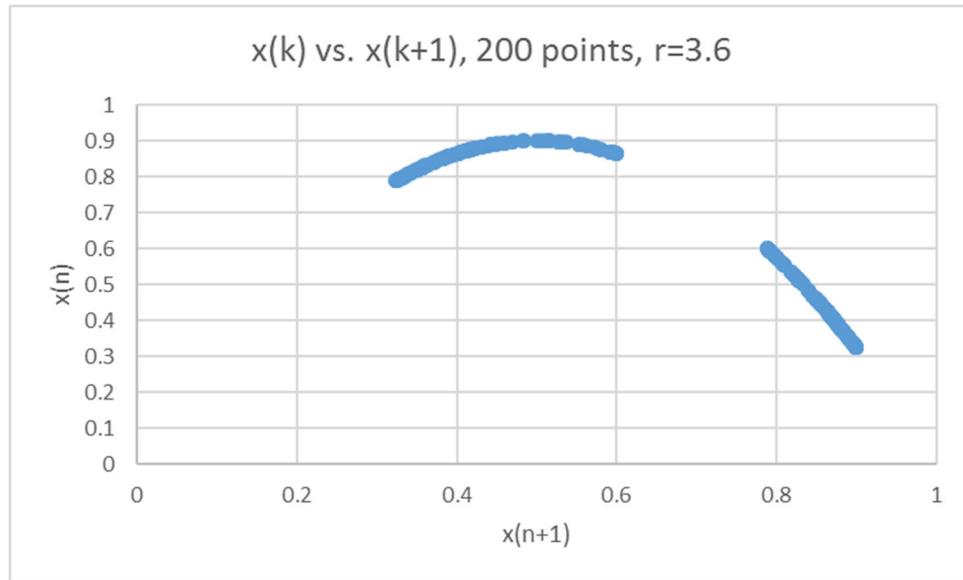
More on the Logistics Map

Recall: $x_{k+1} = rx_k(1 - x_k)$

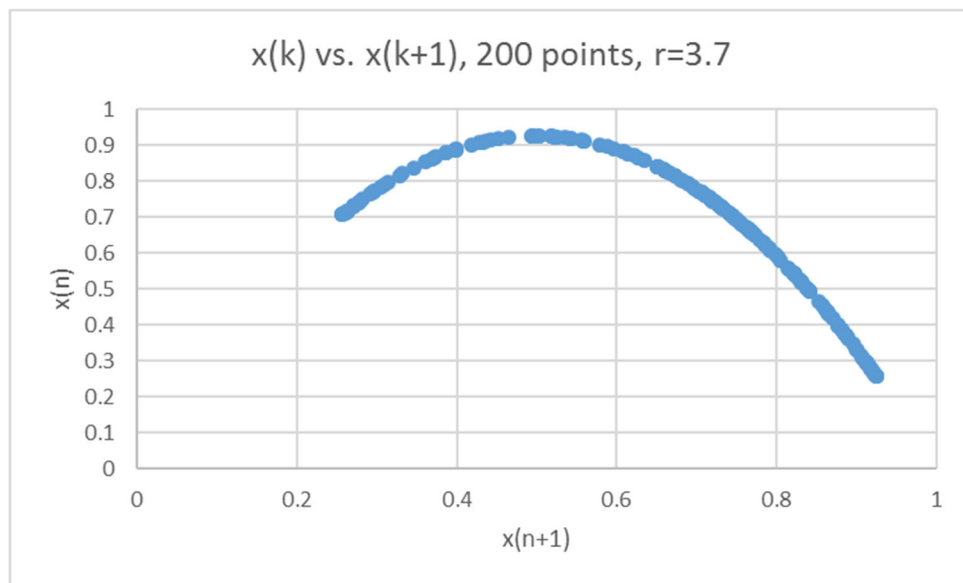
Consider a comparison of the bifurcation diagram with plots of x_k vs. x_{k+1} (phase diagrams for this system)



