Groundwater Hydraulics

"Conceptual" Model

What is groundwater?
- water available underground

Water in the Earth's Environment

Water Balance
97% water is in oceans (salt water)
3% fresh water (2.14% in ice caps)
- 98% of available fresh water is groundwater! (0.61%)
- only 2% in lakes and rivers (0.009%)
Water Zones in the Subsurface

- Unsaturated zone has 3 phases:
  - Soil solids
  - Water
  - Air
- Saturated zone has 2 phases:
  - Water
  - Soil solids

Take a soil sample from an unsaturated zone and divide it into all 3 phases.

1) Porosity of soil \( \eta = \frac{V_g + V_w}{V} = \frac{V_v}{V} \) (typically \( \approx 0.4 \))

2) Volumetric water content \( \Theta_v = \frac{V_w}{V} \) when zone becomes saturated \( \Theta_v = \eta \Rightarrow V_w = V_v \)

Note, \( \emptyset \leq \Theta_v < \eta \)

3) Saturation ratio \( R_s = \frac{V_w}{V_v} \)

\( \emptyset \leq R_s \leq 1 \)
4) dry bulk density \( \rho_b = \frac{M_s}{V} \) (kg/m³)  \[ \text{typical values: 1600 kg/m³ or 1.6 g/cm³} \]

Compare to water @ 1000 kg/m³

5) soil particle density \( \rho_m = \frac{M_s}{V_s} \) (rock density)

\[ \text{typical values: 2600 kg/m³ or 2.66 g/cm³} \]

→ when soil is saturated, \( V_g = \emptyset \) and we only have 2 phases

\[ \text{porosity } \eta = \frac{V_g + V_w}{V} , \text{ since } V_g = \emptyset \Rightarrow \eta = \frac{V_w}{V} \]

\[ \text{saturation ratio } R_s = \frac{V_w}{V_v} , \text{ since } V_w = V_v \Rightarrow R_s = 1 \]

Additional Formulas:

- porosity \[ \eta = \left[ 1 - \frac{V_s}{V} \right] \]  \[ (1) \]

- we know \( \rho_b = \frac{M_s}{V} \)

\[ V = \frac{M_s}{\rho_b} \]  \[ (2) \]

- also, \( \rho_m = \frac{M_s}{V_s} \)

\[ V_s = \frac{M_s}{\rho_m} \]  \[ (3) \]

- put eqn (2) and (3) into eqn (1)

\[ \eta = 1 - \frac{M_s}{\rho_m} \cdot \frac{\rho_b}{M_s} \]

\[ \eta = 1 - \frac{\rho_b}{\rho_m} \]
Example: \( \rho_w @ 20^\circ C = 0.998 \text{ g/cm}^3 \)

\[ V = 75 \text{ cm}^3 \quad \text{mass} = 150.79 \text{ g (natural)} \quad 153.67 \text{ g (sat)} \quad 126.34 \text{ g (dry)} \]

Find: \( \eta, \Theta_v, R_s, \rho_b, \rho_m \)

\( \eta = \frac{V_v}{V} = \frac{27.4 \text{ cm}^3}{75 \text{ cm}^3} = 0.365 \)

\( \rightarrow \text{Volume of voids (} V_v \text{) is the volume of water @ saturation (} V_{w,sat} \text{)} \)

\( V_{w,sat} = 153.67 \text{ g} - 126.34 \text{ g} = 27.33 \text{ g} \)

\( V_v = \frac{V_{w,sat}}{\rho_w} = \frac{27.33 \text{ g}}{0.998 \text{ g/cm}^3} = 27.4 \text{ cm}^3 = V_v \)

\( \Theta_v = \frac{V_w}{V} = \frac{24.4 \text{ cm}^3}{75 \text{ cm}^3} = 0.325 \)

\[ W_w = 150.79 \text{ g} - 126.34 \text{ g} = 24.36 \text{ g} \]

\[ V_w = \frac{W_w}{\rho_w} = \frac{24.36 \text{ g}}{0.998 \text{ g/cm}^3} = 24.4 \text{ cm}^3 \]

\( R_s = \frac{V_w}{V_v} = \frac{24.4 \text{ cm}^3}{27.4 \text{ cm}^3} = 0.891 \)

\( \rho_b = \frac{M_s}{V} = \frac{126.34 \text{ g}}{75 \text{ cm}^3} = 1.68 \text{ g/cm}^3 \quad \text{(compare to typical value: 1.6 g/cm}^3 \text{ V)} \)

\( \rho_m = \frac{M_s}{V_s} = \frac{126.34 \text{ g}}{47.6 \text{ cm}^3} = 2.65 \text{ g/cm}^3 \quad \text{(value of 2.66 g/cm}^3 \text{ V)} \)

\[ V_s = V - V_v = 75 \text{ cm}^3 - 27.4 \text{ cm}^3 = 47.6 \text{ cm}^3 \]

2) Values to know:

\[ \eta = 0.41 \quad (0.1 - 0.48) \]

\[ \rho_w = 1000 \text{ kg/m}^3 = 1 \text{ g/cm}^3 = 1 \text{ kg/l} \]

\[ \rho_s = 2600 \text{ kg/m}^3 = 2.6 \text{ g/cm}^3 = 2.6 \text{ kg/l} \]

\[ \rho_b = 1600 \text{ kg/m}^3 = 1.6 \text{ g/cm}^3 = 1.6 \text{ kg/l} \]

\( \text{For idealized systems, porosity is independent of particle size} \)

\( \text{Natural systems do not have a single particle size (GSD curve)} \)

if soil is uniformly mixed, the porosity will be lowered

\( \text{Sorted, porosity will be same as the big box items}\)

\( \text{for ball problem} \)

8/18/12
Cubic packing of uniform spheres of equal diameter yield porosity of 47-65%, regardless of the size of the particle.

Porosity of poorly sorted porous media (PM) is less because small particles will block pores.

Typical porosity values:
- Well sorted sand/gravel: 0.25 - 0.5
- Mixed sand/gravel: 0.2 - 0.35
- Glacial till: 0.1 - 0.2
- Silt: 0.35 - 0.5
- Clay: 0.3 - 0.6

Fractured rocks can have 'dual porosity':
- Rock porosity (fracture)
- Matrix porosity

Bernoulli Eqn:
\[ \Sigma F = ma \quad - \quad (1) \]
\[ F = F_g + F_p \quad - \quad (2) \]
\[ F_p = P_1 A - P_2 A = (P_1 - P_2) A \quad - \quad (3) \]
If \( \frac{dp}{ds} \) is the rate of pressure change along \( s \),
\[ \frac{P_2 - P_1}{ds} = \frac{dp}{ds} \]
\[ P_2 = P_1 + \frac{dp}{ds} ds \quad - \quad (4) \]
Substitute eqn (4) into eqn (3)

\[ F_p = P_i A - \left[ P_i + \frac{dp}{ds} \right] A \]

\[ F_p = -\frac{dp}{ds} ds A \quad \text{- (5)} \]

Component of \( F_g \) along \( \theta \) direction

\[ \sin \theta = \frac{F_{gs}}{F_g} \rightarrow F_{gs} = F_g \sin \theta \quad \text{- (6)} \]

\[ \sin \theta = -\frac{dz}{ds} \quad \text{- (7)} \]

(6) can be written as

\[ F_{gs} = F_g \sin \theta = mg \sin \theta \quad \text{where} \ m = \rho dV \]

\[ F_{gs} = \rho dV g \sin \theta \]

\[ F_{gs} = -\rho g dV \frac{dz}{ds} = -g \frac{dz}{ds} \rho A \frac{dV}{dA} \quad \text{- (8)} \]

Putting all forces into eqn (1) we get,

\[ F_g + F_p = ma \]

\[ -g \frac{dz}{ds} \rho A \frac{dV}{dA} - \frac{dp}{ds} \frac{dA}{ds} A = \rho A \frac{dV}{dA} \cdot a \]

divide by \( \rho A \frac{dV}{dA} \)

\[ g \frac{dz}{ds} + \frac{1}{\rho} \frac{dp}{ds} + a = \Phi \quad \text{- (9)} \]

Acceleration for fluid flow has two components

\[ a = \frac{dV}{dt} + V \frac{dV}{ds} \quad \text{- (10)} \]

Substitute eqn (10) into eqn (9)

\[ g \frac{dz}{ds} + \frac{1}{\rho} \frac{dp}{ds} + V \frac{dV}{dt} + V \frac{dV}{ds} = \Phi \]

**Euler Eqn** - (11)
Assume steady-state, \( \frac{\partial}{\partial t} = 0 \)

\[
g \frac{\partial^2 z}{\partial s^2} + \frac{1}{\rho} \frac{\partial p}{\partial s} + v \frac{\partial v}{\partial s} = 0
\]  

- (12)

\[V \frac{\partial v}{\partial s} = \frac{1}{2} \frac{\partial v^2}{\partial s} \]  

- (13)

Substitute eqn (13) into eqn (12)

\[
\frac{1}{2} \frac{\partial}{\partial s} (v^2) + g \frac{\partial}{\partial s} (z) + \frac{1}{\rho} \frac{\partial p}{\partial s} = 0
\]

divide by \( g \)

\[\frac{\partial}{\partial s} \left( \frac{v^2}{2g} + z + \frac{p}{\rho g} \right) = 0 \]

Bernoulli Eqn

\[
\frac{v^2}{2g} + \frac{p}{\rho g} + z = \text{constant}
\]

energy eqn includes frictional losses

---

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Three types of energy

- elevation or gravitational potential energy
- energy due to fluid pressure
- kinetic energy (fluid movement)

• Elevation Energy

Gravitational Potential Energy

\[ E_g = mgz \]

we want to evaluate the energy per unit volume, hence we will use \( \rho \) (mass/unit volume)

Units

\[
\rho = \frac{kg}{m^3}, \quad g = \frac{m}{s^2}, \quad z = m
\]

\[
\rho g z = \left( \frac{kg}{m^3} \right) \left( \frac{m}{s^2} \right) m = \frac{N-m}{m^3} = \frac{J}{m^3}
\]
we want to express energy per unit weight
\[ \frac{E_g}{\rho g} = z \]  (elevation head)

*Pressure Energy*

![Diagram of pressure energy calculation]

\[ P = \rho gh_p \]

unit \( \rightarrow \) energy/unit volume

\[ \frac{P}{\rho g} = h_p \]  (pressure head)

*Kinetic Energy*

- moving water has extra energy

\[ \rho gh = \frac{1}{2} \rho v^2 \]

flow \( \rightarrow \) velocity

\[ \text{total energy} = \text{k.e.} + \text{elev.} + \text{p.e.} \]

\[ \text{total head} = \frac{v^2}{2g} + z + \frac{P}{\rho g} = \text{constant} \]  (Bernoulli's Eqn)

\[ \rightarrow \text{For typical GW flow problems,} \ v \ \text{is very small and} \ k.e. \ \text{is ignored} \]

\[ h = z + \frac{P}{\rho g} = \text{constant} \]
• \( \frac{P}{\rho g} \) is negative in unsaturated soil
• \( \frac{P}{\rho g} \) is positive in saturated soil

### Groundwater Flow

- Water flows from higher 'total head' to lower 'total head' (H)

\[
H = \frac{P}{\gamma} + Z
\]

\[
H_1 = Z_1 + h_{p1}
\]

\[
H_2 = Z_2 + h_{p2}
\]
GW Flow

- Direction
- Amount of Flow

Head gradient \( \Delta x = \frac{\Delta H}{L} = \frac{H_2 - H_1}{L} \)

Darcy's Law

Henri Darcy - French Civil Engineer (1803)
- Interested in designing water supply for Dijon, France
- Used sand filters

Darcy Experiment

\[ Q \rightarrow \]

\[ x = 0 \rightarrow x_A \rightarrow x_B \]

\[ \Delta x = x_B - x_A = L \]
Darcy found:
\[ Q \propto -\Delta H \]
\[ Q \propto \frac{1}{L} \text{ or } \frac{1}{\Delta x} \]
\[ Q \propto A \]

Based on these "empirical" observations, he formulated
\[ Q = -kA \left( \frac{H_b - H_a}{X_b - X_a} \right) \]
\[ Q = -kA \frac{\Delta H}{\Delta x} \]
\[ Q = -kA \frac{dH}{dx} \]

\( \Theta \) need negative sign to make \( Q \) positive
\( \text{bc } H_b < H_a \)

\( K \) is a proportionality constant \( \Rightarrow \) hydraulic conductivity

\( \text{negative flow} \rightarrow \text{positive flow} \)

- If we repeat Darcy's experiment (with ideal soil) we will find:
  \[ Q \propto d^2 \] (soil particle diameter)
  \[ Q \propto \rho g \]
  \[ Q \propto \frac{1}{\mu} \]

Now, Darcy's Law:
\[ Q = \frac{C \cdot d^2 \rho g A}{\mu} \frac{\Delta H}{\Delta x} \]

\( k_i = C d^2 \left[ \text{cm}^2 \text{ m}^2 \right] \) Intrinsic permeability (just material)

\[ Q = -\frac{k_i \rho g}{\mu} A \frac{\Delta H}{\Delta x} \]

\( k = \frac{k_i \rho g}{\mu} = \text{hydraulic conductivity} \) \( \left[ \frac{L}{t} \right] \) \( \text{m/d} \text{y/day} \) \( \text{cm/sec} \)

\( \text{larger particle size allows larger flow} \)

\( \uparrow Q \)

\( \uparrow \text{surface friction} \)
\( \uparrow \text{energy loss} \)

\( \text{heat} \)
Darcy’s Law

\[ Q = -kA \frac{\Delta H}{\Delta x} \]

- Sand (coarse) \( 100 - 1000 \text{ m}/\text{day} \)
- Fine sand (beach) \( 1 - 100 \text{ m}/\text{day} \)
- Silty sand \( 0.01 - 1 \text{ m}/\text{day} \)
- Clay \( 10^{-6} - 10^{-3} \text{ m}/\text{day} \)

**Example**

\[ H_a = 2\text{ m} \quad H_b = 1\text{ m} \]
\[ A = 5\text{ m}^2 \quad L = 2\text{ m} \]
\[ Q = 1.25 \text{ m}^3/\text{day} \]

\[ Q = -kA \frac{\Delta H}{\Delta x} \]

\[ 1.25 \text{ m}^3/\text{day} = -k(5\text{ m}^2)(\frac{1\text{ m} - 2\text{ m}}{2\text{ m}}) \]

\[ k = 0.5 \text{ m}/\text{day} \]

\[ H_a = 1\text{ m} \quad H_b = 2\text{ m} \quad \text{flow direction would be reversed} \]
\[ A = 5\text{ m}^2 \quad L = 2\text{ m} \]
\[ Q = 1.25 \text{ m}^3/\text{day} \]

4) Water flux = \( \frac{m^3 \text{ water}}{\text{unit area} \cdot \text{time}} \)

Velocity is a flux

\[ V = \frac{Q}{A} \quad \text{m}^3/\text{day} \cdot \text{m}^{-2} \]

Q: Darcy "Flux"
Porous Media

\[ q = \frac{Q}{A} \quad \text{Darcy Flux} \quad \left[ \frac{m^3}{m^2 \cdot \text{day}} \right] \]

use this unit!

\[ \rho \cdot \eta \]

"true" area

\[ A \cdot \eta \text{\scriptsize{porosidy}} \]

\[ V = \frac{Q}{\eta \cdot A} = \frac{q}{\eta} \quad \left[ \frac{m}{\text{day}} \right] \]

**Example**

\[ 1 \text{m}^3/\text{day} \rightarrow 0.1 \text{m}^2 \]

\[ 1 \text{m}^3/\text{day} \rightarrow 2 \text{m} \]

- **Open pipe**

\[ q = \frac{Q}{A} = \frac{1 \text{m}^3/\text{day}}{0.1 \text{m}^2} = 10 \text{ m}^3/\text{m}^2 \cdot \text{day} \]

\[ V = q = 10 \text{ m/day} \]

transport time: how long will it take a particle to travel this pipe?

\[ \frac{dx}{dt} = \frac{2m}{10 \text{ m/day}} = 0.2 \text{ days} \]

- **Porous media**

\[ 1 \text{m}^3/\text{day} \rightarrow 0.1 \text{ m}^2 \]

\[ 1 \text{m}^3/\text{day} \rightarrow 2 \text{ m} \]

\[ q = \frac{Q}{A} = \frac{1 \text{m}^3/\text{day}}{0.1 \text{m}^2} = 10 \text{ m}^3/\text{m}^2 \cdot \text{day} \]

\[ V = \frac{Q}{\eta \cdot A} = \frac{q}{\eta} = \frac{10}{0.4} = 25 \text{ m/day} \]

transport time: \[ t = \frac{2m}{25 \text{ m/day}} = 0.08 \text{ days} \]

*Homework #2 Problem 5*

\[ k = \frac{k_i \rho g}{\mu} \quad ; \quad q = -k \frac{\Delta H}{\Delta x} \quad \text{(formula hints)} \]

- made up constant parameter defined over a representative volume

\[ q = -k \frac{\Delta h}{\Delta x} \quad \text{Darcy's Law} \]

- conceptual model

- "model" of a reality.

