Tuesday 10/14/25

Chaos Theory

Definition: a branch of mathematics that deals with the behavior of nonlinear dynamical systems that:

(1) Are deterministic.
(2) Have a long-term aperiodic response.
(3) Have an extreme sensitivity to initial conditions.
(4) Have topological mixing.
(5) Have a spread spectrum frequency response.

(1) Deterministic – The performance is completely described by the governing nonlinear differential equations, and not stochastic events.

(2) Long term aperiodic response – no periodicity.

(3) Extreme sensitivity to initial conditions – very small changes in initial conditions quickly result in an extreme divergence in the two responses (trajectories). Observing this characteristic led to the study of chaos theory.

(4) Topological mixing – in the phase space plot, some finite areas of the plot will eventually get filled as the system runs for a long time.

(5) Spread spectrum frequency response – the aperiodic characteristic results in a wide bandwidth frequency spectrum.

Historical Perspective

Edward Lorenz was a meteorology professor at MIT. In 1961, he was developing a simplified computer model of the weather using 12 variables. One day he ran a simulation, and the results interested him. So, he reran it and got completely different results. He discovered that the first time he ran it, he had used an initial condition of 0.506127. The second time he ran it, he had used an initial condition of 0.506. The two initial conditions differed by just 0.0251%! This would have had little effect on the performance of a linear system. His result was shocking and led to the development of chaos theory. This extreme sensitivity to initial conditions led to this statement in 1972: "*Does the flap of a butterfly's wings in Brazil set off a tornado in Texas*" – Philip Merilees.

Eventually, Lorenz simplified the model down to these three equations:

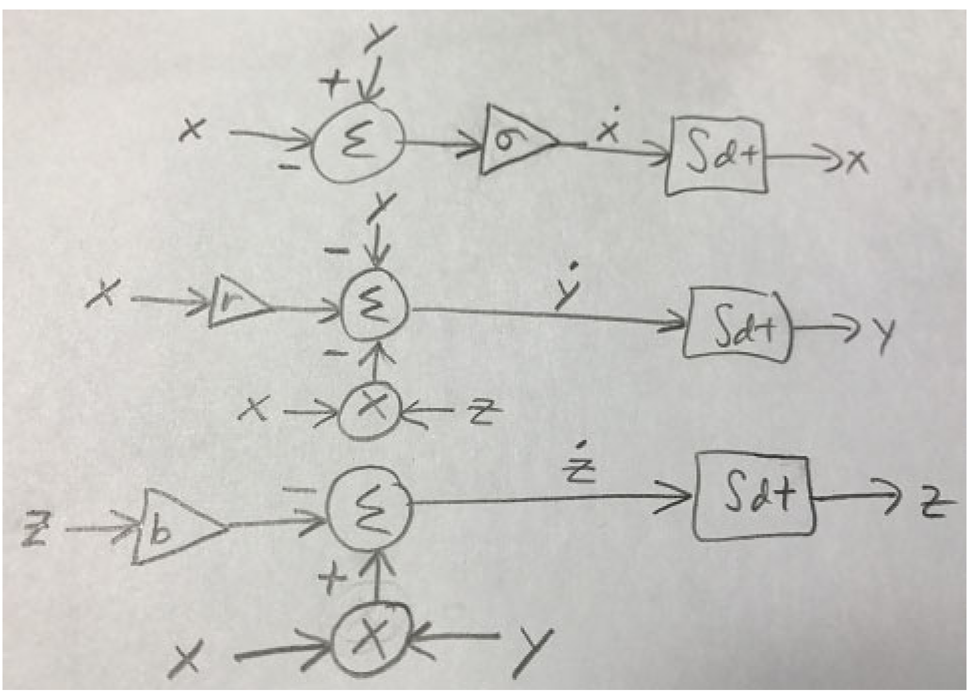
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - bz$$

σ, r and b are constants.

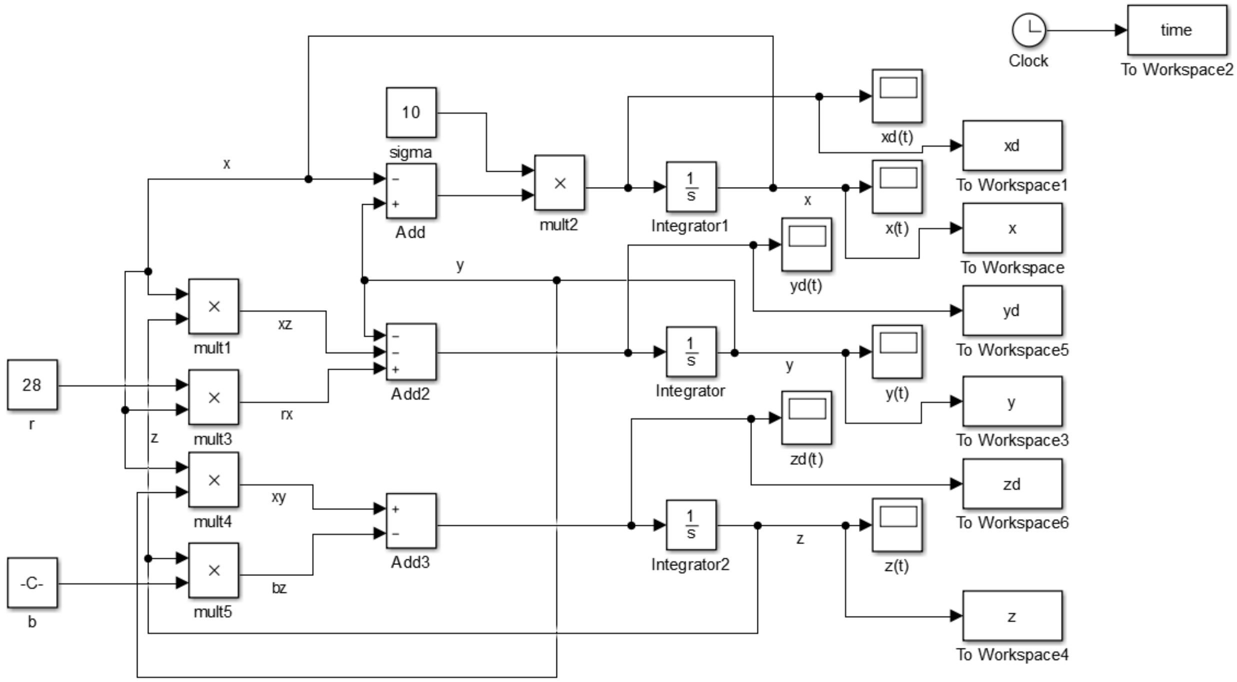
The only nonlinear terms are –xz and xy: two multiplications.