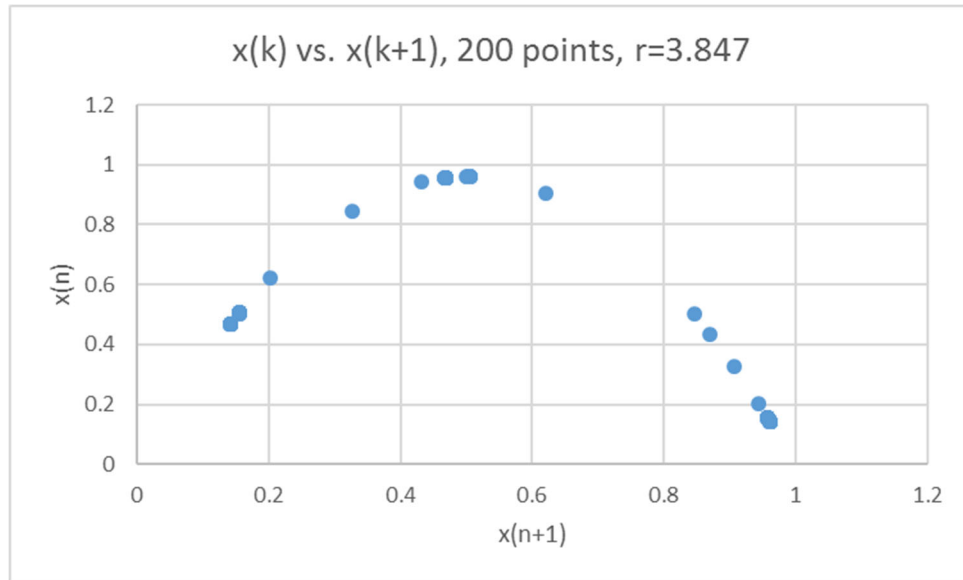
Bifurcation diagram for this system.



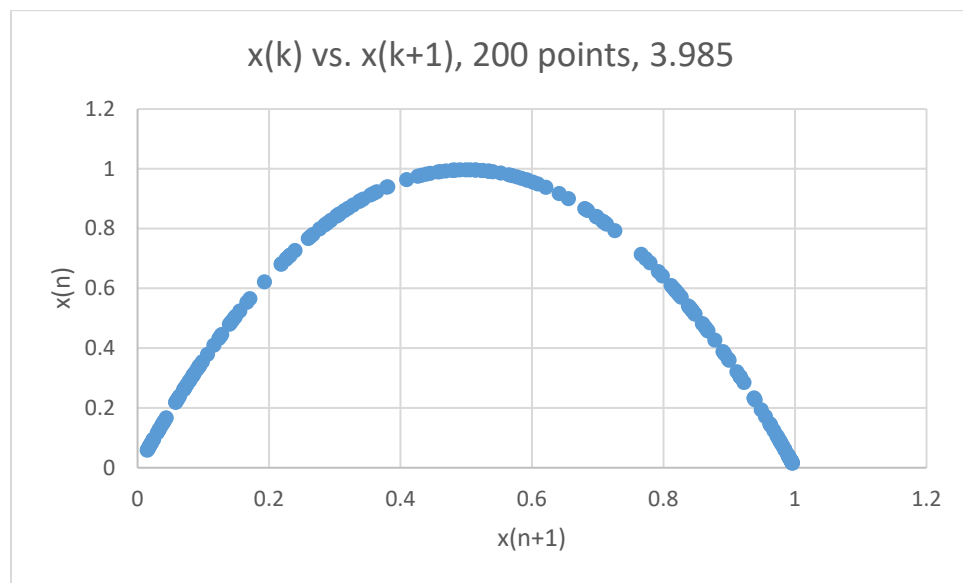
Phase diagram plot for $r = 3.6$



Phase diagram plot for $r = 3.7$, with a fuller region than when $r = 3.6$.



A sparse region.



A dense region.