Come up with laws by doing experiments -> best description

ALL THINGS ARE OPEN TO QUESTION

Simplified reality (models are always wrong, but useful)
\[ q = \frac{\text{volume}}{\text{time}} \rightarrow \text{measure flow} \]

\[ q = -k \frac{\Delta H}{\Delta x} \]

\[ Q = qA \]

\[ A = \pi r^2 \]

"Constant Head Permeameter"

\[ Q = -k \frac{\Delta H}{\Delta x} \pi r^2 \rightarrow \text{calculate } k \]

"Falling Head Permeameter"

\[ Q_{in}(t) = -A_c \frac{dH}{dt} \]

\[ Q_{out}(t) = kA_c \frac{H}{L} \]

\[ Q_{in} = Q_{out} \]

\[ -A_t \frac{dH}{dt} = kA_c \frac{H}{L} \]

\[ k = -\frac{A_t L}{A_c H} \frac{dH}{dt} \quad @t = 0, \quad H(t) = H_0 \]

\[ k = -\frac{A_t L}{A_c H} \ln \left( \frac{H(t)}{H_0} \right) \quad @t = t, \quad H(t) = H \]

\[ k t = -\frac{A_t}{A_c} L \ln (H) + c_0 \Rightarrow \text{use I.C.}@t: \]

\[ c_0 = \frac{A_t}{A_c} L \ln \left[ \frac{H_0}{H} \right] \]

\[ k = \frac{d_e^2 L}{d_e^2 t} \ln \left( \frac{H_0}{H} \right) \]

\[ \text{takes an extremely large amount of static head to push flow through clay \}

\[ \text{pervious (sand)} \]
Find: \( Q \)

\[
q = -k \frac{\Delta H}{\Delta x} = -k \frac{H_2 - H_1}{L}
\]

\( Q = q \cdot A \)

\[
H_A = \frac{P_A}{\rho} + Z_A = \frac{7357.5 \text{ N/m}^2}{1000 \text{ kg/m}^3 \cdot 9.81 \text{ m/s}^2} + 3 \text{ m} = 3.75 \text{ m}
\]

\[
H_B = \frac{P_B}{\rho} + Z_B = \frac{19620 \text{ N/m}^2}{1000 \text{ kg/m}^3 \cdot 9.81 \text{ m/s}^2} + 2 \text{ m} = 4 \text{ m}
\]

\[
Q = -2 \text{ m/day} \left( \frac{4 \text{ m} - 3.75 \text{ m}}{2 \text{ m}} \right) = 0.25 \frac{\text{m}^3}{\text{m}^2 \text{-day}}
\]

\[
Q = q \cdot A = 0.25 \frac{\text{m}^3}{\text{m}^2 \text{-day}} \cdot 0.5 \text{ m}^2 = 0.125 \text{ m}^3/\text{day}
\]

\[
V = \frac{Q}{\eta A} = \frac{q}{\eta} = \frac{0.25}{0.4} = 0.625 \text{ m/day}
\]

1 atm = 101,325 N/m² (Pascal)

\[
9810 \text{ N/m}^2 (10.3 \text{ m})
\]

\rightarrow can get approximately 10 m of head \( P/\rho = 10 \text{ m} \) (max)

\rightarrow using suction

\( \square \) can push any amount of water upward, but vacuum pump can only push up to 10 m.
Problem Solving

1. Horizontal 1-D Flow (ex: 1D column)

\[ \mathbf{Q} = -k \frac{\partial h}{\partial x} \quad q = \frac{m^3}{m^2 \cdot \text{day}} \]

\[ V = \frac{q}{n} \]

Example: Confined aquifer is buried 100 m below land surface. Thickness of the unit is 30 m. A well located at the center of the aquifer recorded 120 m of head. Another well in the eastern boundary located along the centerline recorded 110 m. Find the flow in the eastern direction. Location of 2nd well is 200 m away from 1st well. Drilling data indicated the aquifer is a sandy unit.

\[ (k = 1-100 \text{ mm/day}) \]

\[ (\text{use } 50 \text{ mm/day}) \]

Example: Cartridge is 15 cm long, 5 cm radius. Inlet pressure is \( P_1 \), outlet pressure is \( P_2 \). Oil viscosity is \( \mu \). Find the flow rate. The filter is installed horizontally. Intrinsic permeability is \( k_i \).

\[ K = \frac{k_i \rho g}{\mu} \]

\[ \frac{P_1}{g} = H_1 \]

\[ \frac{P_2}{g} = H_2 \]
Drilling data indicated that there is an unconfined and confined units below Auburn. Water level in the unconfined unit is 3 m below ground. Drillers found an aquitard 10 m below ground. Further drilling indicated a bedrock unit at 30 m below ground. Conductivity of the aquitard is 0.01 m/day and its thickness is 1 m. Piezo measurements in the lower confined unit indicated that the head in the lower unit is 25 m above the bedrock level. Find the amount of water leaking via aquitard if the area of aquifer is 10 km². Find the direction of flow.

\[ H_1 = \frac{P}{g} + z_1 \]
\[ H_2 = \frac{P}{g} + z_2 \]

If datum moved to bedrock level \( (\frac{P}{g} + z_1) \)

\[ H_2 > H_1 \]
\[ H_2 = \frac{P}{g} + z_2 \]
\[ H_2 = 7m + 2m = 27m \]
3. **Darcy's Law in Radial Flow**

- Conceptual model for confined units

Q = ?

5 m

\( \frac{Q}{r} \)

Pumped steady state

water level

\( H = 5 \) m

\( B = 10 \) m

radius of well

\( r = 10 \) cm

5 m

Lake

\[ Q_r = -k \frac{dh}{dr} \]

\[ Q_r = -k \left( \frac{H_2 - H_1}{r_2 - r_1} \right) \]

\( H_1 = 2 \)

\( H_2 = 5 \)

\( r_1 = 10 \) cm

\( r_2 = 100 \) m

Q = ?

\[ A = 2\pi r B \]

\[ Q = -k \left( \frac{H_2 - H_1}{r_2 - r_1} \right) \frac{A}{2\pi r B} \]

- Plan view

- Section view

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\[ \text{area is the right area and why?} \]
Groundwater Aquifers

- Groundwater - Definition:
  - Water is found in the pores between particles of rock and other material.
  - The more pores there are, the more water can be held.

- Groundwater in soil and in fractured rock

  - Soil flow - (porous media)

  - Fractured flow

  - Successful well accessing GW via joints or fractures
    - Unsuccessful well did not intersect joints or fractures

- Two Important Properties of GW Systems
  - Porosity - related to the capacity to store water
  - Hydraulic Conductivity - related to capacity to transmit water

<table>
<thead>
<tr>
<th>Porosity (%)</th>
<th>Strata</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 - 60</td>
<td>Clay</td>
</tr>
<tr>
<td>35 - 50</td>
<td>Silt</td>
</tr>
<tr>
<td>10 - 35</td>
<td>Sand and gravel (mixed)</td>
</tr>
<tr>
<td>25 - 30</td>
<td>Well sorted sand and gravel</td>
</tr>
<tr>
<td>5 - 30</td>
<td>Sandstone</td>
</tr>
<tr>
<td>&lt;1 - 20</td>
<td>Carbonates</td>
</tr>
<tr>
<td>5 - 50</td>
<td>Karst limestone and fractured basalt</td>
</tr>
<tr>
<td>0 - 10</td>
<td>Shale and fractured crystalline rock</td>
</tr>
</tbody>
</table>
• What is the source of ground water?
  
  **Rainfall**

• Classification of Aquifers

<table>
<thead>
<tr>
<th>Unconfined Aqurifer</th>
<th>Confined Aqurifer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confining Layer</td>
<td>Confined Aqurifer</td>
</tr>
</tbody>
</table>

- Like open channel
- Like pipe flow (pressurized)

• Perched Aqurifer

- Rainfall

• Sedimentary Deep Aquifer

  - Generally large
  - Scale productive aquifers

  - Unconfined Aquifer in fractures of impermeable rock
- Fractured Basement Rocks
- Basement Rock
- Impermeable Basement Rock
- Artesian Wells - high pressure, no pump needed

Aquifer - a saturated permeable unit that can transmit significant amount of water

Aquiclude - saturated, low K (ex: clay)

Aquitard - rather impervious formation, but can transmit water at a very slow rate

Aquifer:
- confined *
- unconfined *
- leaky confined
- perched

Confined Aquifer

Unconfined Aquifer

For cases when head is above GL, it is an artesian aquifer.
Leaky Confined Aquifer

- Water is "leaking" from unconfined aquifer because $H_u > H_c$

Conceptual Model

- Confined

- Unconfined

Storage Properties of Aquifers

- Basic 3-Dimensional for defining storage in an aquifer is $S_s \rightarrow$ specific storage

$S_s =$ amount of water released or stored per unit volume of saturated formation per unit change in head

$$\frac{\text{Vol water (released/stored)}}{\text{Unit Vol of Aquifer - Unit head drop/gain}} = \frac{m^3 \text{ of water}}{m^3 \text{ per unit media} - m \text{ (head change)}}$$

Adding water to system compresses PM + H2O molecules together

- Confined aquifer

$$S = S_s \cdot B$$

$S =$ amount of water released or stored per unit area (x-y area) per unit head drop

$$S = \frac{m^3}{m^2 \cdot m}$$
Fig. 4.1. Definition sketches for storativity: (a) confined aquifer; (b) phreatic aquifer (unconfined)
Figure 5.6. Unit prisms of unconfined and confined aquifers illustrating differences in storage coefficient. For equal declines in head, the yield from an unconfined aquifer is much greater than that from a confined aquifer. (After Heath and Trainer, 1968)
Matrix compression + Water compression

\[ S_s = \frac{\rho_w g \alpha}{\rho_w g N B} \]

Typical Values of \( S_s \): \( 10^{-4} \) to \( 10^{-6} \left[ \frac{m^3}{m^3 \cdot m} \right] \)

Typical Values of \( B \): 1 to 100 m

Typical Values of \( S \): \( 10^{-2} \) to \( 10^{-4} \left[ \frac{m^3}{m^2 \cdot m} \right] \)

Example

1) Confined aquifer below AU is 10 m thick. Drop piezometric head by 2 m over entire area (~ 4 km²). What will be the volume (m³) of water released from the system?

\[ S_s = 10^{-4} \frac{m^3}{m^3 \cdot m} \]

\[ S = S_s \cdot B \]

\[ S = 10^{-4} \frac{m^3}{m^3 \cdot m} \cdot 10 m \]

\[ V_w = S \cdot A \cdot \Delta H \]

\[ V_w = 10^{-3} \frac{m^3}{m^2 \cdot m} \cdot (4 \times 10^6 \ m^2)(2 m) \]

\[ V_w = 8,000 \ m^3 \]

Storage in Unconfined Aquifer

Unconfined aquifer

Unconfined = \( S_y \)

2D Property

\[ S_{unconfined} = S_s B + S_y \]

So small that it is typically ignored (ideally, \( S_y = \eta \))

\[ \eta = S_y + S_r \]

\( S_r = 0.1 - 0.2 \)
example (continued)

2) Further drilling indicated presence of an unconfined formation 10 m thick. Drop the head by 2 m, how much water is released by unconfined system? $S_y = 0.1$

$$V_w = S \cdot A \cdot c \cdot H$$

$$V_w = 0.1 \left(4 \times 10^6 \text{ m}^2\right) \left(2 \text{ m}\right)$$

$$V_w = 800,000 \text{ m}^3$$ (draining entire aquifer)

---

7) Darcy's Law

$$\frac{dH}{dx} \Rightarrow \text{Darcy flux} \quad \frac{\text{m}^3}{\text{m}^2 \cdot \text{day}}$$

$$Q = k \cdot A$$

$$V = \frac{Q}{A} \quad \text{water velocity}$$

Aquifers

- Confined - deeper
- Unconfined - shallow

Confined

$$S_s \rightarrow \text{specific storage coefficient (3D)}$$

$$S_s = \frac{m^3}{m^3 \cdot \text{m}} \quad \left(10^{-4} \text{ to } 10^{-6} \frac{m^3}{m^3 \cdot \text{m}}\right)$$

$$S \rightarrow \text{storage coefficient (2D)}$$

$$S = S_s \cdot B \quad \frac{m^3}{m^2 \cdot \text{m}}$$

Unconfined

$$S \rightarrow \text{storage coefficient}$$

$$\frac{S = S_s \cdot B + S_y}{\text{very small} \quad \text{specific yield}} \quad \Rightarrow \quad S \approx S_y \quad \frac{m^3}{m^2 \cdot \text{m}}$$

$$S_r = \text{specific retention}$$

$$n = S_y + S_r \quad \left(\text{ideally, } S_y = n\right)$$

8/28/12
Figure 11

1) Compute Gw discharge into the river along the 7 Mile Reach

2) Average gradient

3) Darcy flux

\[ 1 \text{ km} = 0.62 \text{ mi} \]

\[ 1 \text{ m} = 3.28 \text{ ft} \]
Ground Water Movement

- Recharge area
- Water table
- Flowing well
- Unconfined aquifer
- Confined aquifer
- Height to which water will rise
- Impermeable bottom
Dropping the head in a confined aquifer does not release as much water as dropping the head in an unconfined aquifer (b/c of drainage) aka "phreatic"

Homework 3, Problem 8

1) average gradient
\[
\frac{dh}{dx} = \frac{390 \text{ ft}}{11.25 \text{ km}} = \frac{390 \text{ m}}{8000 \text{ m}} = 0.04875 \text{ m/m}
\]

2) darcy flux
\[
q = -k \frac{dh}{dx}
\]

3) total groundwater discharge into river \(51450 \text{ m}^3/\text{day}\)

\[
Q = q \cdot A
\]

4) how long it will take for spill discharged at point X to reach river \(57.5 \text{ yrs}\)
Homogeneity and Heterogeneity

- Properties the same everywhere
- Properties (k) vary over space

Types of Heterogeneities (Layering)

- Spatial Heterogeneity
  - Slip
  - \( k_1 \)
  - \( k_2 \)
  - \( k_3 \)

- Directional Heterogeneity
  - Isotropic and anisotropic

  \[ k_y \uparrow \quad k_x \downarrow \quad k_y \downarrow \quad k_x \rightarrow \]

  \( k_y \neq k_y \Rightarrow \text{anisotropic} \)

  \( k_x = k_y \Rightarrow \text{isotropic} \)

Equivalent \( k \) in layered systems

- \( Q_i = b_i k_i \frac{\Delta H}{\Delta x} w_i^{n-1} \)
- \( Q_2 = b_2 k_2 \frac{\Delta H}{\Delta x} w_2^{n-1} \)
- \( \vdots \)
- \( Q_n = b_n k_n \frac{\Delta H}{\Delta x} w_n^{n-1} \)

\[ Q = \sum_{i=1}^{n} b_i k_i \frac{\Delta H}{\Delta x} \]

Equivalent 'k' system

\[ Q = k_h \cdot b \cdot \frac{\Delta H}{\Delta x} w_1^{n-1} \]

\[ k_h b \frac{\Delta H}{\Delta x} = \sum_{i=1}^{n} b_i k_i \frac{\Delta H}{\Delta x} \]

\[ k_h = \frac{n}{i=1} b_i k_i \text{ arithmetic mean} \]

\[ \text{governed by largest \#} \]
**Vertical Flow**

- $k_1$
- $k_2$
- $k_3$
- $k_4$

$Q \uparrow$

- In horizontal systems, largest $K$ dominates b/c it is where all the flow goes.
- In vertical systems, smallest $K$ dominates b/c of its resistance.