A solution is chaotic for (σ, r, b) = (10, 28, 8/3)

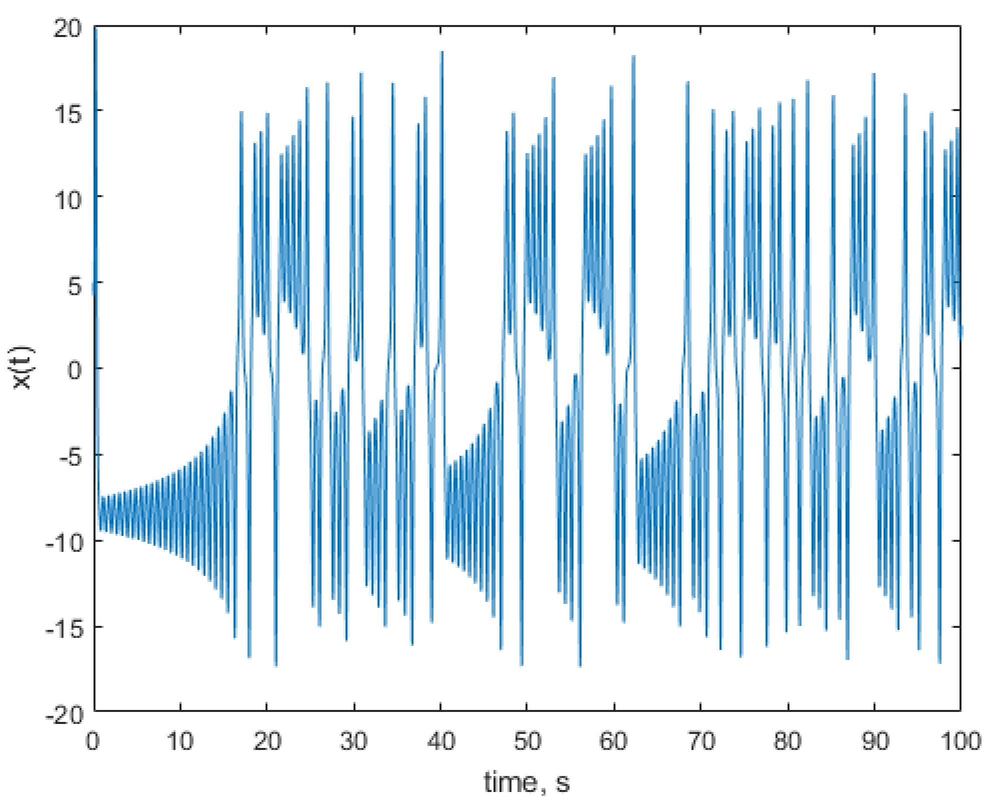
Building the system in hardware or simulation:



Simulink model below (run for 100s):



time
To Workspace2
Clock
10
sigma
x
Add
mult2
Integrator1
xd(t)
xd
To Workspace1
x(t)
x
To Workspace
y
yd(t)
yd
To Workspace5
xz
mult1
28
r
rx
mult3
z
Add2
Integrator
y(t)
y
To Workspace3
zd(t)
zd
To Workspace6
xy
mult4
Add3
Integrator2
z(t)
-C-
b
bz
mult5
z
To Workspace4



x(t)
time, s

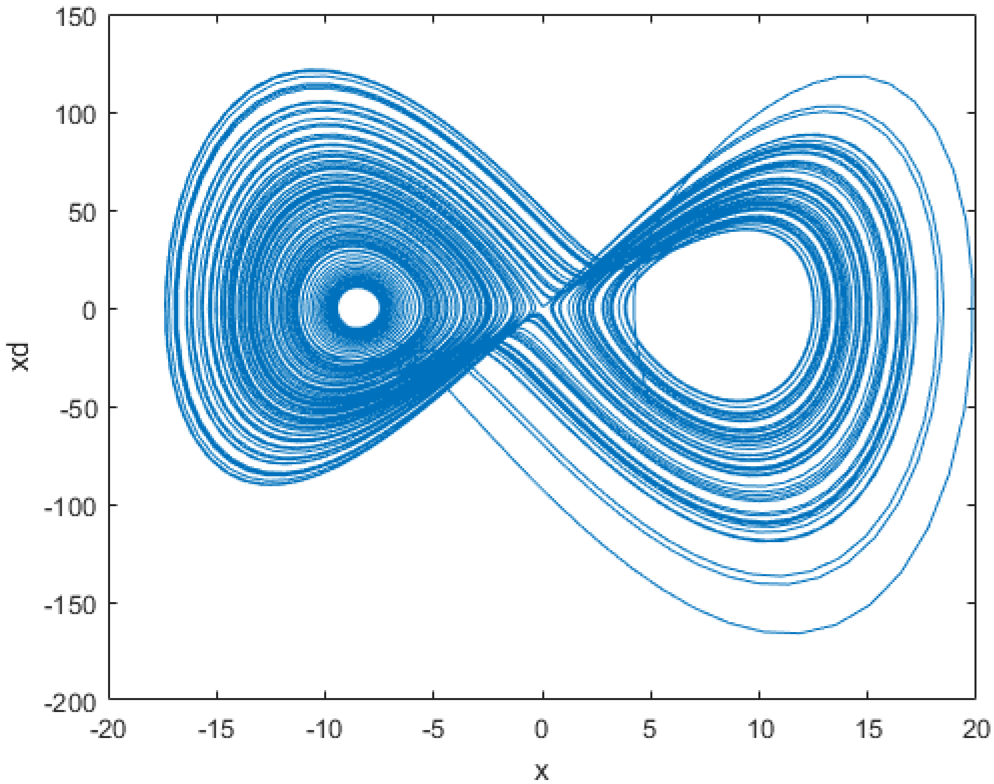
Observe that x(t) appears to oscillate about one point for some period of time, with the amplitude of the oscillation increasing over time. Then at some point in time, x(t) jumps to some new point about which it oscillates for a while before jumping back to the first point. This "jump" happens because some "guard condition" was met that allows the trajectory to then move from orbiting about some point in the phase space to orbiting about some other point in the phase space, and vice versa. So over time, x(t) jumps back and forth between these two points at which it oscillates for some amount of time. These two points by which x(t) oscillates are called "attractors", and specifically due to the chaotic nature of the system they are called "strange attractors." A phase plot of xd vs. x clearly shows this <below>.