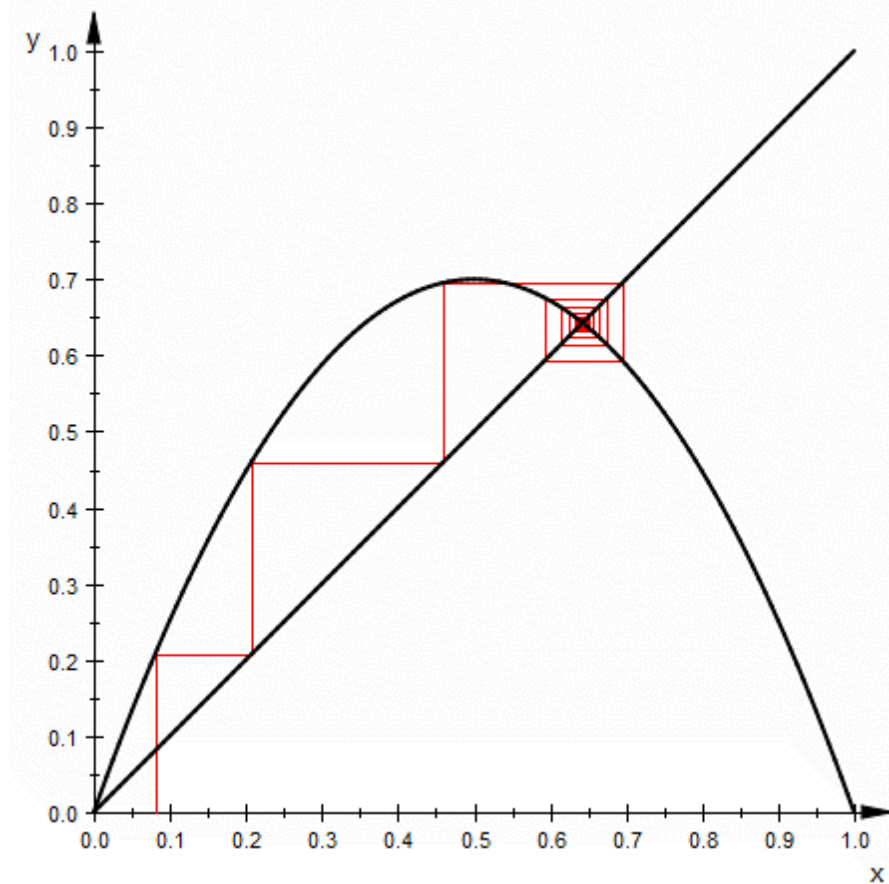
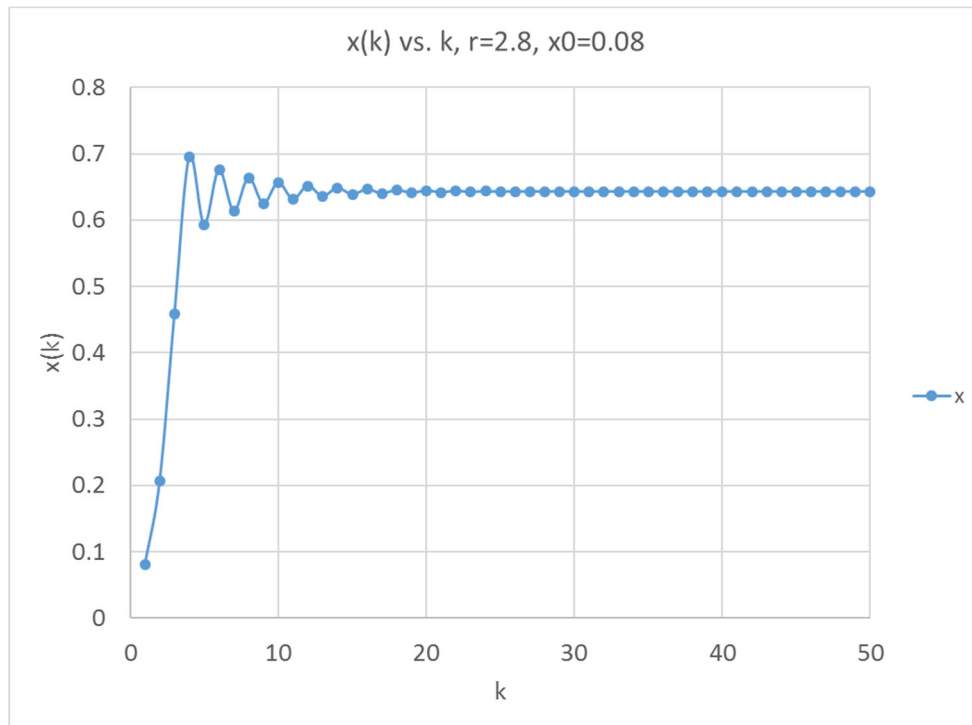
Another qualitative technique for analyzing dynamical systems is to use a “cobweb plot.” Here, $f(x) = y$ is plotted versus x , along with the straight line $f(x) = y = x$. Then other lines link the discrete values of x_k to $f(x_k) = x_{k+1}$. The plot of $f(x)$ vs. x is a curve called the function graph, and the straight line is called the diagonal.

The intersections for $y = f(x)$ and $y = x$ are called “fixed points.” Stable fixed points (inward spirals) are attractors (as shown in first cobweb plot below). Unstable fixed points (outward spirals) are repellers. Stable fixed points can also be attractors in a weaker sense, where they are referred to as having “asymptotic stability.” The center of spirals will occur at fixed points, i.e. where the diagonal and the function graph cross. If the system is chaotic, a region of the plot will get filled in, indicating an infinite number of non-repeating values.