---

**Equivalent System**

$\Delta h = \Delta h_1 + \Delta h_2 + \Delta h_3 + \ldots$

$$Q = A \frac{\Delta h_1}{b_1} \implies \Delta h_1 = \frac{Qb_1}{AK_1}$$

Total

$$\Delta h = \frac{Q}{A} \left[ \frac{b_1}{k_1} + \frac{b_2}{k_2} + \ldots \right]$$

$$\frac{Qb}{A K_v} = \frac{Q}{A} \left[ \frac{b_1}{k_1} + \frac{b_2}{k_2} + \ldots \right]$$

$$K_v = \frac{b}{\sum_{i=1}^{n} \frac{b_i}{k_i}}$$

harmonic mean (governed by lowest #)

---

**Homework 3, Problem 9**

Not uniform movement

- $q_x \rightarrow k_1$
- $q_x \rightarrow k_2$
- $q_x \rightarrow k_3$

1 m/day

16 m/day

5 m/day

$K_{av}$ is not a good use for time estimate b/c layer 2 is transporting contaminants faster.

---

**General Form of Darcy's Law** (3D, heterogeneous, anisotropic)

In a 3D domain, there will be 3 fluxes:

$\dot{q} = q_x k_x + q_y k_y + q_z k_z$

For an anisotropic system, general form of Darcy's Law is

$\dot{q} = \mathbf{k} \cdot \mathbf{j}$

2nd rank tensor

gradient vector
\[ Q_x = k_{xx} J_x + k_{xy} J_y + k_{xz} J_z \]
\[ Q_y = k_{yx} J_x + k_{yy} J_y + k_{yz} J_z \]
\[ Q_z = k_{zx} J_x + k_{zy} J_y + k_{zz} J_z \]

In general
\[ \vec{q} = \vec{k} \cdot \vec{J} \]

\[ \vec{k} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \quad \vec{J} = \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} \]

In a general 3D system, if we assume the coordinate \( x, y, z \) is aligned with the layering direction, then
\[ \vec{k} = \begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{zz} \end{bmatrix} \quad \vec{q} = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix} \]
\[ Q_x = k_{xx} J_x \quad J_x = \frac{-\partial h}{\partial x} \]
\[ Q_y = k_{yy} J_y \quad J_y = \frac{-\partial h}{\partial y} \]
\[ Q_z = k_{zz} J_z \quad J_z = \frac{-\partial h}{\partial z} \]

For isotropic systems, \( k_{xx} = k_{yy} = k_{zz} = k \)
\[ Q_x = -k \frac{\partial h}{\partial x} \]
\[ Q_y = -k \frac{\partial h}{\partial y} \]
\[ Q_z = -k \frac{\partial h}{\partial z} \]

\[ \vec{q} = -k \cdot \nabla h \]

\[ \nabla \rightarrow \text{gradient operator} \]
\[ \nabla = \hat{i} \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \]
\( q_x = -K \frac{\partial h}{\partial x} \)

\[ q = q_x \hat{i} + q_y \hat{j} + q_z \hat{k} \]

\[ \bar{q} = -K \nabla h \]

* don't have to use Cartesian coordinate system

\( \nabla \rightarrow \) gradient operator = \( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \)

\[ q = -\left[ K \frac{\partial h}{\partial x} \hat{i} + K \frac{\partial h}{\partial y} \hat{j} + K \frac{\partial h}{\partial z} \hat{k} \right] \]

* radial flow

\[ \nabla \rightarrow \text{for cylindrical} \]

\[ \nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \]

\[ \rightarrow \text{For pumping well} \]

\[ \nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_z \frac{\partial}{\partial z} \]

\[ \rightarrow \text{Radial direction} \]

\[ \nabla = \hat{e}_r \frac{\partial}{\partial r} \]

\[ \bar{q}_r = -K \frac{\partial h}{\partial r} \]

---

**Homework 4, Problem 1**

typically we make pumping flow positive instead of negative

**600 m**

**r**

**Q = 0.3 m**

**V = \frac{dr}{dt} \text{ or } \frac{d\theta}{dt}**

**\int \frac{Q}{2\pi r B} dr = \text{Som}**

**Velocity is not constant!**

**it varies radially**

\[ V(r) = \frac{Q}{2\pi r B} \]

**1) How long will it take a pollutant introduced 1000 m away from the well to reach it?**

\[ q = K \frac{\partial h}{\partial r} \]

\[ Q = q \frac{A}{A} \]

\[ Q = K \frac{\partial h}{\partial r} 2\pi r B \]

\( dx = \int V \, dl \)
Homework 4, Problem 2

compare velocity functions for radially flow and linear flow

Homework 4, Problem 3

- For the pipe itself, apply Darcy's Law
  \[ Q = k \frac{h_1 - h_2}{L} A_{pipe} \]

- For tank 1
  \[ Q = -A_1 \frac{dh_1}{dt} \]

For the pipe itself, apply Darcy's Law
\[ Q_{pipe} = Q_{tank 1} \]

\[ k \frac{h_1 - h_2}{L} A_{pipe} = -A_1 \frac{dh_1}{dt} \]  \[ \text{1 eqn} \]

\[ \frac{dh_1}{dt} = -kA_{pipe} \frac{1}{LA_1} (h_1 - h_2) \]

\[ @ t=0 \quad h_1 = h \]
\[ h_2 = \] \[ h \]

\[ \Rightarrow \text{must express } h_2 \text{ in terms of } h_1 \text{ to "Close" System} \]

\[ A_1 (h_0 - h_1) = A_2 (h_2 - h_0) \]

\[ A_1 h_0_1 - A_1 h_1 = A_2 h_2 - A_2 h_0 \]

\[ \Rightarrow \text{what is lost by first tank is gained by second tank} \]

\[ h_2 = f(h_1, h_0, h_0) \]

\[ \text{constant known value} \]

\[ h_2 = \frac{A_1 h_0 - A_1 h_1 + A_2 h_0}{A_2} \]
\[ \phi(x, y) = a + bx + cy \]

\( x = 0 \quad y = 0 \quad h_1 = 10 \text{ m} \)

\( \Rightarrow a = 10 \text{ m} \)

\( x = 300 \text{ m} \quad y = 0 \quad h = 11.5 \text{ m} \)

\[ a + b(300) = 11.5 \text{ m} \]

\( \Rightarrow b = 0.005 \)

\( x = 0 \quad y = 200 \text{ m} \quad h = 8.4 \text{ m} \)

\[ a + c(200) = 8.4 \text{ m} \]

\( \Rightarrow c = -0.008 \)

\[ \phi(x, y) = 10 + 0.005x - 0.008y \]

\[ \frac{\partial \phi}{\partial x} = b \quad \Rightarrow 0.005 \]

\[ \frac{\partial \phi}{\partial y} = c \quad \Rightarrow -0.008 \]

\[ \nabla \phi(x, y) = 0.005i - 0.008j \]

\[ |\nabla \phi| = \sqrt{0.005^2 + (-0.008)^2} = 0.0094 \]

\[ \theta = \tan^{-1} \left( \frac{-0.008}{0.005} \right) = -57.99^\circ \]
\[ v_x = k \left( \frac{\partial \phi}{\partial x} \right) \quad \text{Darcy's Law} \]

\[ v_x = -15 \text{ m/day} \quad (0.005) = -0.075 \text{ m/day} \]

\[ v_y = -k \frac{\partial \phi}{\partial y} = -15 \text{ m/day} \quad (-0.008) \]

\[ = 0.12 \text{ m/day} \]

\( \theta = -58^\circ \text{C} \)

So 'C' is in 'clockwise' from x axis

(h) total discharge

\[ Q = KA (-\nabla \phi) \]

If new magnitude of discharge alone then

\[ Q = KA |\nabla \phi| \]

\[ \frac{Q}{w} = (15 \text{ m/day}) \quad (20 \text{ m}^2) \quad (0.0014) \text{ m/m} \]

\[ = 0.0282 \text{ m}^2/\text{day} \quad \text{or} \quad \frac{\text{m}^3}{\text{m-day}} \]
Homework 4, Problem 4

\( H(x,y) = ? \)

\[ \begin{align*}
\hat{a} &+ \hat{b}x + \hat{c}y \\
\text{Find } a, b, c.
\end{align*} \]

3 eqn. 3 unknowns

To find direction of flow, or gradient:

\[ \nabla H = \hat{i} \frac{\partial H}{\partial x} + \hat{j} \frac{\partial H}{\partial y} \]

\[ \nabla H = \hat{i} b + \hat{j} c \quad \Rightarrow \quad \Theta = \tan^{-1} \left( \frac{b}{c} \right) \]

\[ |\nabla H| = \sqrt{b^2 + c^2} \]

\[ q = -k |\nabla H| \quad \Rightarrow \quad Q = q \cdot A \quad \text{remember that } A \text{ is} \quad \perp \text{ to flow direction} \]

\[ \text{HW 5} \quad \text{Single space 12 pt} \]

1) 1 page. Review MODFLOW.

- What does code do?
- What is it for?

2) 1 page.

Application of MODFLOW (LUGURU)

Scott Bar
OH 100 ST.

Derivation of GW Flow Equation

We will use the following two general principles:

- Conservation of mass
- Conservation of momentum
Consider the following control volume.

- Flux is 'stuff' crossing unit area per unit time.

Some notation:

- \( q \): Darcy Flux  \( \Rightarrow \) volume of water per unit area - time [\( \text{m}^3 \text{ m}^2 \text{-day} \)]
- \( \phi \): Hydraulic Head (potential that drives flow)
- \( \rho \): Density [\( \text{kg/m}^3 \)]
- \( \rho q \): Mass Flux [\( \text{kg/m}^2 \text{-day} \)]
- \( S_s \): Specific Storage Coefficient [\( \text{m}^3 \text{ m}^{-1} \)]

- Amount of water stored or released per unit volume of aquifer per unit change in head.

Note, the control volume is an arbitrary 3D CV taken deep inside an aquifer.

- Law 1: Conservation of Mass

\[ \text{mass}_{\text{in}} - \text{mass}_{\text{out}} = \text{accumulation} \quad \text{(mass addition is positive)} \]

We will look at a case when \( \text{mass}_{\text{out}} < \text{mass}_{\text{in}} \) and hence, \( \phi \) is increasing.
Total input mass in the x-direction in time $\Delta t$

$$\left( \rho g \right)_x \cdot \Delta y \cdot \Delta z \cdot \Delta t$$

Total output mass in the x-direction in time $\Delta t$

$$\left( \rho g \right)_{x+\Delta x} \cdot \Delta y \cdot \Delta z \cdot \Delta t$$

Balance in x-direction

$$\left[ \left( \rho g \right)_x - \left( \rho g \right)_{x+\Delta x} \right] \cdot \Delta y \cdot \Delta z \cdot \Delta t$$

$$- \left[ \left( \rho g \right)_{x+\Delta x} - \left( \rho g \right)_x \right] \Delta y \Delta z \Delta t \quad (1a)$$

Balance in y-direction

$$- \left[ \left( \rho g \right)_{y+\Delta y} - \left( \rho g \right)_y \right] \Delta x \Delta z \Delta t \quad (1b)$$

Balance in z-direction

$$- \left[ \left( \rho g \right)_{z+\Delta z} - \left( \rho g \right)_z \right] \Delta x \Delta y \Delta t \quad (1c)$$

Finding accumulation term $\Delta M$

- Assume mass in $> \text{mass out}$

$$\Phi_{e+\Delta t} > \Phi_e$$

- Define $\Delta \Phi = \left( \Phi_{e+\Delta t} - \Phi_e \right)$

The mass of water needed to be accumulated within C.V. to increase the potential by $\Delta \Phi$ can be computed from the definition of $S$s.

$$\Delta M = Ss \Delta x \Delta y \Delta z \rho \left( \Phi_{e+\Delta t} - \Phi_e \right) \quad (2)$$
Mass in - Mass out = Accumulation

→ Combine Eqn 1 and 2 and divide all terms by Δx Δy Δz Δt

\[
- \left[ \frac{Q_{x+\Delta x} - Q_x}{\Delta x} \right] - \left[ \frac{Q_{y+\Delta y} - Q_y}{\Delta y} \right] - \left[ \frac{Q_{z+\Delta z} - Q_z}{\Delta z} \right] = S_s \left[ \frac{\phi_{t+\Delta t} - \phi_t}{\Delta t} \right]
\]

Note: \(Q_{x+\Delta x} = Q_x + \frac{\partial Q_x}{\partial x} \Delta x\) (Taylor Series)

→ If we limit Δx, Δy, Δz, Δt → 0

\[
- \frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} - \frac{\partial Q_z}{\partial z} = S_s \frac{\partial \phi}{\partial t} \quad - (3)
\]

mass balance equation

→ In a lake \(S_s\) becomes = 0

\[
0 = \nabla \cdot \left( \nabla \phi \right) - \nabla \cdot \left( \nabla \phi \right) = 0
\]

Continuity Eqn

External Sources / Sinks

\[
W = \text{Volume of water added per unit volume of porous media per day} \quad \left[ \frac{m^3}{m^3 \text{-day}} \right]
\]

→ Input mass from external source

\[
W \cdot \Delta x \Delta y \Delta z \cdot \rho \Delta t
\]

→ Adding this term to input for Eqn 3

\[
- \frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} - \frac{\partial Q_z}{\partial z} + W = S_s \frac{\partial \phi}{\partial t} \quad - (4)
\]
**Law 2: Conservation of Momentum**

In GW, it is Darcy's Law (comes from N.S. eqns)

\[
q_x = -k_x \frac{\partial \phi}{\partial x}; \quad q_y = -k_y \frac{\partial \phi}{\partial y}; \quad q_z = -k_z \frac{\partial \phi}{\partial z}\quad (5)
\]

Darcy's Law for Flow in 3D Anisotropic Domains

In 3D, \( q_x, q_y, \) and \( q_z \) are the three components of Darcy Flux vector \( \vec{q} \).

\[
\vec{q} = \hat{i} q_x + \hat{j} q_y + \hat{k} q_z
\]