Each complete loop around an attractor on the phase plot represents one oscillation of x(t) about the point of oscillation. If the radius of the orbit about an attractor grows, the amplitude of oscillation is increasing.

As shown in the x(t) vs. t plot, the xd vs. x phase plot also shows that after some number of oscillations about one attractor, the trajectory jumps to oscillating about a second attractor.

Observe that the trajectory is only allowed in a certain region of the phase plot, but in that region it can go anywhere. This boundedness of the trajectory shows that the system is globally stable.

The "Basin of Attraction" is the region of the phase space where if a trajectory began at a point via an initial condition, the resulting trajectory would eventually be pulled into orbits oscillating about the attractor(s).


150
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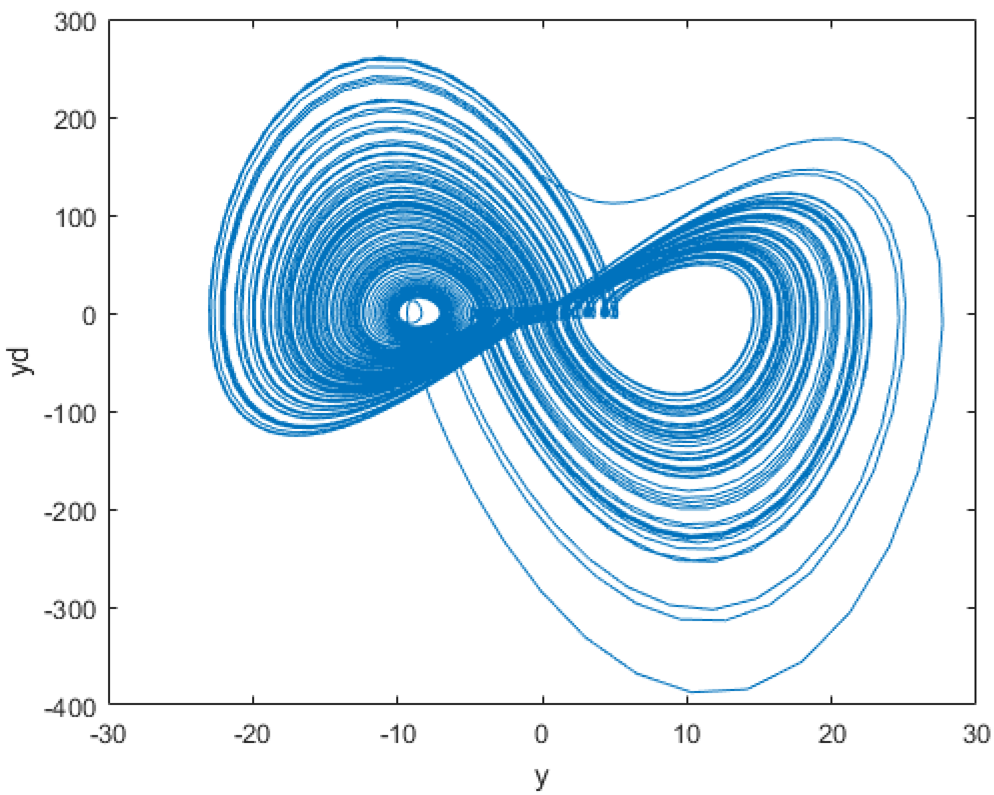
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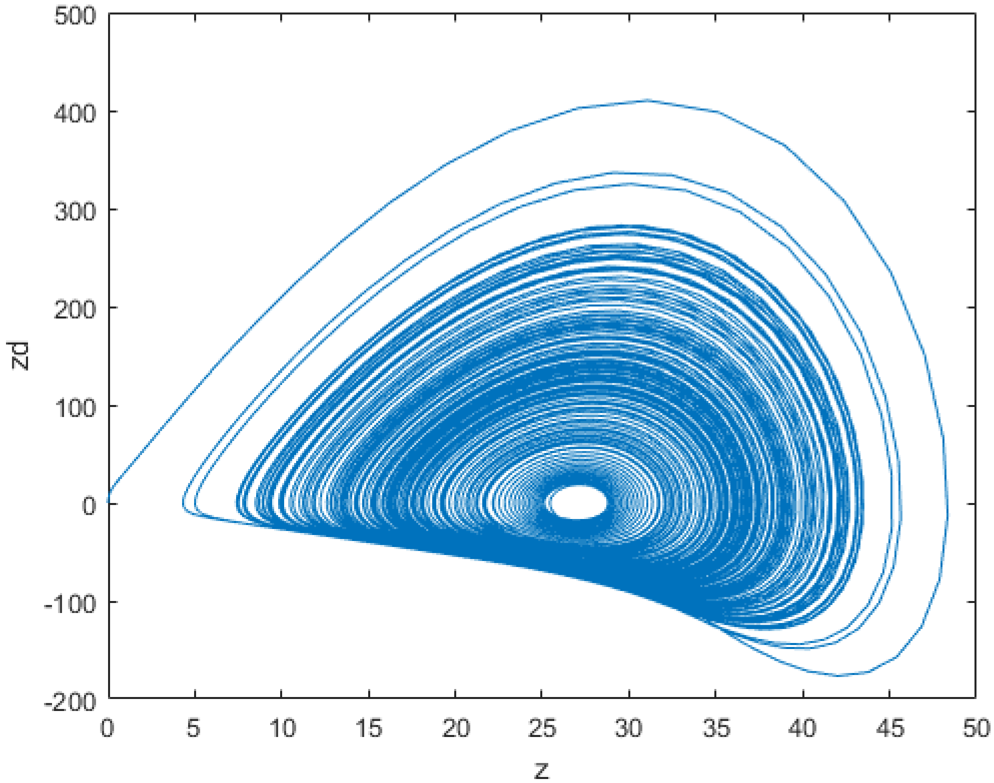
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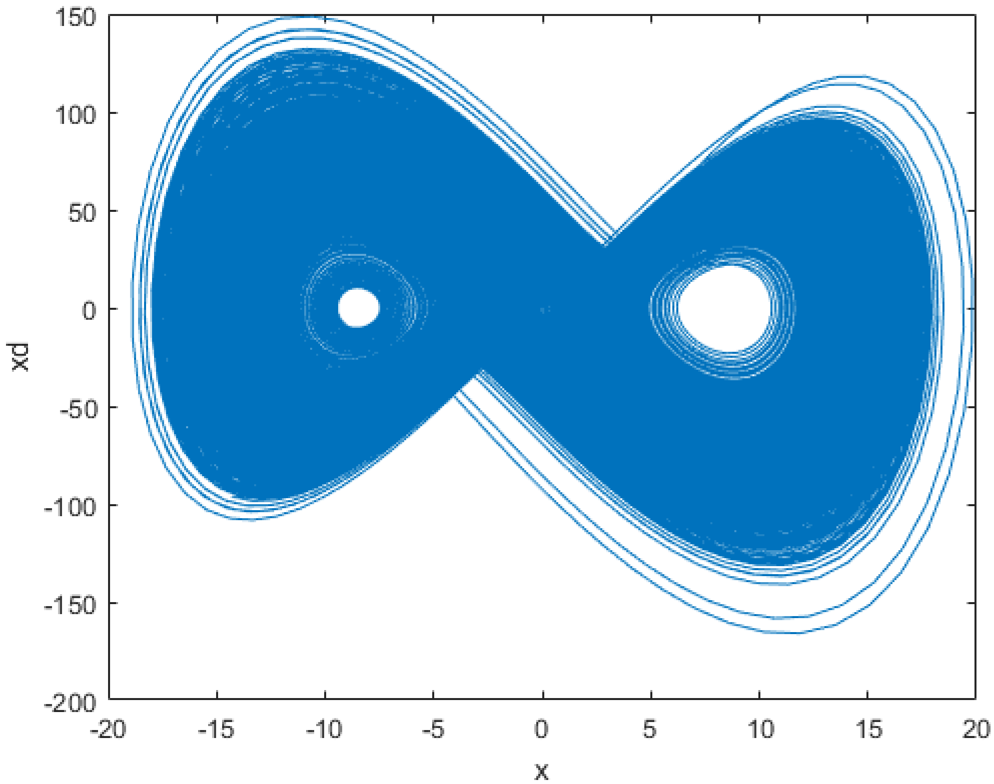
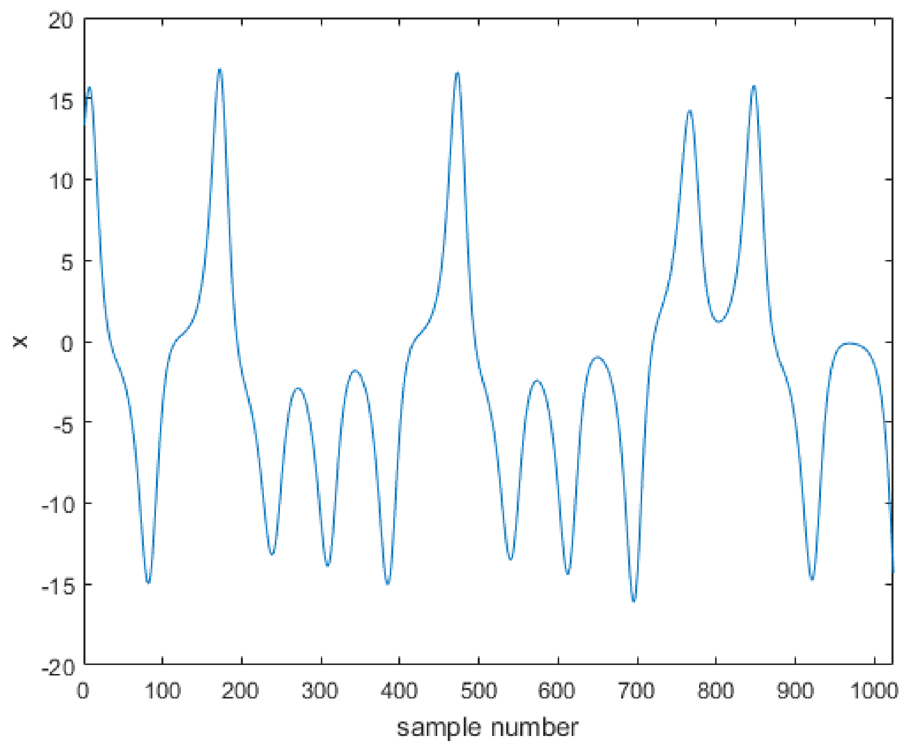
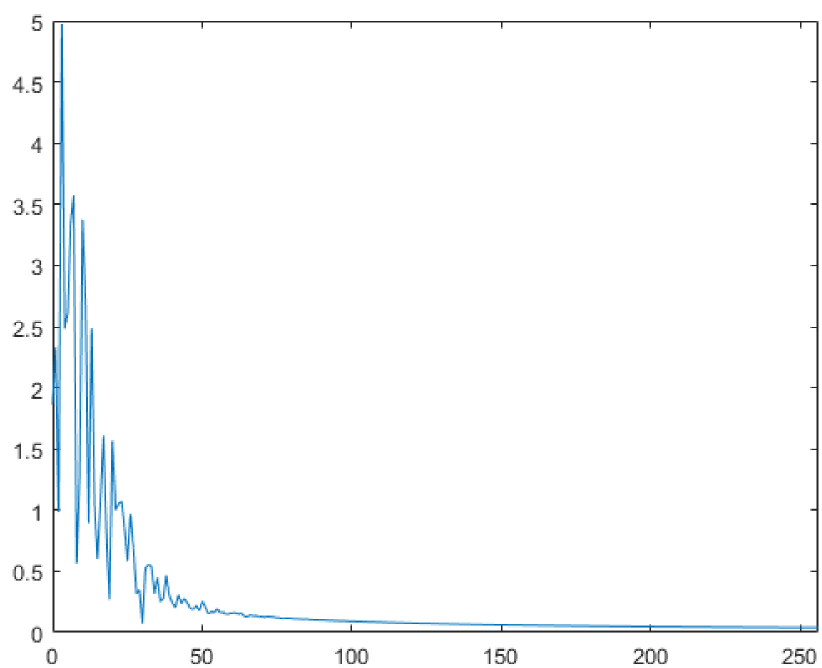