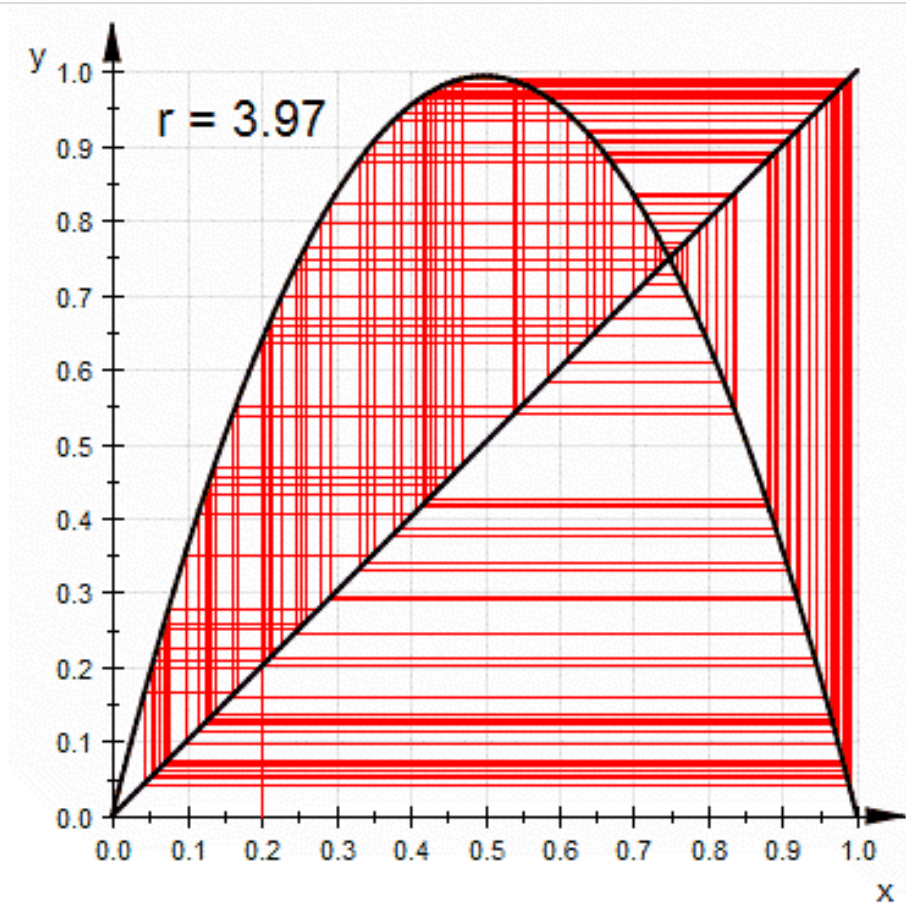
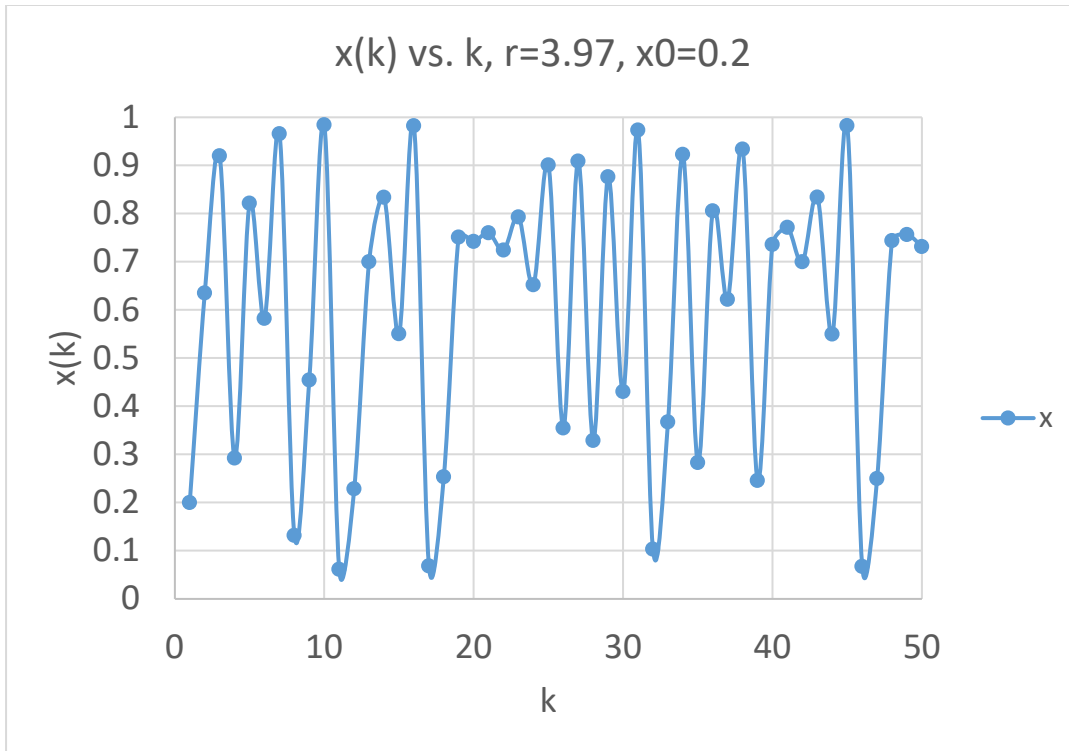
Animated cobweb plot of the logistics map:

https://en.wikipedia.org/wiki/Cobweb_plot#/media/File:LogisticCobwebChaos.gif

Cobweb Plot Illustration



Cobweb plot for $r=2.8$



Cobweb plot in chaotic region.