For an anisotropic porous medium, the general expression of \( \vec{q} \) is:

\[
\vec{q} = \vec{k} \cdot \vec{J}
\]

\[
\begin{bmatrix}
q_x \\
q_y \\
q_z
\end{bmatrix} =
\begin{bmatrix}
k_{xx} & k_{xy} & k_{xz} \\
k_{yx} & k_{yy} & k_{yz} \\
k_{zx} & k_{zy} & k_{zz}
\end{bmatrix}
\begin{bmatrix}
J_x \\
J_y \\
J_z
\end{bmatrix}
\]

\( \bullet \) Note, we have assumed diagonal \( \vec{k} \) tensor (see Lecture 7 notes)

→ Substituting Eqn 5 into Eqn 4

\[
\frac{\partial}{\partial x} \left[ k_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y \frac{\partial \phi}{\partial y} \right] + \frac{\partial}{\partial z} \left[ k_z \frac{\partial \phi}{\partial z} \right] + \mathbf{W} = S_g \frac{\partial \phi}{\partial t}\quad (6)
\]

General GW Flow Equation (3D systems)

→ Simplifications to Eqn 6

1. Homogeneous, 3D system

→ \( k_x, k_y, k_z \) are constant (homogeneous)

\[
k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} + k_z \frac{\partial^2 \phi}{\partial z^2} + \mathbf{W} = S_g \frac{\partial \phi}{\partial t}
\]

2. Homogeneous and Isotropic, 3D system

→ \( k_x = k_y = k_z = k \) (isotropic)

\[
k \frac{\partial^2 \phi}{\partial x^2} + k \frac{\partial^2 \phi}{\partial y^2} + k \frac{\partial^2 \phi}{\partial z^2} + \mathbf{W} = S_g \frac{\partial \phi}{\partial t}
\]
3. Steady-state, homogeneous, isotropic, 3D system
\[ \frac{\partial^2 \phi}{\partial t^2} = 0 \] (steady-state)
\[ K \frac{\partial^2 \phi}{\partial x^2} + K \frac{\partial^2 \phi}{\partial y^2} + K \frac{\partial^2 \phi}{\partial z^2} + W = 0. \]

\[ \text{Steady-state, homogeneous, isotropic, 3D system } \Rightarrow \text{ NO WELLS} \]
\[ \nabla^2 \phi = 0 \]

LaPlace Eqn

4. Steady-state, heterogeneous, isotropic, 3D system
\[ \frac{d}{dx} \left[ k \frac{d \phi}{dx} \right] + \frac{d}{dy} \left[ k \frac{d \phi}{dy} \right] + \frac{d}{dz} \left[ k \frac{d \phi}{dz} \right] + W = 0 \]

5. Homogeneous, isotropic, no well, 1-D in x-direction
\[ \Rightarrow \frac{d}{dy} = 0, \quad \frac{d}{dz} = 0 \] (1-D)
\[ \frac{K}{S_b} \frac{d^2 \phi}{dx^2} = S_s \frac{d \phi}{dt} \]

Multiply by \( B \)
\[ \frac{K B}{T} \frac{d^2 \phi}{dx^2} = S_s B \frac{d \phi}{dt} \]

\( T = \text{transmissivity} = k B \left[ \frac{m^2}{\text{day}} \right] \)
\( S = S_s B \left[ \frac{m^3}{m^2 \cdot m} \right] \)

1-D Confined Flow Eqn in Transient Systems

Solving this eqn
\[ \phi(x, t) \]

HW 5: Problem 6 \( \Rightarrow \) analytical solution to this eqn
### 3D GW Flow Equation

\[
\frac{\partial}{\partial x} \left[ k_x \frac{\partial q}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y \frac{\partial q}{\partial y} \right] + \frac{\partial}{\partial z} \left[ k \frac{\partial q}{\partial z} \right] = -q_x - q_y - q_z \quad W = S \frac{\partial q}{\partial t}
\]

Q = q_x \cdot B \cdot L

Q = q_x \cdot L \quad \Rightarrow \quad \text{Darcy Flux}

T_x = k_x \cdot B \quad \text{transmissivity}

Q = \frac{m^3}{m^2 \cdot \text{day}} \quad \text{flow per unit area}

Q = \frac{m^3}{m \cdot \text{day}} \quad \text{flow per unit width}

### 2D GW Flow Equation

N = \frac{m^3}{m^2 \cdot \text{day}} \quad \text{(rainfall)}

\Rightarrow \text{Mass Balance}

\text{In} - \text{Out} = \text{Accumulation}

- \text{x-direction}

\[(\rho q^*_x \bigg|_x - \rho q^*_x \bigg|_{x+\Delta x}) \Delta y \cdot \Delta t\]

- \text{y-direction}

\[-(q_y^* \bigg|_{y+\Delta y} - q_y^* \bigg|_{y}) \rho \Delta x \Delta t\]
Accumulation $\Delta M$

$$\Delta M = S \left( \sum_{t} \frac{m^3}{m^2 \cdot \text{day}} \right)$$

$$\Delta M = S \Delta x \Delta y \rho \Delta \phi$$

Source/Sink in 2D?

- Recharge/rainfall

$$N = \left[ \frac{m^3}{m^2 \cdot \text{day}} \right]$$

total mass coming in through rainfall = $N \Delta x \Delta y \rho \Delta t$

$$- \left( \frac{q_x^* |_{x+\Delta x} - q_x^* |_x}{\Delta x} \right) \rho \Delta y \Delta t - \left( \frac{q_y^* |_{y+\Delta y} - q_y^* |_y}{\Delta y} \right) \rho \Delta x \Delta t + N \Delta x \Delta y \rho \Delta t$$

$$= S \Delta x \Delta y \rho \Delta \phi$$

Divide by $\rho \Delta x \Delta y \Delta t$

$$- \frac{\frac{\partial q_x^*}{\partial x}}{\Delta x} - \frac{\frac{\partial q_y^*}{\partial y}}{\Delta y} + N = S \frac{\partial \phi}{\partial t}$$

$\Delta x, \Delta y, \Delta t \to 0$

$$- \frac{\partial q_x^*}{\partial x} - \frac{\partial q_y^*}{\partial y} + N = S \frac{\partial \phi}{\partial t}$$

$q_x^* = q_x \cdot B$

$$= -k \frac{\partial \phi}{\partial x} \cdot B$$

$q_x^* = -T_x \frac{\partial \phi}{\partial x}$

$q_y^* = -T_y \frac{\partial \phi}{\partial y}$

Integrating 3D eqn

$$\frac{\partial}{\partial x} \left[ T_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ T_y \frac{\partial \phi}{\partial y} \right] + N = S \frac{\partial \phi}{\partial t}$$

$\Rightarrow$ 2D Confined Flow Eqn
\[ \frac{\partial}{\partial x} \left[ k_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y \frac{\partial \phi}{\partial y} \right] + \frac{\partial}{\partial z} \left[ k_z \frac{\partial \phi}{\partial z} \right] + W = \int_0^b S_s \frac{\partial \phi}{\partial t} \quad \text{3D Eqn Confined Aquifer} \]

\[ B \frac{\partial}{\partial x} \left[ k_x \frac{\partial \phi}{\partial x} \right] + B \frac{\partial}{\partial y} \left[ k_y \frac{\partial \phi}{\partial y} \right] + N = S \phi \frac{\partial \phi}{\partial x} \]

\[ \frac{\partial}{\partial x} \left[ T_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left( T_y \frac{\partial \phi}{\partial y} \right) + N = S \frac{\partial \phi}{\partial t} \quad \text{(same!)} \]

\[ \text{2D Confined Flow} \]

\[ \frac{\partial}{\partial x} \left[ T_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ T_y \frac{\partial \phi}{\partial y} \right] + N = S \frac{\partial \phi}{\partial t} \]

\[ \rightarrow \text{For 1D Confined Flow} \]

\[ \frac{\partial}{\partial y} = 0 \]

\[ \frac{\partial}{\partial x} \left[ T_x \frac{\partial \phi}{\partial x} \right] + N = S \frac{\partial \phi}{\partial t} \quad \text{1D Confined Flow Eqn} \]

\[ \rightarrow \text{For homogeneous systems} \]

\[ T_x \frac{\partial \phi}{\partial x} + N = S \frac{\partial \phi}{\partial t} \]

\[ \text{1D Confined Flow} \]

\[ \phi \quad \text{dependent variable} \]

\[ T_x \frac{\partial \phi}{\partial x} + N = S \frac{\partial \phi}{\partial t} \quad x, t \quad \text{independent variables} \]

\[ \rightarrow \text{Find } \phi(x, t) \]

\[ \begin{align*}
BC_1 & \quad \phi(0) = \phi_1 \\
BC_2 & \quad \phi(L) = \phi_2 \\
IC & \quad \phi(0) = \phi_1
\end{align*} \]

\[ \text{SOLUTION!} \]

\[ \rightarrow \text{To solve, what do we need?} \]

\[ - \text{Boundary Conditions (2)} \]

\[ - \text{Initial Condition} \]
→ Solutions

- Analytical ✓
- Numerical ....

1D Confined Flow

\[ T_x \frac{\partial \phi}{\partial x^2} + N = S \frac{\partial \phi}{\partial t} \]

Steady-state condition \( \frac{\partial \phi}{\partial t} = 0 \)

Boundary Conditions (2)
Initial Condition? No!
(steady-state same @ all t)

\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{-N}{T_x} \]

\[ \begin{align*}
\text{BC}_1 \ & \ x = 0, \ \phi = \phi_1 \quad \text{[these are specific]} \\
\text{BC}_2 \ & \ x = L, \ \phi = \phi_2
\end{align*} \]

\[ \begin{align*}
\frac{\partial \phi}{\partial t} &= -\alpha \phi + c_1 \\
\phi(x) &= -\alpha \frac{x^2}{2} + c_1 x + c_2 \end{align*} \]

Steady State Solution

When \( N = 0 \) (no recharge)

\[ \alpha = 0 \]

\[ \phi(x) = \left( \frac{\phi_2 - \phi_1}{L} \right) x + \phi_1 \]

- How to find \( x_{\text{max}} \), where \( \phi_{\text{max}} \) occurs?

\[ \begin{align*}
\frac{\partial \phi}{\partial x} &= 0 \\
\frac{\partial \phi}{\partial x} &= -2 \frac{\alpha x}{2} + \frac{\phi_2 - \phi_1}{L} + \frac{\alpha L}{2} = 0
\end{align*} \]

\[ X_{\text{max}} = \frac{\phi_2 - \phi_1 + \alpha L}{\alpha} \]

\[ X_{\text{max}} = \frac{\phi_2 - \phi_1 + L}{\alpha L} \]

↑ location of a groundwater divide
Governing Equations of GW Flow

$\phi (x, y, z, t) \leftarrow \text{Head}$

3-D

$$\frac{\partial}{\partial x} \left[ k_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y \frac{\partial \phi}{\partial y} \right] + \frac{\partial}{\partial z} \left[ k_z \frac{\partial \phi}{\partial z} \right] + W = S \frac{\partial \phi}{\partial t}$$

$\uparrow$ Source/sink (well)

2-D Confined (Linear)

$$\frac{\partial}{\partial z} = 0 \quad \text{and multiply by} \quad B \quad \text{(aquifer thickness)}$$

$$\frac{\partial}{\partial x} \left[ T_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ T_y \frac{\partial \phi}{\partial y} \right] + N = S \frac{\partial \phi}{\partial t}$$

$\uparrow$ Recharge (rain)

1-D Confined

$$\frac{\partial}{\partial y} = 0$$

$$\frac{\partial}{\partial x} \left[ T_x \frac{\partial \phi}{\partial x} \right] + N = S \frac{\partial \phi}{\partial t}$$

2D Unconfined (Non-linear)

$T = KB$

$S = SsB \begin{cases} \text{2D properties} \\ \text{properties} \end{cases}$

Recall:

- Confined: $q \rightarrow \frac{\partial \phi}{\partial n}$
  
  $q^* = qB$ \begin{cases} \text{depth average flux} \\ \text{depth average flux} \end{cases}$

  $$q = \frac{m^3}{m^2 \cdot \text{day}} \quad q^* = \frac{m^3}{m \cdot \text{day}}$$

- Unconfined: $q \rightarrow h$
  
  $q^* = qh$ \begin{cases} \text{depth average flux} \\ \text{depth average flux} \end{cases}$

  $$q = \frac{m^3}{m^2 \cdot \text{day}} \quad q^* = \frac{m^3}{m \cdot \text{day}}$$
Dupuit's Assumptions:

1) Hydraulic gradient is equal to the slope of the water table
2) For small water table gradients, the streamlines are horizontal (or potential lines are vertical)

→ Assumption 1
* we ignore vertical component
\[ q_x = -k_x \frac{\partial h}{\partial x} \]

Similarly, \( q_y = -k_y \frac{\partial h}{\partial y} \)

→ Assumption 2

Assumption allows to define depth average Darcy Flux
\[ q_x^* = q_x h \quad [\text{m}^3/\text{m-clay}] \]

By using Dupuit assumptions, we have replaced
\[ \phi (x, y, z, t) \rightarrow h(x, y, t) \]
and
\[ q_x = -k_x \frac{\partial h}{\partial x} \quad q_y = -k_y \frac{\partial h}{\partial y} \]
\[ q_x^* = q_x h \quad q_y^* = q_y h \]
Cross-sectional view

\[ q_x^* \bigg|_x = q_x h \bigg|_x \]
\[ h(x,y,t) \]
\[ q_x^* \bigg|_{x+\Delta x} = q_x h \bigg|_{x+\Delta x} \]

→ Conservation of Mass

\[ \text{In - Out} = \text{Accumulation} \]

\[ \left[ \rho q_x^* \bigg|_x - \rho q_x^* \bigg|_{x+\Delta x} \right] \Delta y \Delta t + \left[ \rho q_y^* \bigg|_y - \rho q_y^* \bigg|_{y+\Delta y} \right] \Delta x \Delta t \]

\[ + N \rho \Delta x \Delta y \Delta t = S_y \Delta x \Delta y \rho \frac{(h_{t+\Delta t} - h_t)}{\Delta h} \]

- Divide by \( \Delta x \Delta y \Delta t \) and take limit as \( \Delta x \Delta y \Delta t \to 0 \)

\[-\frac{\partial}{\partial x} (q_x^*) - \frac{\partial}{\partial y} (q_y^*) + N = S_y \frac{\partial h}{\partial t} \]

- From Dupuit assumptions

\[ q_x^* = -k_x h \frac{\partial h}{\partial x} \quad q_y^* = -k_y h \frac{\partial h}{\partial y} \]

\[ \frac{\partial}{\partial x} \left[ k_v h \frac{\partial h}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y h \frac{\partial h}{\partial y} \right] + N = S_y \frac{\partial h}{\partial t} \]

\( \rho \)h causes non-linearity inside brackets

Grounding equation for 2-D unconfined flow
• For homogeneous, isotropic systems
\[
\frac{\partial}{\partial x} \left( h \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( h \frac{\partial h}{\partial y} \right) + \frac{N}{K} = \frac{S_y}{K} \frac{\partial h}{\partial t}
\]
"Boussinesq Eqn"

• 1-D unconfined flow eqn \((\frac{\partial}{\partial y} = 0)\)
\[\frac{\partial}{\partial x} \left[ k_x h \frac{\partial h}{\partial x} \right] + N = S_y \frac{\partial h}{\partial t}\]

• At steady state, \(\frac{\partial}{\partial t} = 0\)
\[\frac{\partial}{\partial x} \left[ k_x h \frac{\partial h}{\partial x} \right] + N = 0\] Steady-state
Uncoupled flow equation

Example Problem

River A
\[\begin{array}{c}
A \\
A'
\end{array}\]

• A-A' Cross-Section

River B

• B-B' Cross-Section

Assume 1-D flow, also homogeneous

\[k \frac{\partial}{\partial x} \left( h(x) \frac{\partial h}{\partial x} \right) + N = S_y \frac{\partial h}{\partial t} \quad (1)\]