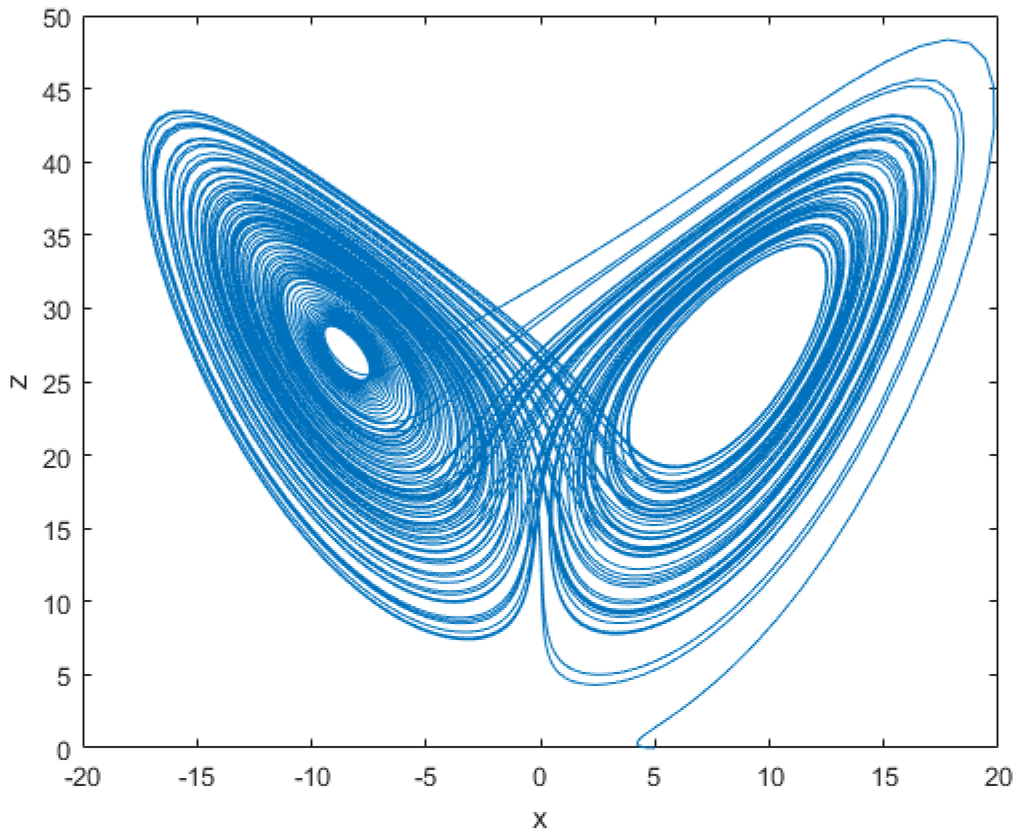
A plot of xd vs. x after running 1000s, illustrating topological mixing.



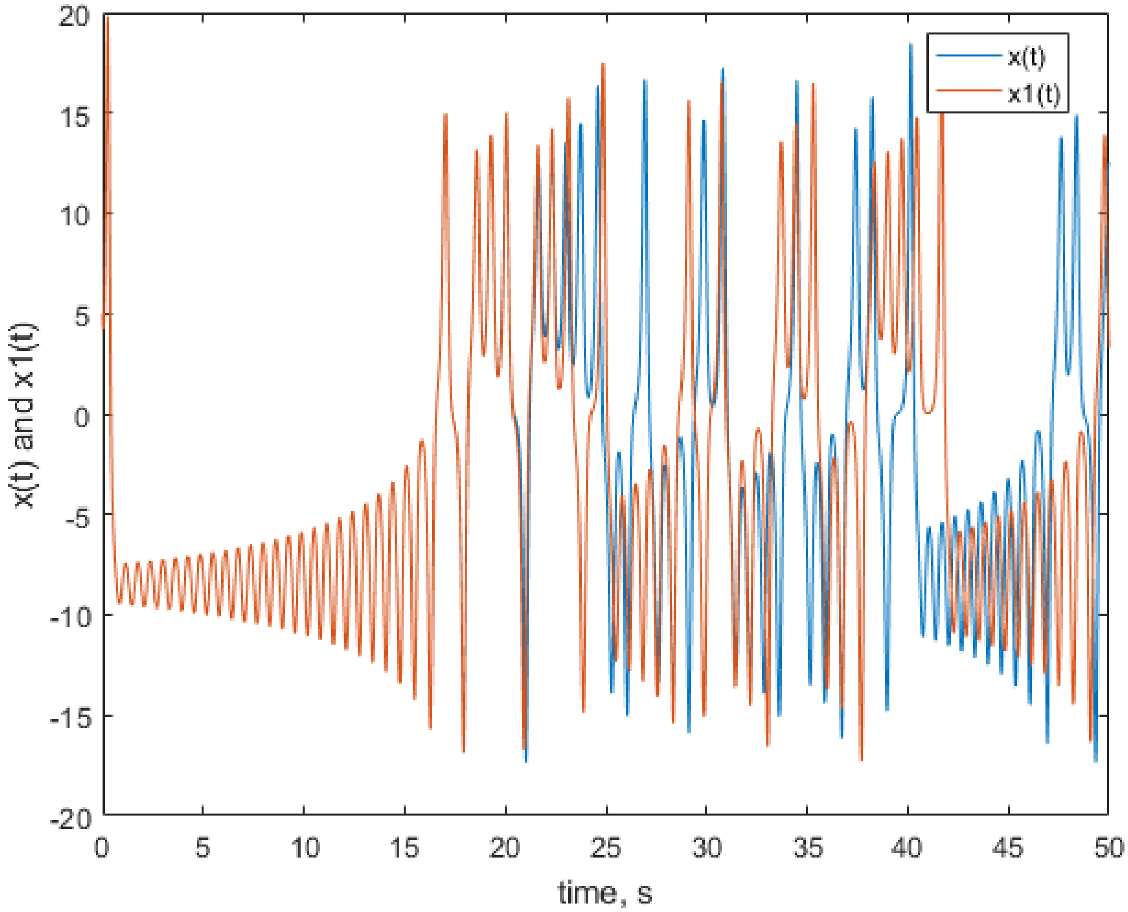
Last 1024 data points of 1000s of x(t)



Illustrative 1024 point FFT of last 1024 data points for 1000s of x(t). Note: only first 256 bins shown. Illustrates wide bandwidth characteristic.