Note, the equation is non-linear because \(h\) is a function of \(x \rightarrow h(x)\)

Note, \[\frac{\partial}{\partial x} (h^2) = 2h \frac{\partial h}{\partial x}\] \(\{\) use these
in eqn \(1\)

\[h \frac{\partial h}{\partial x} = \frac{1}{2} \frac{\partial h^2}{\partial x}\]
\[
\frac{1}{2} k \frac{\partial h^2}{\partial x^2} + N = S y \frac{\partial h}{\partial t} \rightarrow \frac{\partial}{\partial x} (h^2) = \frac{-2N}{k} x + C_1
\]

@ steady-state \( \left( \frac{\partial}{\partial t} = 0 \right) \)

\[
\frac{\partial^2}{\partial x^2} (h^2) = -\frac{2N}{k}
\]

solve this...

\[ \rightarrow \text{For our problem} \]

BC's:  
\( @ x = 0 \quad h = h_1 \)
\( @ x = L \quad h = h_2 \)

\[
h^2 = -\frac{N}{k} x^2 + C_1 x + C_2
\]

- Apply BC's:
  - \( h_1^2 = C_2 \quad \Rightarrow \text{BC 1} \)
  - \( h_2^2 = -\frac{NL^2}{k} + C_1 L + h_1^2 \quad \Rightarrow \text{BC 2} \)
  - \( C_1 = \frac{h_2^2 - h_1^2}{L} - \frac{NL}{k} \)

\[
h^2 = -\frac{N}{k} x^2 + \left( \frac{h_2^2 - h_1^2}{L} + \frac{NL}{k} \right) x + h_1^2
\]

\[
h^2 = h_1^2 - \frac{x}{L} \left( h_2^2 - h_1^2 \right) + \frac{NL}{k} (L-x)
\]

If recharge, \( N = 0 \)

\[
h^2 = h_1^2 - \frac{(h_2^2 - h_1^2)x}{L}
\]

\[ \rightarrow \text{We can get the Dufour discharge} \ q_x^* \text{ by differentiating eqn (2) or (3)} \]

(4) Note, \( q_x^* = -k h \frac{\partial h}{\partial x} \)

- From (2)
  - \( 2h \frac{\partial h}{\partial x} = -(h_2^2 - h_1^2) + \frac{NL}{k} - \frac{2N x}{k} \)
\[ h \frac{\partial h}{\partial x} = -\left( \frac{h^2 - h_z^2}{2} \right) + \frac{N}{2k} (L - 2x) \]
\[ h \frac{\partial h}{\partial x} = -\left( \frac{h^2 - h_z^2}{2L} \right) + \frac{N}{k} \left( \frac{L}{2} - x \right) \]

\[ \Rightarrow \text{we know } q_x^* = -k h \frac{\partial h}{\partial x} \text{ (multiply by } -k) \]
\[ q_x^* = k \left[ \frac{h^2 - h_z^2}{2L} \right] - N \left( \frac{L}{2} - x \right) \]

where will \( \max h(x) \)

\[ x = \frac{L}{2} \text{ if } h_1 = h_2 \]

\[ \max h(x) \Rightarrow q_x^* = 0 \]

\[ 0 = k \left[ \frac{h_1^2 - h_2^2}{2L} \right] - N \left( \frac{L}{2} - x \right) \]

Example Problem

1-D Problem
Steady-State
Homogeneous
Isotropic

\[ (h_4 = 94 \text{ m}) \]

a) What is the \( N \) required to keep the lake/wetland wet?
b) Find \( h_{\text{max}} \) between river and lake.
c) Find \( x_{\text{max}} \) between river and lake.
d) What is water level below the wetland if \( N = 0 \)?
\[
\frac{\partial}{\partial x} \left[ kh \frac{\partial h}{\partial x} \right] + N = S_t \frac{\partial h}{\partial t} \to 0
\]

\[
\frac{2}{2x} \left( kh \frac{\partial h}{\partial x} \right) + N = 0 \quad (1)
\]

* Integrate

\[
k h \frac{\partial h}{\partial x} + N_x + C_1 = 0 \quad (2)
\]

\[
\frac{K h^2}{2} + \frac{N_x^2}{2} + C_1 x + C_2 = 0 \quad (3)
\]

* Apply BC's (problem specific)

12. Why is there no 3-D flow equation for unconfined aquifers?

By Definition: 2-D problems \( \Rightarrow \) depth-averaged

![Diagram showing ground surface and bedrock as boundary conditions]

Rather than blindly plugging in values to formulas, we will derive the solution for each problem from the governing equation:

* 1-D, Unconfined, Steady-State

\[
\frac{\partial}{\partial x} \left[ kh \frac{\partial h}{\partial x} \right] + N = S_t \frac{\partial h}{\partial t} \to 0
\]

\[
- \left[ kh \frac{\partial h}{\partial x} \right] + N = 0
\]

\[
\frac{d}{dx} (h^2) = 2h \frac{dh}{dx}
\]

\[
h \frac{dh}{dx} = \frac{1}{2} \frac{d}{dx} h^2
\]

\[
K \frac{d}{dx} \left[ \frac{1}{2} \frac{d}{dx} h^2 \right] + N = 0
\]

\[
K \frac{d^2 h^2}{dx^2} + N = 0
\]

\[
\frac{cl^2(h^2)}{dx^2} = - \frac{2N}{K}
\]

Unconfined steady-state uniform \( k \) \( \to \) Starting equation
\[
\frac{d}{dx} \left( h^2 \right) = -\frac{2N}{k} \quad \text{and} \quad h^2 = -\frac{2N}{k} \quad \text{knowns}
\]

\[
\frac{d}{dx} \left( h^2 \right) = -\frac{2N}{k} \quad x + C_1
\]

\[
h^2 = -\frac{N}{k} \quad x^2 + C_1 \quad x + C_2
\]

\[
h^2 = -\frac{N}{k} \quad x^2 + C_1 \quad x + C_2
\]

\[
c_1 \quad \text{and} \quad c_2 \quad \text{determined from boundary conditions}
\]

Problem Solving Steps

1) Draw a nice diagram
2) Write governing equation \( \frac{d^2 h}{dx^2} = -\frac{2N}{k} \)
3) Simplify
4) Identify BCs and other known conditions
5) Identify unknowns
   - Check unknowns = knowns
6) Derive the solution
7) Plug in numbers

\[
Q^+ = \int \frac{dh}{dx}
\]

Homework 6

\#1 \[
\frac{d^2 (h^2)}{dx^2} = -\frac{2N}{k} \quad \text{-- (1)}
\]

\[
\frac{d}{dx} (h^2) = -\frac{2N}{k} \quad x + C_1 \quad \text{-- (2)}
\]

\[
h^2 = -\frac{N}{k} \quad x^2 + C_1 \quad x + C_2 \quad \text{-- (3)}
\]

<table>
<thead>
<tr>
<th>Unknowns</th>
<th>Knowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1, c_2, N )</td>
<td>( 2 \text{ BCs} )</td>
</tr>
<tr>
<td>@ ( x = 700 ), ( h = 94 )</td>
<td></td>
</tr>
<tr>
<td>@ ( x = 0 ), ( h = 100 \text{ m} )</td>
<td>( \text{-- (4)} )</td>
</tr>
<tr>
<td>@ ( x = L ), ( h = 80 \text{ m} )</td>
<td>( \text{-- (5)} )</td>
</tr>
<tr>
<td>@ ( x = 700 ), ( h = 94 \text{ m} )</td>
<td>( \text{-- (6)} )</td>
</tr>
</tbody>
</table>

\[
N + A = \text{total flow in system}
\]

\[
\% \text{ goes to river } A
\]

\[
\% \text{ goes to river } B
\]

\[
N + A \quad \text{flow balance}
\]

Only \( c_1, c_2 \)

\( k \), and \( N \)

can be unknown

Can you use well as a BC? not for this problem

river level (BC) won't change but well level will

BC's head will never change
• plug (4) into (3)
  \[100^2 = C_2\]
  \[C_2 = 10,000 \quad \text{— (7)}\]

• plug (5) into (3)
  \[80^2 = \frac{-N}{0.5} (1000)^2 + 1000C_1 + 10,000 \quad \text{— (8)}\]

• plug (6) into (3)
  \[94^2 = \frac{-N}{0.5} (700)^2 + 700C_1 + 10,000 \quad \text{— (9)}\]

Solve (8) and (9) \(\Rightarrow C_1 = ? \quad N = ?\)

\(\text{\(\Rightarrow\)} \text{To find flows?}\)

\[
\frac{dh}{dx} = -\frac{2N}{K}x + C_1 \quad \text{— (2)}
\]

\[k2h \frac{dh}{dx} = -2Nx + KC_1\]

\[-kh \frac{dh}{dx} = Nx - \frac{k}{2}C_1\]

\[
q^* = Nx - \frac{k}{2}C_1
\]

\[
q^* \text{ at } x = 0 \quad \text{multiply by} \quad 2000 \text{ m to get flow}
\]

\[
q^* \text{ at } x = 1000
\]

Answers to check:

\[
\begin{align*}
Q @ x = 1000 & = 5030 \text{ m}^3/\text{day} \\
Q @ x = 0 & = -1430 \text{ m}^3/\text{day} \\
\text{flow to the left}
\end{align*}
\]
3-D Saturated Flow | 2-D Confined | 2-D Unconfined
\[ \begin{align*}
q^* &= kB \\
q_r &= -k \frac{\Delta h}{\Delta r} \\
q_r^* &= -kB \frac{\Delta h}{\Delta r}
\end{align*} \]

Radial Flow (Depth-averaged, confined flow)

\[ S \rightarrow \left[ \frac{m^3}{m^3 - m} \right] \]

\[ q_r = \frac{2m \pi r B \rho \Delta \phi}{r} \]

\[ q_r^* = \frac{2m \pi r B \rho \Delta \phi}{r} \]

In - Out = Accumulation

\[ \text{Inflow} = q_r 2\pi r B \rho \Delta \phi | r \]

\[ \text{Outflow} = q_r 2\pi r B \rho \Delta \phi | r + \Delta r \]

Accumulation = \text{Area} \cdot \rho \cdot \Delta \phi = \rho S \pi \Delta r [\Delta r + 2r] \Delta \phi

\[ \begin{align*}
\pi \left( r + \Delta r \right)^2 - \pi r^2 \\
\pi \left[ (r + \Delta r)^2 - r^2 \right] \\
\pi \left[ r^2 + 2r \Delta r + 2r \Delta r - r^2 \right] \\
\pi \Delta r \left[ 2r + \Delta r \right]
\end{align*} \]

\[ \frac{\text{In} - \text{Out}}{\rho \Delta \phi} \quad \text{divide by } \rho \]

\[ \begin{align*}
\frac{d}{dt} \left( \frac{\Delta h}{\Delta r} \right) &= \frac{\Delta h}{\Delta r} \\
\frac{\Delta h}{\Delta r} &= \frac{\Delta h}{\Delta r}
\end{align*} \]
Flow Nets

Problem Solving Steps
1) Draw a diagram - identify BC's and other conditions
2) Write governing equations
3) Solve
4) Plug-in numbers

Homogeneous, Isotropic

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \]

2D Laplace equation

Homework 7 - Problem 1

\[ \frac{\partial^2 (h^2)}{\partial x^2} = -\frac{2N}{K} \]  
(uncounfined flow)

<table>
<thead>
<tr>
<th>Unknown</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( c_1, c_2, L )</td>
<td>( x=0, h=2.9 \text{ m} ) (BC1)</td>
</tr>
<tr>
<td></td>
<td>( x=L, h=2.9 \text{ m} ) (BC2)</td>
</tr>
<tr>
<td></td>
<td>( h_{\text{max}} = 4 \text{ m} ) (internal)</td>
</tr>
</tbody>
</table>

Answer:
\[ L = 50 - 100 \text{ m} \]

Homework 7 - Problem 2

\[ Q_v = -K_{\text{shale}} \left( \frac{h_2 - h_1}{\Delta z} \right) \]
\[ Q_v = 3.5 \times 10^{-6} \left( \frac{0 - (1000 + 50)}{1000} \right) \approx -1 \]

Answer:
\[ Q_h = -K_{\text{shale}} \frac{dh}{dz} \]
Homework 7 - Problem 4

M recharge rate
Plan from aquifer into stream

\[ N = 1 \times 10^{-3} \frac{m^3}{m^2 \cdot s} \]

<table>
<thead>
<tr>
<th>Unknown</th>
<th>known</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1, c_2, N )</td>
<td>( @x=0, h=2m )</td>
</tr>
<tr>
<td></td>
<td>( @x=100, h=3m )</td>
</tr>
<tr>
<td></td>
<td>( @x=925, q^* = 0 \Rightarrow kh \Delta x = 0 )</td>
</tr>
</tbody>
</table>

\( \Rightarrow \) No Problem 3 on HW 7

Groundwater Wells
- allows to pump water out
- serve as monitoring points

1) Monitoring well
    - Slotted PVC pipe w/ screening @ bottom

2) Pumping well
    - Concrete
    - Fine sand
    - Silt & clay
    - Coarse sand & gravel
    - Sandstone

\[ S = \phi - \phi_c \text{ "drawdown"} \]

\[ \text{must balance recharge} \]
\[ \text{and pumping} \]

\( \Rightarrow \) when cone stops expanding, steady - state!
Monitoring Bore Design

- End Cap
- Lockable Steel Protective Cap
- Concrete Block
- Ground Surface
- Blank UPVC Pipe to surface
- Backfill
- Bentonite Seal
- Machine Slotted UPVC Pipe
  - 50 mm nominal diameter
  - Class 9 - 0 to 60 m
  - Class 12 - 60 to 120 m
- Graded Filter Pack
  - eg. 0.8 - 1.6 mm
  - 1.6 - 3.2 mm
  - 3.2 - 6.4 mm
- Base of Hole
and the kelly is allowed to slide down through the rotary table as the hole is deepened. When the full length of the kelly has slid through the rotary table in this way, the drilling string is pulled up by an amount equal to one length of drill pipe and the kelly is unscrewed. An extra length of drill pipe is then added before the kelly is reconnected, the drilling string being held meanwhile by steel wedges called 'slips' which fit into the rotary table. The drilling fluid is pumped down the kelly through a hose and a swivel arrangement which permits the mud to enter the kelly while the kelly is rotating.

Rotary rigs range in size and complexity from units capable of drilling holes about 100 m deep, which are usually mounted on tractors or lorry trailers, to the giant machines with masts about 60 m or 70 m high which are used for drilling oil and gas wells. Tall masts permit the use of long individual lengths of drill pipe, thus reducing the time spent in adding or removing lengths. They similarly speed up the process of installing casing. Any rig, whether rotary or percussion, must have the lifting ability to handle the long and heavy total lengths of drill pipe or casing which may have to be lowered into, or pulled from, the well at total depth; this lifting ability is usually what decides the depth of well that can be drilled by a particular rig.

Variations on the basic rotary method include the top-drive rig, in which the rotary power is transmitted directly to the top of the drill pipe by a hydraulic turbine which is lowered down the mast by the draw works as drilling progresses. Another variation uses reverse circulation, in which the drilling fluid (which in this case is usually water) is pumped down the annulus and returns up the drill pipe, which is of large diameter. This method is especially suitable for drilling large-diameter wells in gravels, as quite large cobbles can be brought up the drill pipe without being broken up.