Phase plot of z vs. x, the classic Lorenz attractor “butterfly.”



Exact same simulations except σ=10 for the x(t) system and σ=10.001 for the x1(t) system. The two trajectories quickly diverge, illustrating the extreme sensitivity to initial conditions, a hallmark of chaotic systems.

Building this system in electronic circuitry:

1) Integrators: one op amp, one R and one C per negative integrator
2) Summers: one op amp, R's per negative summer
3) Fixed gain amplifiers: one op amp, R's per amplifier (inv or non-inv)
4) Multipliers: low frequency four quadrant multiplier IC's available

For low frequency operation, it is relatively easy to implement this chaotic system as an electronic circuit.

For high frequency operation, electronic circuit implementation issues arise:

1) Four quadrant multiplier ICs are generally low frequency devices.
2) Required bandwidth increases with operating speed. Chaotic systems are often very sensitive to nonlinear phase responses of op amps. Many high speed op amps are generally not designed for extremely wide bandwidth operation where a linear phase response over the entire bandwidth is required.
3) Op amp gain roll off and phase delay change the realized system governing differential equations.
4) EMI issues such as electromagnetic coupling can adversely affect circuit performance (extreme sensitivity to initial conditions)
5) Noise issues can also adversely affect circuit performance (extreme sensitivity to initial conditions). The thermal noise floor is proportional to the square root of the bandwidth, and therefore increases as the speed of the chaotic system increases.

Circuits for Implementing Nonlinear Functions

1) 4 Quadrant multiplier chips

Analog Devices makes several four quadrant multiplier (and related) chips <below>. Some are pretty fast now.

Analog Devices Multiplier/Divider chips:

Part Number	Transfer Function	Vin min | V	Vin max | V	Vin Range max |	BW -3 dB typ | Hz	Output Range	Total Error max | %	Price (1000+) $ US
Filter Parts 11 parts	11 Values Selected HIDE	-12.5 - -1 HIDE	1 - 12.5 HIDE	7 Values Selected HIDE	400k - 2G HIDE	8 Values Selected HIDE	0.25 - 2.7 HIDE	4.8 - 44.88 HIDE
ADL5391	a (XPLS-XMNS)(YPLS-YMNS)/1V + (ZPLS-ZMNS)	-1	1	2Vpp	2G	±2V	2.7	$4.95 (ADL5391ACPZ-R7)
AD835	[(X1–X2)(Y1–Y2)/U] + Z	-1	1	±1V	250M	±2.5V	0.3	$11.47 (AD835ARZ)
AD633	[(X1–X2)(Y1–Y2)/10] + Z	-10	10	±10V	1M	±11V	2	$4.80 (AD633ARZ)
AD734	[(X1–X2)(Y1–Y2)/(U1–U2)] + (Z1–Z2)	-12.5	12.5	±12.5V	10M	12V	0.4	$19.01 (AD734ANZ)
AD834	(4 mA)(XY)	-1	1	±1V	500M	±4.04mA	2	$22.85 (AD834JRZ)
AD538	[y(z/x)]^m	-10	10	±10V	400k	±11V	1	$44.88 (AD538ADZ)
AD539	-(vx*vy)/vu	-4.2	4.2	±4.2V	25M	±2.8mA	2.5	$29.94 (AD539JNZ)
AD632	[(X1–X2)(Y1–Y2)/10] + Z2	-12	12	±12V	1M	±11V	0.5	$17.34 (AD632AHZ)
AD534	[(X1-X2)(Y1-Y2)/10 V] + Z2	-10	10	±10V	1M	±11V	0.25	$26.38 (AD534JHZ)
AD534S	[(X1-X2)(Y1-Y2)/10 V] + Z2	-10	10	±10V	1M	±11V	0.25	-
AD532	(X1-X2)(Y1-Y2)/10 V	-10	10	±10V	1M	±10V	1	$36.54 (AD532JHZ)

ANALOG DEVICES

250 MHz, Voltage Output, 4-Quadrant Multiplier

Data Sheet **AD835**

FEATURES

Simple: basic function is W = XY + Z
Complete: minimal external components required
Very fast: Settles to 0.1% of full scale (FS) in 20 ns
DC-coupled voltage output simplifies use
High differential input impedance X, Y, and Z inputs
Low multiplier noise: 50 nV/√Hz

APPLICATIONS

Very fast multiplication, division, squaring
Wideband modulation and demodulation
Phase detection and measurement
Sinusoidal frequency doubling
Video gain control and keying
Voltage-controlled amplifiers and filters

FUNCTIONAL BLOCK DIAGRAM

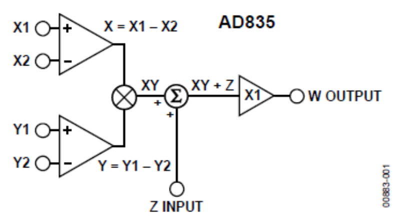


Figure 1.

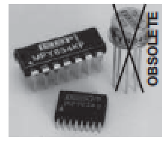
GENERAL DESCRIPTION

The AD835 is a complete four-quadrant, voltage output analog multiplier, fabricated on an advanced dielectrically isolated complementary bipolar process. It generates the linear product of its X and Y voltage inputs with a −3 dB output bandwidth of 250 MHz (a small signal rise time of 1 ns). Full-scale (−1 V to +1 V) rise to fall times are 2.5 ns (with a standard R_L of 150 Ω), and the settling time to 0.1% under the same conditions is typically 20 ns.

PRODUCT HIGHLIGHTS

1. The AD835 is the first monolithic 250 MHz, four-quadrant voltage output multiplier.
2. Minimal external components are required to apply the AD835 to a variety of signal processing applications.
3. High input impedances (100 kΩ||2 pF) make signal source loading negligible.
4. High output current capability allows low impedance loads to be driven.

Burr-Brown Products from Texas Instruments



MPY634

SBFS017A – DECEMBER 1995 – REVISED DECEMBER 2004

Wide Bandwidth PRECISION ANALOG MULTIPLIER

FEATURES

- WIDE BANDWIDTH: 10MHz typ
- ±0.5% MAX FOUR-QUADRANT ACCURACY
- INTERNAL WIDE-BANDWIDTH OP AMP
- EASY TO USE
- LOW COST

APPLICATIONS

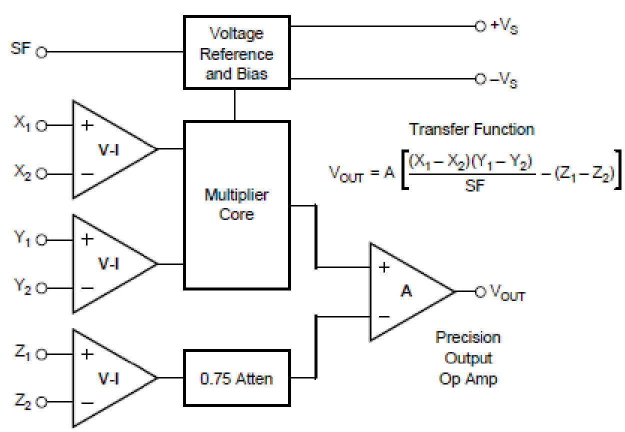
- PRECISION ANALOG SIGNAL PROCESSING
- MODULATION AND DEMODULATION
- VOLTAGE-CONTROLLED AMPLIFIERS
- VIDEO SIGNAL PROCESSING
- VOLTAGE-CONTROLLED FILTERS AND OSCILLATORS

DESCRIPTION

The MPY634 is a wide bandwidth, high accuracy, four-quadrant analog multiplier. Its accurately laser-trimmed multiplier characteristics make it easy to use in a wide variety of applications with a minimum of external parts, often eliminating all external trimming. Its differential X, Y, and Z inputs allow configuration as a multiplier, squarer, divider, square-rooter, and other functions while maintaining high accuracy.

The wide bandwidth of this new design allows signal processing at IF, RF, and video frequencies. The internal output amplifier of the MPY634 reduces design complexity compared to other high frequency multipliers and balanced modulator circuits. It is capable of performing frequency mixing, balanced modulation, and demodulation with excellent carrier rejection.

An accurate internal voltage reference provides precise setting of the scale factor. The differential Z input allows user-selected scale factors from 0.1 to 10 using external feedback resistors.



Texas Instruments
www.ti.com

![ANALOG DEVICES]

10 MHz, Four-Quadrant Multiplier/Divider

AD734

FEATURES

High accuracy
- **0.1% typical error**

High speed
- **10 MHz full power bandwidth**
- **450 V/μs slew rate**
- **200 ns settling to 0.1% at full power**

Low distortion
- **−80 dBc from any input**
- **Third-order IMD typically −75 dBc at 10 MHz**

Low noise
- **94 dB SNR, 10 Hz to 20 kHz**
- **70 dB SNR, 10 Hz to 10 MHz**

Direct division mode
- **2 MHz BW at gain of 100**

APPLICATIONS

High performance replacement for AD534
Multiply, divide, square, square root
Modulators, demodulators
Wideband gain control, rms-to-dc conversion
Voltage-controlled amplifiers, oscillators, and filters
Demodulator with 40 MHz input bandwidth

FUNCTIONAL BLOCK DIAGRAM

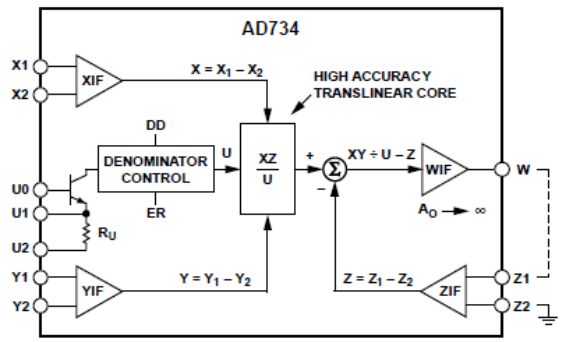


Figure 1.

GENERAL DESCRIPTION

The AD734 is an accurate high speed, four-quadrant analog multiplier that is pin compatible with the industry-standard AD534 and provides the transfer function W = XY/U. The AD734 provides a low impedance voltage output with a full power (20 V p-p) bandwidth of 10 MHz. Total static error (scaling, offsets, and nonlinearities combined) is 0.1% of full scale. Distortion is typically less than −80 dBc and guaranteed. The low capacitance X, Y, and Z inputs are fully differential. In most applications, no external components are required to define the function.

The internal scaling (denominator) voltage, U, is 10 V, derived from a buried-Zener voltage reference. A new feature provides the option of substituting an external denominator voltage, allowing the use of the AD734 as a two-quadrant divider with a 1000:1 denominator range and a signal bandwidth that remains 10 MHz to a gain of 20 dB, 2 MHz at a gain of 40 dB, and 200 kHz at a gain of 60 dB, for a gain-bandwidth product of 200 MHz.

The advanced performance of the AD734 is achieved by a combination of new circuit techniques, the use of a high speed complementary bipolar process, and a novel approach to laser trimming based on ac signals rather than the customary dc methods. The wide bandwidth (>40 MHz) of the AD734's input stages and the 200 MHz gain-bandwidth product of the multiplier core allow the AD734 to be used as a low distortion demodulator with input frequencies as high as 40 MHz as long as the desired output frequency is less than 10 MHz.

The AD734AQ and AD734BQ are specified for the industrial temperature range of −40°C to +85°C and come in a 14-lead CERDIP and a 14-lead PDIP package. The AD734SQ/883B, available processed to MIL-STD-883B for the military range of −55°C to +125°C, is available in a 14-lead CERDIP.