Some rotary rigs use compressed air as the drilling fluid. Another use of compressed air is the method known as downhole hammer drilling. For this a small rotary rig is used, but the drilling action is essentially percussive. The drill pipe carries a tool rather like a roadworker's pneumatic drill; compressed air pumped down the pipe operates the cutting tool and carries the cuttings to the surface. The technique is especially suited to hard formations such as basalts, like the air-flush rotary method, its disadvantage is that most compressors are unable to provide air at sufficient pressure to operate the downhole tool and overcome a large head of water. It is therefore not possible to drill to any great depth below the water table with this method, unless the permeability is so low that the air-lift action of the returning air is able to keep the hole effectively pumped dry.

A relatively recent innovation is the downhole motor. In this drilling method the drill pipe remains stationary, serving merely to convey drilling fluid to a turbine which sits at the bottom of the drilling string, coupled
d, particularly where it is saturated with water, keeps collapsing into.

Above the water table in a well, this problem can be overcome. Shafts can be lined with masonry, with pre-cast concrete rings, or with concrete poured in place. Boreholes are usually supported by inserting lengths of pipe, called casing or lining tubes, of diameter slightly smaller than the drilled diameter of the borehole. The space between the outside of the casing and the borehole wall is usually filled with a thin concrete called cement grout. In addition to providing support, this prevents dirty or polluted surface water or soil water from running into the well and the casing after heavy rain; wells in consolidated rocks are usually given this sanitary protection for a few metres below the surface.

Below the water table the insertion of unperforated linings would support the surrounding aquifer but would also prevent the entry of water - not a desirable state of affairs! The well-sinkers who dug some of Britain's older wells got around this problem neatly, if laboriously, by constructing a lining of stones without mortar, like a dry-stone wall. The technique used in modern boreholes is to insert a lining tube that is perforated in such a way as to permit groundwater to flow from the aquifer into the well and at the same time to prevent aquifer particles such as sand grains from entering the well.

The choice of lining will depend on the aquifer. If the aquifer is consolidated but fractured, requiring support to prevent large blocks of rock from falling into the well, then casing with large circular perforations or crude slots will be adequate. If on the other hand the aquifer is loose sand, then a special lining called a screen is used. Screens have fine slots, and the appropriate slot size must be specified to suit the particular aquifer. Samples of the aquifer are passed through a series of sieves to determine the sizes of the sand particles, and the screen is chosen so that the slots are as large as possible (to permit the water to flow through the screen with minimum head loss) while being small enough to prevent the aquifer particles from moving into the well. Some of the finest particles will enter the well when it is first pumped - a process called development - but thereafter the well should produce sand-free water.

It sometimes happens that the aquifer is composed of particles which are so fine that to exclude them the screen slots would have to be impractically narrow. In such cases coarser slots are used, and an envelope of coarse sand or gravel, called a filter pack, is poured into the annular space between the aquifer and the screen. The filter pack retains the aquifer particles outside its outer boundary, and the screen slot size is chosen to prevent the filter pack itself from passing through the slots. A filter pack theoretically needs to be only a few grains thick to function effectively, but it is generally impossible to place sand or gravel in an annulus less than about 60 mm to 80 mm wide. Figure 9.4 shows a theoretical design for a well that draws water from three aquifers - a fine sand that requires a filter pack, a coarse sand, and a sandstone that does not need support. Development of the well causes removal of the finer particles from the coarse-sand aquifer, leading to the formation of a coarser zone - sometimes called a natural filter pack - immediately around the screen.
## Lithology

<table>
<thead>
<tr>
<th>Depth (m)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>Sandstone, brown, moderately sorted, fine grained, subrounded to rounded, minor clay.</td>
</tr>
<tr>
<td>10-20</td>
<td>Sandstone, white, soft, medium grained.</td>
</tr>
<tr>
<td>20-30</td>
<td>Sandstone, brown, moderately sorted, fine grained, subrounded to rounded, minor clay.</td>
</tr>
<tr>
<td>30-40</td>
<td>Sandstone, white to grey, poorly sorted, fine grained, subrounded to rounded, minor thin clay beds.</td>
</tr>
<tr>
<td>40-50</td>
<td>Sandstone, brown, moderately sorted, medium grained.</td>
</tr>
<tr>
<td>50-60</td>
<td>Sandstone, white to grey, predominantly fine grained, subrounded to rounded, minor red-brown and green-white clay.</td>
</tr>
<tr>
<td>60-70</td>
<td>Sandstone, white to grey, moderately sorted, predominantly fine grained, subrounded to rounded, minor shale.</td>
</tr>
</tbody>
</table>

## Geophysical Logs

<table>
<thead>
<tr>
<th>Depth (m)</th>
<th>Gamma Counts per Second</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>0-20</td>
</tr>
<tr>
<td>10-20</td>
<td>20-30</td>
</tr>
<tr>
<td>20-30</td>
<td>30-40</td>
</tr>
<tr>
<td>30-40</td>
<td>40-50</td>
</tr>
<tr>
<td>50-60</td>
<td>60-70</td>
</tr>
<tr>
<td>70-80</td>
<td>80-90</td>
</tr>
<tr>
<td>90-100</td>
<td>100-110</td>
</tr>
<tr>
<td>110-120</td>
<td>120-130</td>
</tr>
</tbody>
</table>

## Hydrology

<table>
<thead>
<tr>
<th>Depth (m)</th>
<th>T.D.S. mg/l</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>0-20</td>
</tr>
<tr>
<td>10-20</td>
<td>20-30</td>
</tr>
<tr>
<td>20-30</td>
<td>30-40</td>
</tr>
<tr>
<td>30-40</td>
<td>40-50</td>
</tr>
<tr>
<td>50-60</td>
<td>60-70</td>
</tr>
<tr>
<td>70-80</td>
<td>80-90</td>
</tr>
</tbody>
</table>

## State Energy Commission of W.A. Shire of Upper Gascoyne

**Dampier to Perth Natural Gas Pipeline Project**

**Access Road Watering Water Supply Bore**

## Composite Bore Log

**Bore UG5**

(Sheet 1 of 2)

### Bore Data
- **Location:** 1km S Pinbee Homestead on A.P. Road.
- **Reduced Level of Collar:** Not determined.
- **Static Water Level below Collar:** 59-13m.
- **Depth Drilled:** 160m.
- **Date Completed:** 27/4/82.
- **Status:** Production here, eased & developed.
- **Height of collar rel. ground:** 0-60m.

### Geophysical Data
- **Date Run:** 20/4/82.
- **Logger:** S.I.E. T450 E.
- **Casing:** 0-5-4m, 225mm I.D. steel.
- **Hole Size:** 222mm.
- **Fluid Type:** Mud.
- **T.C.:** 2 seconds.
- **Recorded by:** P Horroch.

### Geological Reference
- **Sand.**
- **Silt.**
- **Clay.**
- **Gravel.**
- **Radiolarite.**

*Rockwater*  
*November 0*  
*511C/82/3-3*
Wells

→ Steady-state well flows - confined

<table>
<thead>
<tr>
<th>Monitoring well #1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pumped well</td>
</tr>
<tr>
<td>Static water level</td>
</tr>
</tbody>
</table>

\[ Q_r = +K \frac{\partial \phi}{\partial r} \]

\[ Q = Q_r A \]

\[ Q = 2\pi r B Q_r \]

\[ Q = 2\pi r B K \frac{\partial \phi}{\partial r} \frac{dr}{T} \]

\[ Q = 2\pi r T \frac{dr}{dr} \rightarrow d\phi = \frac{Q}{2\pi T} \frac{dr}{r} \]

\[ \int_{\phi_1}^{\phi_2} d\phi = \frac{Q}{2\pi T} \int_{r_1}^{r_2} \frac{dr}{r} \]

\[ \phi_2 - \phi_1 = \frac{Q}{2\pi T} \ln \left( \frac{r_2}{r_1} \right) \]

\[ T = \frac{Q}{2\pi(\phi_2 - \phi_1)} \ln \left( \frac{r_2}{r_1} \right) \]

Conventions

→ Standard practice:
  - \( Q \) → pump out
  + \( Q \) → pump in

→ Book:
  + \( Q \) → out
  - \( Q \) → in

\[ Q_r = +K \frac{\partial \phi}{\partial r} \]

(We will use book notation)

From Darcy's law
From Darcy's Law

\[ Q_r = \pi h \left( \frac{d h}{d r} \right) \]

\[ Q = Q_r A \]

\[ Q = 2\pi r h Q_r \]

\[ Q = 2\pi r h k \frac{dh}{dr} \]

\[ \int_{h_i}^{h_2} h \, dh = \frac{Q}{2\pi K} \int_{r_i}^{r_2} \frac{dr}{r} \]

\[ h_2^2 - h_i^2 = \frac{Q}{\pi K} \ln \left( \frac{r_2}{r_i} \right) \]

\[ K = \frac{Q}{\pi \left( h_2^2 - h_i^2 \right) \ln \left( \frac{r_2}{r_i} \right)} \]

\[ K = \frac{Q}{2\pi (h_2 - h_i) \ln \left( \frac{r_2}{r_i} \right)} \]

\[ K = \frac{Q}{2\pi \text{avg} \left( h_2 - h_i \right) \ln \left( \frac{r_2}{r_i} \right)} \]

Equivalent for unconfined flow

\[ \text{compare w/ confined flow eqn} \]
Unsteady-state well flows - confined (transient)

Governing eqn:\n\[
\frac{2}{r^2} \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{S}{T} \frac{\partial \phi}{\partial t}
\]

"Solve" this \( \phi (r, t) \)

- Write governing eqn in terms of drawdown:

\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} = \frac{S}{T} \frac{\partial s}{\partial t}
\]

- Initial Condition
  \( @ t = 0 \quad s = 0 \)

- Boundary Conditions
  1) \( r \to \infty \quad s = 0 \)
  2) @ well: \( t \leq 0 \quad Q_w = 0 \)
  \( t > 0 \quad Q_w = \) constant

Define \( W(u) \) well function as:

\[
W(u) = \int_u^\infty \frac{e^{-x}}{x} \, dx
\]

\[
s(r, t) = s(u) = \frac{Q}{4\pi T} W(u)
\]

where \( u = \frac{r^2 S}{4T t} \)
Series approximation to \( W(u) \)
\[
W(u) = \left[ -0.577216 + \ln u + u - \frac{u^3}{2.2!} + \frac{u^3}{3.3!} - \ldots \right]
\]

How to calculate \( W(u) \)?

1) Program in Excel/VB for \( \approx 10 \) terms
2) page 535 \( \rightarrow \) Appendix 1 (linear interpolation between values)
3) page 170 \( \rightarrow \) Figure 5.6

* Typical problem:

Compute drawdown at \( r = 10 \) m
\( t = 100 \) min
\( T = 100 \) m\(^3\)/day
\( S = 0.001 \)

→ Example

Find \( s : \)
\( r = 824 \) ft
\( t = 100 \) min = 0.0644 day
\( T = 1400 \) ft\(^3\)/day
\( S = 2.4 \times 10^{-5} \)
\( Q = 42,000 \) ft\(^3\)/day

\[
U = \frac{r^2S}{41Tt} = \frac{(824)^2(2.4 \times 10^{-5})}{4(1400)(0.0644)} = 0.0419 \approx 4.2 \times 10^{-2}
\]

\( W(u) = 2.68 \)

\[
S(824, 100 \text{ min}) = \frac{Q}{41\pi T} W(u) = \frac{42,000}{41\pi (1400)} 2.68 = \boxed{6.40 \text{ ft}}
\]

For any \( r \)
\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{S}{T} \frac{\partial \phi}{\partial t} \quad \text{(in terms of head)}
\]
\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} = \frac{S}{T} \frac{\partial s}{\partial t} \quad \text{(in terms of drawdown)}
\]

I.C. \quad t = 0, \quad s = 0
B.C. \quad \text{pumping at the well}, \quad r \to \infty, \quad s = 0

\[
S(r, t) = \frac{Q}{4\pi T} \left[ \ln 0.56147 - \ln u \right]
\]

If \( u \ll 0.01 \), then we can approximate

\[
W(u) = -0.577216 - \ln u
\]

Copper-Jacob Approximation: \( u = \frac{r^2 S}{4\pi T} \)

\( s \) is fixed

\( \uparrow t, \) approach steady-state

\( u \ll 0.01 \)

First applicable close to well (relatively quickly)

\[ s(r, t) = \frac{Q}{4\pi T} \ln \left[ \frac{0.56147}{u} \right] \]

\[ s(r, t) = \frac{Q}{4\pi T} \ln \left[ \frac{0.56147}{u} \right] \]

\[ s(r, t) = \frac{Q}{4\pi T} \ln \left[ \frac{0.56147}{u} \right] \]

\[ s(r, t) = \frac{Q}{4\pi T} \ln \left[ \frac{2.25 T t}{r^2 S} \right] \]

Convert \( \ln \to \log \)

\[ s(r, t) = \frac{2.3 Q}{4\pi T} \log \left( \frac{2.25 T t}{r^2 S} \right) \]

\( S(r, t) \) must use Theis solution \( W(u) \) computation
**Pump Tests** - field based methods to find \( T \) and \( S \) \( k \) and \( S \)

_Theis Eqn_

\[
S(r, t) = \frac{Q}{4\pi T} W(u) \quad U = \frac{r^2 S}{4Tt}
\]

\[
W(u) = \left[ -0.57716 - \ln u + u - \frac{u^2}{2!} - \frac{u^3}{3!} - \ldots \right]
\]

- How to find \( T \) and \( S \) based on field data?

  → trial and error! change \( T \) and \( S \) to fit curve (Solver)

1) _Theis Test Method_ ( _Theis Type Curve Matching Method_)

\[
S = \frac{Q}{4\pi T} W(u) \quad U = \frac{r^2 S}{4Tt}
\]

Fit both equations to real data

**Fitting procedure**

1. Plot \( S \) versus \( t \) on log-log paper
2. Do curve matching w/ a type curve (transparency)
   - Can only move the type curve
   ← → ↑↓ (cannot rotate!)
3. Select a match point
   - Any point is okay, but it is easy to work with \( W(u) = 1 \) and \( \frac{1}{u} = 1 \) on type curve
4. Find \( S \) and \( t \) at the match point
5. Convert everything to proper units
6. Find \( T \) and \( S \)

\[
T = \frac{Q}{4\pi S_{\text{match point}}} W(u)_{\text{match point}}
\]

\[
S = \frac{4T U_{\text{match point}}}{r^2} \quad t_{\text{match point}}
\]
Semi-log graph showing data with annotations:

- $t_0 = 5.2$ min
- $C_0 = 100$
- $C_{final} = 1$
- $t_f = 1000$ min
- $A = 5.5$ ft

Graph notes:
- Darcy's Law
- Drawdown
- 5 feet (one log cycle)
2) Jacob Straight line method

1. plot S vs. t on semi-log paper
2. fit straight line portion of data w/ straight line
3. find \( \Delta s \) over one log cycle
4. find \( T \) and \( S \)

\[
T = \frac{2.3 Q}{4\pi \Delta s} \\
S = \frac{2.25 T t_0}{r^2}
\]

→ How does this method work?

\[
S(r, t) = \frac{2.3 Q}{4\pi T} \log \left( \frac{2.25 T t_0}{r^2} \right) + \log (B, t)
\]

\[
S(r, t) = A_1 \log (B, t)
\]

\[
S(r, t) = A_1 \log t + A_1 \log B_1 
\]

(y = mx + c)

- \( m = A_1 \)
- \( x = \log t \)
- \( c = A_1 \log B_1 \)

To find slope, apply (1) at two points

\[
s_1(t_1) = A_1 \log (t_1) + A_1 \log B_1 
\]

\[
s_2(t_2) = A_1 \log (t_2) + A_1 \log B_1
\]

Take 1 log cycle

\[
\log (t_2) - \log (t_1) = 1
\]

\[
S_2 - S_1 = A_1 \\
\Delta s = \frac{2.3 Q}{4\pi T} \quad \Rightarrow \quad T = \frac{2.3 Q}{4\pi \Delta s}
\]

when \( s = 0 \), (1) becomes

\[
0 = A_1 \log (t_0) + A_1 \log B_1
\]

\[
\log (t_0) = \log \left( \frac{1}{A_1} \right) \quad \Rightarrow \quad t_0 = \frac{r^2 S}{2.25 \pi} \quad \Rightarrow \quad S = \frac{2.25 T t_0}{r^2}
\]

Excel plot S vs. log t
**Well Flows**

\[ s(r, t) = \frac{Q}{4\pi T} W(u) \]

\[ u = \frac{r^2 s}{4\pi T} \]

---

*How to predict \( s(t) \) for multiple wells?*

\[ S_1(r_1, t) = \frac{Q_1}{4\pi T} W(u_1) \]

\[ S_2(r_2, t) = \frac{Q_2}{4\pi T} W(u_2) \]

\[ S_{\text{total}}(r, t) = ? \]

**B.C.** System is linear, net drawdown is summation of drawdown due to each well.

---

*How to model presence of river as a B.C.?* \( \Phi = \Phi_{\text{river}} \)

\[ s(r, t) = \frac{Q}{4\pi T} W(u) \]

---

*Analytical/Superposition Solution*

\[ S_1 \Rightarrow \text{real well } (r_1, u_1) \]

\[ S_2 \Rightarrow \text{hypothetical well } (r_2, u_2) \]

\[ S_{\text{total}} = S_1 + S_2 \]

(goes rid of river)
Groundwater and Well Hydraulics

Thus,

\[ h_{\text{obs}} = \frac{550 \text{ m}^3/\text{d}}{4\pi(300 \text{ m}^3/\text{d})} (12.43) \]

Well 1 causes 1.81 m of drawdown at the observation well, while well 2 causes 3.46 m, for a total drawdown of 5.27 m. Under these circumstances, we say that the drawdown cones of the wells overlap, or that the wells interfere with each other.