2) Diodes

Small signal: $i = I_s\left(e^{\frac{v}{nV_T}} - 1\right)$

Large signal: useful as a switch

3) MOSFET

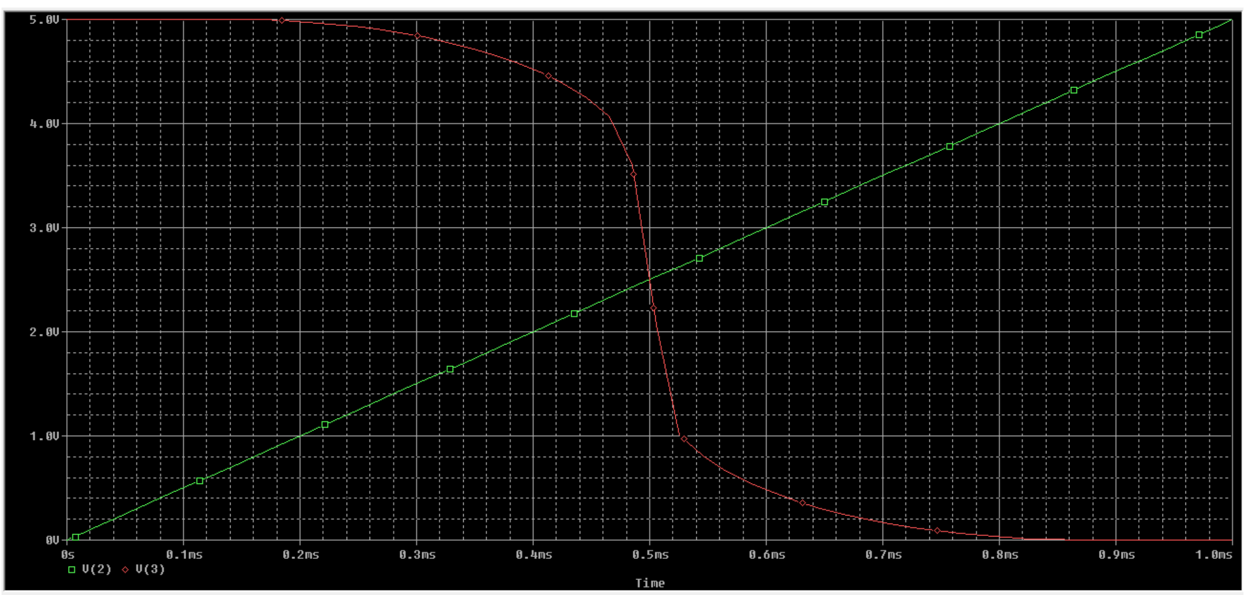
Triode region: useful as a nonlinear voltage controlled resistance

Small signal: in saturation, the MOSFET is a square law device:
$I_D \approx \frac{\beta_n}{2}(V_{GS} - V_T)^2$

Larger signal: useful as an analog switch (analog comparator and digital logic used to trip it)

4) CMOS Inverter

Amplifier with nonlinear saturation



V_o vs. V_{in} for approximately balanced CMOS inverter. Green: V_{in}, red: V_o.

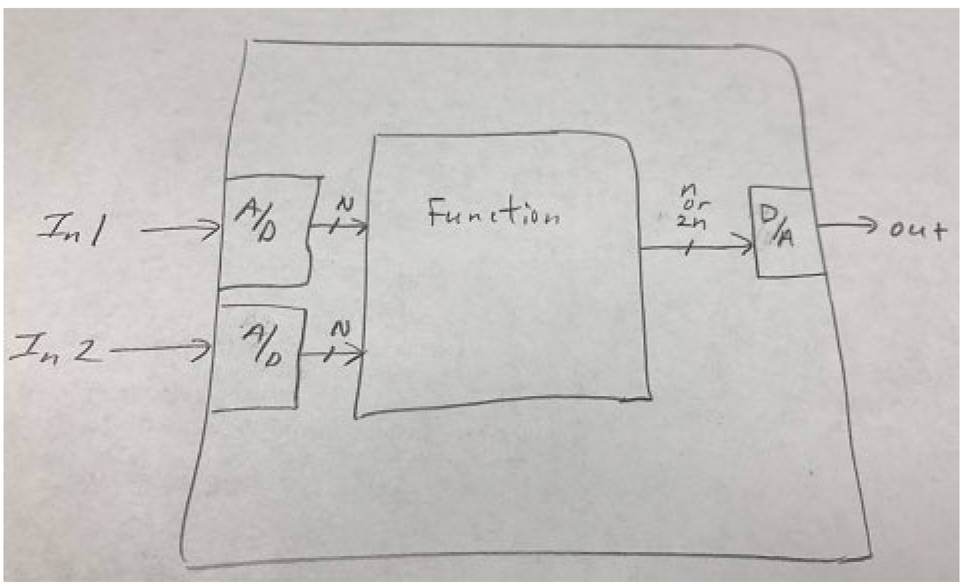
5) OP amp and related circuits

Useful as a saturating amplifier

Useful as a comparator (signum function): $sgn(x) = \begin{cases} -1 \ if\ x < 0 \\ 0\ if \quad x = 0 \\ 1\ if\ x > 0 \end{cases}$

Useful in hysteresis circuits (positive feedback amplifier)

6) Digital Nonlinearity



The input could be one, two, or even more A/D's. Could be made from discrete components or a single microcontroller.

The function could be a memory lookup table, a multiport SRAM for a time delay, or a microcontroller arithmetic processing unit.

It could be a simple multiplication, or almost any mathematical function or nonlinearity, even time varying.

It would need to be clocked at a considerably higher frequency than the BW of the rest of the system. A LPF on the D/A output might be required.

Issues to consider: operating speed, converter resolution, propagation delay, filtering requirements, added noise to the system.

7) Other components possibly useful for circuit nonlinearity generation:
    a. BJTs
    b. jFETs
    c. SCR's and triacs
    d. Electrolytic capacitors
    e. LED with a CdS photocell
    f. MEMS voltage controlled capacitor
    g. High current resistor that heats up due to I^2R losses (light bulb)
    h. Saturating inductor
    i. Current mirror circuits
    j. Switches and relays
    k. Plasma breakdown devices (neon bulb, etc.)