**Multiple Pumping Rates**

For a problem with multiple pumping rates, the principle of superposition says:

The drawdown due to a changing pumping rate is equal to the sum of the drawdown values due to the "incremental changes" in rates.

The following example will help explain what incremental changes and incremental duration mean.

**EXAMPLE 3.3.5**  **Superposition with Multiple Pumping Rates**

A well with an effective radius of 0.3 m produces from an aquifer with \( T = 300 \text{ m}^3/\text{d} \) and \( S = 0.005 \). A well pumps water at a rate of 150 m\(^3\)/d for 30 days, and then the rate is decreased to 50 m\(^3\)/d for an additional 30 days. The well is then shut off. Calculate the residual drawdown after 80 days from the start of pumping (20 days after the well was shut off).

The table below shows what is meant by incremental changes. Each change in rate is implemented by a new well function with the incremental discharge added to the existing set of well functions. The solution for this problem is represented as the result of three wells pumping at different rates and for different durations at the same location. The rule is that "Once a well function has been turned on, it cannot be turned off."

<table>
<thead>
<tr>
<th>Pumping Schedule for Prediction of Drawdown at 80 Days</th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time [days]</td>
<td>0-30</td>
<td>31-60</td>
<td>61-80</td>
</tr>
<tr>
<td>( Q ) [m(^3)/d]</td>
<td>150</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>( AQ ) [m(^3)/d]</td>
<td>550</td>
<td>550</td>
<td>550</td>
</tr>
<tr>
<td>Duration [days]</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>
Chapter Three

For the incremental rates and durations we have

\[ l_1 = \frac{(0.3 \text{ m})^2 \times 0.0006}{4 \times 300 \text{ m}^2/\text{d} \times 30 \text{ d}} = 5.6 \times 10^{-6} \text{ m} \]

\[ l_2 = \frac{(0.2 \text{ m})^2 \times 0.0006}{4 \times 300 \text{ m}^2/\text{d} \times 20 \text{ d}} = 2.2 \times 10^{-4} \text{ m} \]

and the residual drawdown is

\[ z_{\text{rel}} = \frac{250 \text{ m}^2/\text{d}}{4 \times 300 \text{ m}^2/\text{d} \times (0.561 \times 5.6 \times 10^{-6})} = 2.2 \times 10^{-6} \text{ m} \]

Thus, a drawdown of 26 cm remains 20 days after the well has stopped pumping.

3.3.6 Image Well Theory

Image well theory is an application of the principle of superposition that allows us to represent simple types of boundaries for an aquifer. The theory is also applicable for transient problems. For example, if a production well is adjacent to a constant head boundary, the drawdown is given by

\[ z_{\text{rel}} = \frac{Q}{4\pi T} \left( \frac{2.257}{r_1^5} - W(u) \right) \]

(3.3.14)

where \( u \) and \( u_* \) represent the real well and its image well, which is located opposite to the recharge boundary. As time increases, the Jacob approximation becomes valid for the real well \( u_* < 0.01 \), and

\[ z_{\text{rel}} = \frac{Q}{4\pi T} \left( \ln \left( \frac{2.257}{r_1^5} \right) - W(u) \right) \]

Ultimately, the Jacob approximation becomes valid for the image well, and

\[ z_{\text{rel}} = \frac{Q}{4\pi T} \left( \ln \left( \frac{2.257}{r_2^5} \right) - \frac{Q}{2\pi T} \ln \left( \frac{r_1}{r_2} \right) \right) \]

This result is the same result found in Section 3.2.4 for steady flow conditions near a recharge boundary. Thus, by the time the Jacob approximation becomes valid for both the real and image wells, the drawdown has reached its steady value. Near the well, this occurs when \( u_* < 0.01 \), or

3.4 Pumping Tests

The section and those that follow provide a more detailed examination of in situ methods for estimating aquifer and well characteristics. One usually distinguishes two different types of pumping tests: Aquifer tests are used to determine field-scale hydraulic characteristics of the aquifer, particularly its transmissivity and storativity coefficient. Well tests are used to determine yield and well loss information for a particular well. Aquifer tests generally require an observation well or a piezometer in which the water level response to an induced aquifer stress (pumping the well) is recorded, although the aquifer transmissivity can be determined accurately with only pumped well data. Well tests require only pumped well data and are discussed later in this chapter.

3.4.1 Aquifer Tests

Estimation of aquifer characteristics through analysis of aquifer test data is a standard practice in the evaluation of groundwater resources. There is much art to successful aquifer testing analysis interpretation, and an excellent reference is Kruisema and de Ridder (1984). In addition, Walton (1970, 1977) presents various case histories. The basic setup for an aquifer test is to pump one well with a known discharge and measure the drawdown in nearby observation wells that are generally located at distances from 10 m to 100 m from the pumped well. Figure 3.4.1 shows a possible well configuration, in addition to the type of data one typically obtains.

**FIGURE 3.4.1 Aquifer test data**
The method of images applied to a well near a recharge boundary.

Figure 8.25. The method of images applied to a well near an impervious boundary.
How to model presence of rock as a B.C.

\[ \frac{\partial s}{\partial x} = 0 \]

 HW & Prob. 2 (Extra)

Superposition in time

"Once a well function has been turned on, it cannot be turned off"
"Slug" Tests - used for contaminated sites

"bailout" - draws up contaminated water just like a pump test would. BAD!

Hvorslev Method

\[ k = \frac{r_c^2 \ln \left( \frac{L_e}{R} \right)}{2 L_e T_0} \]

Derivation
- Rate at which water storage changes in well bore hole
  \[ Q_b(t) = \pi r_c^2 \frac{dh}{dt} \]
- Rate at which aquifer discharges \( \propto K \) and \( \propto h \)
  \[ Q_a(t) = -Fkh \]

Mass Balance \( Q_b(t) = Q_a(t) \)
- Rate at which water storage decreases with time
  \[ \frac{dh}{dt} = -\frac{Fkh}{r_c^2} \]
\[
\int \frac{dh}{h} = -\frac{F k h}{\pi r_c^2} \quad \text{--- (1)}
\]

\[\ln(h) - \ln(h_0) = -\frac{F k}{\pi r_c^2} t \quad \text{--- (1)}\]

\[\Rightarrow \text{When } \frac{Le}{R} > 8, \text{ Hvorslev did some field work and evaluated } F \]

\[F = \frac{2\pi Le}{\ln(\frac{Le}{R})} \quad \text{empirical formula based on field data} \quad \text{--- (2)}\]

\[\Rightarrow \text{Plug (2) into (1)} \]

\[\ln\left(\frac{h}{h_0}\right) = -\frac{2\pi Le k}{\ln(\frac{Le}{R}) r_c^2} t \quad \text{--- (3)}\]

\[\ln\left(\frac{h_0}{h}\right) = \frac{2Le k}{\ln(\frac{Le}{R}) r_c^2} t\]

\[k = \frac{\ln\left(\frac{Le}{R}\right) r_c^2}{2Le t} \ln\left(\frac{h_0}{h}\right)\]

\[\Rightarrow \text{Define } T_0 \text{ (time lag) as time at which } h(t) = 0.37 h_0 \]

\[\ln\left(\frac{h_0}{h}\right) = \ln\left(\frac{h_0}{0.37 h_0}\right) = \ln\left(\frac{1}{0.37}\right) = \ln(2.7) = 1\]

\[K = \frac{\ln\left(\frac{Le}{R}\right) r_c^2}{2Le T_0}\]

\[T_0 = t @ h(t) = 0.37\]

\[\text{Bower and Rice Method}\]

\[\text{Diagram showing leaching and log reduction over time}\]
Bower and Rice Formula:

\[ K = \frac{r_c^2 \ln \left( \frac{R_e}{R} \right)}{2L_e} \left[ \frac{\ln \left( H_1 \right) - \ln \left( H_2 \right)}{t_2 - t_1} \right] \]

Special cases for analysis:

1) \( t_1 = 0 \) \( H_1 = H_0 \)
   \( t_2 = t \) \( H_2 = H(t) \)

\[ K = \frac{r_c^2 \ln \left( \frac{R_e}{R} \right)}{2L_e} \left[ \frac{\ln \left( H_0 \right) - \ln \left( H(t) \right)}{t - 0} \right] \]

\[ K = \frac{r_c^2 \ln \left( \frac{R_e}{R} \right)}{2L_e} \frac{1}{t} \ln \left( \frac{H_0}{H(t)} \right) \]

--- Textbook 5.91

2) Assume 1 log cycle

\[ \ln \left( \frac{H_1}{H_2} \right) = 2.3 \]

(example: \( \ln \left( \frac{0.1}{0.01} \right) = \ln(10) = 2.3 \))

\[ K = \frac{r_c^2 \ln \left( \frac{R_e}{R} \right)}{2L_e} \frac{2.3}{\Delta t} \]

\[ \ln \left( \frac{R_e}{R} \right) = \left[ \frac{1.1}{\ln \left( \frac{L_0}{R} \right)} + \frac{(A+B) \ln \left( \frac{H+L_0}{R} \right)}{L_e} \right]^{-1} \]

--- 5.92

If \( L_0 = H \) (fully screened well)

\[ \ln \left( \frac{R_e}{R} \right) = \left[ \frac{1.1}{\ln \left( \frac{L_0}{R} \right)} + \frac{C}{L_e} \right]^{-1} \]

--- 5.93

- Partially screened wells (partially penetrating)
- Fully screened wells (fully penetrating)

Slug tests are local.

Pump tests cover large area.
Theim Eqn
\[ T = \frac{Q}{k b} = \frac{Q}{2\pi L e} \ln \left( \frac{r_2}{r_i} \right) \]

\[ k = \frac{Q}{2\pi L e} \ln \left( \frac{r_2}{r_i} \right) \]

\[ Q = \frac{2\pi L e k (s_1 - s_2)}{\ln \left( \frac{r_2}{r_i} \right)} \]

\[ s_2 = \text{drawdown } r_2 \]
\[ s_1 = \text{drawdown } r_1 \]
\[ r_1 = \text{radius of influence where } s_2 = 0 \]

\[ Q = \frac{2\pi L e k (H(t) - 0)}{\ln \left( \frac{r_2}{r_i} \right)} \]

\[ Q(t) = \frac{2\pi L e k H(t)}{\ln \left( \frac{r_2}{r_i} \right)} \]

\[ Q(0) = -\pi r_e^2 \frac{dH}{dt} \]

\[ \int_{H_1}^{H_2} \frac{dH}{H} = -\frac{2\pi L e k}{\ln \left( \frac{r_2}{r_i} \right)} \left( t_2 - t_1 \right) \]

\[ \ln \left( \frac{H_2}{H_1} \right) = \frac{-2\pi L e k}{\ln \left( \frac{r_2}{r_i} \right)} \left( t_2 - t_1 \right) \]

\[ k = \frac{r_e^2}{2\pi L e} \left[ \frac{\ln \left( \frac{r_2}{r_i} \right) \ln \left( \frac{r_2}{r_i} \right)}{t_2 - t_1} \right] \]

\[ k = \frac{Q}{2\pi L e} \ln \left( \frac{r_2}{r_i} \right) \]

\[ Q = \frac{2\pi L e k (s_1 - s_2)}{\ln \left( \frac{r_2}{r_1} \right)} \]

\[ s_2 = (h_0 - h_2) \]
\[ s_1 = (h_0 - h_1) \]

→ From well bore storage:

→ From (1) = (2)

\[ \int_{H_1}^{H_2} \frac{dH}{H} = -\frac{2\pi L e k}{\ln \left( \frac{r_2}{r_i} \right)} \left( t_2 - t_1 \right) \]

\[ \ln \left( \frac{H_2}{H_1} \right) = -\frac{2\pi L e k}{\ln \left( \frac{r_2}{r_i} \right)} \left( t_2 - t_1 \right) \]

\[ k = \frac{r_e^2}{2\pi L e} \left[ \frac{\ln \left( \frac{r_2}{r_i} \right) \ln \left( \frac{r_2}{r_i} \right)}{t_2 - t_1} \right] \]
Why?
- pesticides/nitrogen (agricultural sources)
- nuclear waste (radioactive metals)
- mercury \( \text{Hg} \)
- arsenic (III and V) - wood preservatives
- copper \( \text{Cu} \)
- chromium \( \text{Cr} \)
- organic
  - crude oil
  - TCE (tri chloro ethylene)
  - DCE, PCE
  - VC (vinyl chloride) - nasty carcinogen
- landfills
- saltwater intrusion

Flow Problem ← Find Head Distribution
\[
\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} \left[ k_x \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y \frac{\partial \phi}{\partial y} \right] + \frac{\partial}{\partial z} \left[ k_z \frac{\partial \phi}{\partial z} \right] + W
\]
Solve for \( \phi (x, y, z, t) \)
then \( q = -k \frac{\partial \phi}{\partial x} \) (flux)
then \( v = \frac{q}{n} \) (velocity needed for contaminant transport)

Contaminant Problem
- advection - velocity
- diffusion/ - concentration gradient, temperature dependent dispersion

P. 402 → Figure 10.8: Basic causes of pore-scale longitudinal dispersion
Ground Water Contaminant Transport

- Advection (flux)
- Diffusion/Dispersion (flux) dominant

→ Advective Flux
transported entity is "mass"

\[
\text{Flux} = \frac{\text{mass of contaminant}}{\text{unit area - unit time}} = \frac{\text{unit mass}}{\text{unit area - unit time}} = \frac{\text{kg}}{\text{m}^2 \cdot \text{day}}
\]

\[
F_{\text{adv}} = qC \quad C = \text{concentration of contaminant} \quad \left[ \frac{\text{mg}}{\text{L}} \right]
\]

\[
F_{\text{adv}} = \nabla \cdot \mathbf{v} = q \cdot n
\]

\[
\text{Which area? porous area or total area?}
\]

\[
A_n \quad \text{total area}
\]

→ Disp/Diff Flux

\[
F_{\text{disp}} \propto -\nabla \cdot \text{concentration gradient} \quad (\text{Fick's Law})
\]

For porous media,

\[
F_{\text{diff}} = -\eta \text{Diff} \frac{\partial C}{\partial x}
\]

\[
F_{Hd} = F_{\text{diff}} + F_{\text{disp}}
\]

\[
F_{Hd} = D^* \frac{\partial C}{\partial x} + D_{\text{disp}} \frac{\partial C}{\partial x}
\]

\[
F = \frac{D_{\text{disp}} \frac{\partial C}{\partial x}}{D_{\text{disp}} \frac{\partial C}{\partial x}}
\]

\[
D_{\text{disp}} \propto V_x
\]

\[
D_{\text{disp}} = C_L^* V_x
\]
Fluxes
\[ F_a = \eta V c \]
\[ F_d = -\eta D \frac{\partial c}{\partial x} \]
\[ D = \alpha_L V \]

\[ \text{Mass Balance} \]

- Mass In - Mass Out = Accumulation

\[ c(x) \eta \cdot b \cdot 1 \cdot \Delta t - c(x+\Delta x) \eta \cdot b \cdot 1 \cdot \Delta t \]

\[ -b \cdot 1 \cdot \Delta t \eta D \frac{\partial c(x)}{\partial x} + b \cdot 1 \cdot \Delta t \eta D \frac{\partial c(x+\Delta x)}{\partial x} = \eta \frac{\Delta C}{\Delta t} \]

- Divide by b \Delta x \Delta t

\[ -V \eta \left[ \frac{c(x+\Delta x) - c(x)}{\Delta x} \right] + \eta D \left[ \frac{\partial c(x+\Delta x)}{\partial x} - \frac{\partial c(x)}{\partial x} \right] = \eta \frac{\Delta C}{\Delta t} \]

- \Delta x, \Delta t \to 0

\[ -V \eta \frac{\partial c}{\partial x} + \eta D \frac{\partial^2 c}{\partial x^2} = \eta \frac{\partial c}{\partial t} \]

To Solve:
\[ \frac{\partial c}{\partial t} = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} \]

v = \frac{q}{n}\]
To Solve, Need:

- 2 Boundary Conditions
- 1 Initial Condition
- parameters v, D, etc

Solution is \( C(x,t) \)

Example

\[
\text{constant } 10 \text{ mg/L}
\]

\( \rightarrow V = 0.5 \text{ miles/day} \)

\[ \text{BCs:} \quad @ x=0 \quad c_0 = 10 \text{ mg/L} \]
\[ @ x=\infty \quad c = 0 \text{ mg/L} \]

\[ \text{IC:} \quad @ t=0, \quad c(x,0) = 0 \text{ mg/L} \]

\[
\text{pulse } 10 \text{ mg/L}
\]

\( \rightarrow V = 0.5 \text{ miles/day} \)

\[ D = \alpha L V \]

determined by running experiment and fitting a curve

\[ \text{formula p. 1205} \]
Contaminant Transport (II)

- Advection \( \mathbf{F}_{\text{adv}} = q \mathbf{C} \)
- Dispersion \( \mathbf{F}_{\text{disp}} = -D \frac{\partial \mathbf{C}}{\partial x} \)

\[
\frac{\partial \mathbf{C}}{\partial t} = -V \frac{\partial \mathbf{C}}{\partial x} + D \frac{\partial^2 \mathbf{C}}{\partial x^2}
\]

\( \text{advection} \quad \text{dispersion} \)

\( \mathbf{D} = \mathbf{D}^* + \alpha_L \mathbf{V} \)

\( \mathbf{V} = \frac{q}{n} \)

\( \mathbf{D} = \alpha_L \mathbf{V} \quad \text{dispersivity} \) [1-

B.C. \( @x=0 \quad \mathbf{C} = \mathbf{C}_0 \)

\( @x \rightarrow \infty \quad \mathbf{C} = 0 \)

I.C. \( @t=0 \quad \mathbf{C}(x) = 0 \)

\[
\mathbf{C}(x,t) = \frac{\mathbf{C}_0}{2} \left[ \text{erfc} \left( \frac{x-Vt}{2\sqrt{D}t^{1/2}} \right) + \exp \left( \frac{Vx}{\mathbf{D}} \right) \text{erfc} \left( \frac{x+Vt}{2\sqrt{D}t^{1/2}} \right) \right]
\]

Analytical Solution

\( \text{erf} \rightarrow \text{error function} \quad \text{erfc} \rightarrow \text{complimentary error function} \)

\[ \text{erfc}(x) = 1 - \text{erf}(x) \]

\[ \text{erf}(-x) = -\text{erf}(x) \]

Example Problem p. 40 (HW 11)

\( @ x = 15 \text{ m} \)

What about Adsorption?!

→ Soil particles interacting w/ chemical + reacting

→ Conservative tracer - \( \mathbf{Cl}^- \) no decay

→ \( \mathbf{Br}^- \) no sorption
What about adsorption?

Mass Balance

\[
\text{In - Out} = \text{Accumulation} + \text{Solid/soil phase accumulation} + \text{liquid/water phase accumulation}
\]

\[b \Delta x \times 1 \cdot \eta \cdot \Delta c\]

→ Storage/accumulation w/o sorption (liquid phase)

Notations/Definitions

\(\rho_b\) → dry bulk density of soil \(\frac{\text{mass of solid}}{\text{bulk volume}} = 1.6 \text{ g/cm}^3 = 1600 \text{ kg/m}^3\)

\(Q\) → concentration of soil phase sorbed contaminant \(\frac{\text{mass of contaminant in solid}}{\text{mass of solid}}\)

\(\rho_b Q\) = \(\frac{\text{mass of concentration in solid}}{\text{bulk volume of solid}}\)

\(\eta \cdot c\) = \(\frac{\text{volume of water}}{\text{bulk volume}}, \frac{\text{mass of contaminant}}{\text{volume of water}} = \frac{\text{mass of contaminant}}{\text{bulk volume}}\)

Storage = \(b \Delta x 1 \eta \Delta c + b \Delta x 1 \Delta (\rho_b Q)\)

\[-\nu \eta \left[ \frac{c(x+\Delta x) - c(x)}{\Delta x} \right] + \eta D \left[ \frac{\frac{\partial c}{\partial x} (x+\Delta x) - \frac{\partial c}{\partial x} (x)}{\Delta x} \right] = \eta \frac{\Delta c}{\Delta t} + \frac{\Delta (\rho_b Q)}{\Delta t}\]

\[\Delta x, \Delta t \to 0\]

\[-\nu \eta \frac{\partial c}{\partial x} + \eta D \frac{\partial^2 c}{\partial x^2} = \eta \frac{\partial c}{\partial t} + \rho_b \frac{\partial Q}{\partial t}\]

Solid phase accumulation term
\[ \frac{\partial c}{\partial t} + \frac{\rho c}{\eta} \frac{\partial \bar{Q}}{\partial t} = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} \]

Transport Equation w/ Adsorption

- two dependent variables: \( c, \bar{Q} \)
- independent variables: \( y, t \)

\( \rightarrow \) one equation, two unknowns \( \nabla \)

\( \rightarrow \) Isotherms

\[ \begin{align*}
& \text{C}_1 \quad \text{C}_1^* \\
& \text{Q}_1 \quad \text{Q}_1^* \\
& \text{C}_2 \quad \text{C}_2^* \\
& \text{Q}_2 \quad \text{Q}_2^* \\
& \text{C}_3 \quad \text{C}_3^* \\
& \text{Q}_3 \quad \text{Q}_3^* \\
\end{align*} \]

\( \text{sorbate(liquid)} \)

\( \text{Sorbent(solid)} \)

what is \( K_d \)?

Units:

\[ K_d = \frac{\text{(mass of sorbed cont.)}}{\text{(mass of solid)}} \times \frac{\text{(mass of cont.)}}{\text{(volume of water)}} \]

\[ \frac{\text{mg cont.}}{\text{mg solid}} / \text{vol of water} \]

Linear Isotherm

\[ \bar{Q} = k_d c \]

Linear Equilibrium Isotherm

2nd equation! (Isotherm Plot)
\( \frac{\partial c}{\partial t} + \frac{\rho b}{\eta} \frac{\partial}{\partial x} (kd c) = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} \)

\[ \left[ 1 + \frac{\rho b kd}{\eta} \right] \frac{\partial c}{\partial t} = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} \]

\( R \rightarrow \text{retardation factor (linear case)} \)

\( R \frac{\partial c}{\partial x} = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} \)

\( \frac{\partial c}{\partial t} = -\frac{V}{R} \frac{\partial c}{\partial x} + \frac{D}{R} \frac{\partial^2 c}{\partial x^2} \)

where \( \tilde{V} = \frac{V}{R}, \quad \tilde{D} = \frac{D}{R} \)

\( \rightarrow V = 0.5 \text{ mile/day} \)

\( \tilde{V} = \frac{V}{R} = \frac{0.5 \text{ mile/day}}{2} = 0.25 \text{ mile/day} \)

Slows contaminant transport

\( t = \frac{100 \text{ miles}}{0.25 \text{ m/day}} = 400 \text{ miles} \)

![Graph showing contaminant transport over time](image)
Contaminant Transport

- governing equation

\[ \frac{\partial c}{\partial t} = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} \left\{ \begin{array}{c} \text{no sorption} \\ \text{with sorption} \end{array} \right. \]

\[ V = \frac{q}{n} \quad D = \alpha_e V \quad R = \left( 1 + \frac{\alpha L K_d}{n} \right) \]

\[ Q = k_d C \quad k_d = \frac{Q}{C} = \frac{\text{mass of } c \text{ in solid}}{\text{mass of solid}} \frac{\text{mass of } c \text{ in liquid}}{\text{volume of liquid}} \]

\[ \left[ \frac{\text{mg of } c}{\text{mg of } s} \frac{\text{mg of } c}{\text{cm}^3 \text{ water}} \right] \]

\[ \text{Do NOT cancel out units like textbook!} \]

Why do we toss out kinetic data when finding K_d when governing eqn is transient?

In batch tests, it takes a relatively short time to reach steady-state, but groundwater transport times are very large.

GW >> batch time scale

test time scale

Quiz

- How much water volume needs to be removed to get rid of contaminant tracer? (No sorption)

\[ V_{cl} = \pi r^2 h n \]

\[ V_{cl} = \frac{\pi}{6} (2m)^3 (4m)(0.5) \]

\[ V_{cl} = 25.1 \text{ m}^3 \]

How much volume needs to be removed for sorbing contaminant?

\[ R = (1 + \frac{(1.88 \text{ g/cm}^3)(3.9 \text{ ml})}{0.5}) = 15.7 \]

\[ V_{TCE} = 15.7 \text{ V}_{cl} = 394.6 \text{ m}^3 \]

\( \text{pump slowly so that equilibrium assumption is not violated (soap in clothes example)} \)
\[ V = 1 \text{ m/day} \quad \text{chloride} \quad V = 1 \text{ m/day} \]

Continuous injection of Cl

\[ L = 30 \text{ m} \]

Breakthrough

\[ V = 1 \text{ m/day} \]

Continuous injection of TCE

\[ (R = 15) \]

\[ \text{TCE} \quad \frac{V}{R} = \frac{1 \text{ m/day}}{15} = 0.067 \text{ m/day} \]

---

→ What happens to mass balance with retardation?

\[ V = 1 \text{ m/day} \]

10 days

10 m³/L

30 m

---

**Input**

\[ C (\text{mg/L}) \]

10

\[ \text{impulse} \]

\[ t \text{ (days)} \]

---

**Input**

\[ C (\text{mg/L}) \]

10

Continuous

\[ t \text{ (days)} \]

---

**Input**

\[ C (\text{mg/L}) \]

10

\[ \text{mass} \]

After 30 days

---

**Input**

\[ C (\text{mg/L}) \]

10

\[ \text{mass} \]

After 30 days
Contaminant Transport

\[ R \frac{dc}{dt} = -V \frac{dc}{dx} + D \frac{d^2c}{dx^2} \]

\[ \begin{bmatrix} \Delta Q \left/ \Delta t + \beta_b \frac{\Delta Q}{\Delta t} \end{bmatrix} = RHS \]

\[ \begin{bmatrix} \Delta c \left/ \Delta t + \rho_b \frac{dQ}{dt} \end{bmatrix} = RHS \]

\[ \begin{bmatrix} \Delta c \left/ \Delta t + \rho_b \kappa_{el} \frac{dC}{dt} \end{bmatrix} = RHS \rightarrow \eta \frac{dC}{dt} \left[ 1 + \frac{Q}{\kappa_{el}} \right] \rightarrow R \eta \frac{dC}{dt} \]

→ Freundlich Isotherm
\[ \bar{Q} = Kc^z \]

→ Langmuir Isotherm
\[ \bar{Q} = \frac{\alpha \beta c}{1 + \alpha c} \]

α and β are constants

→ makes R non-linear
(must solve numerically)
\[ \eta \frac{\partial c}{\partial t} + \rho \frac{\partial \Theta}{\partial t} = -q \frac{\partial c}{\partial x} + \eta D \frac{\partial^2 c}{\partial x^2} \quad \Rightarrow \quad \text{What's missing?} \quad q \]

\[ + \eta \frac{\partial c}{\partial t} \text{ rxn} \quad \Rightarrow \quad \text{Source/} \quad \text{Sink term} \]

\[ \rightarrow \text{Radioactive Contaminant} \]

\[ \frac{dC}{dt} \propto \frac{dC}{dt} \quad \text{first-order kinetics} \]

\[ \frac{dC}{dt} = -k_r C \quad C = C_0 \quad \text{at} \quad t = t_0 \]

\[ c(t) = C_0 \exp(-k_r t) \]

\[ R \frac{\partial c}{\partial x} = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} + \text{rxn} \]

\[ \Rightarrow \frac{\partial c}{\partial x} \text{ rxn} \]

\[ R \frac{\partial c}{\partial x} = -V \frac{\partial c}{\partial x} + D \frac{\partial^2 c}{\partial x^2} - k_r C \]

\[ \text{first order decay} \]

\[ \frac{d(TCE)}{dt} = -k_T [TCE] \quad \Rightarrow \quad R_T \frac{\partial c}{\partial x} = -V \frac{\partial c_T}{\partial x} + D \frac{\partial^2 c_T}{\partial x^2} - k_T T \]

\[ \text{A first-order model for TCE} \]

\[ \begin{array}{c}
\text{Cl} & \text{PCE} & \text{Cl} \\
\text{Cl} & \text{C} = \text{C} & \text{Cl} \\
\end{array} \]

\[ \begin{array}{c}
\text{Cl} & \text{TCE} & \text{H} \\
\text{Cl} & \text{C} = \text{C} & \text{Cl} \\
\end{array} \]

\[ \begin{array}{c}
\text{Cl} & \text{DCE} & \text{H} \\
\text{Cl} & \text{C} = \text{C} & \text{Cl} \\
\end{array} \]

\[ \text{BAD} \quad \text{STUFT} \quad \text{ethene} \]

\[ \text{TCE} \quad \text{DCE} \quad \text{VC} \]

\[ \text{E} \quad \text{E} \]
Coupled Kinetics in batch

\[
\frac{dT}{dt} = -k_T T
\]
\[
\frac{dD}{dt} = +k_T T - k_D D
\]
\[
\frac{dV}{dt} = +k_D D - k_V V
\]
\[
\frac{dE}{dt} = +k_V V
\]

\[\text{rxns}\]

\[
R_T \frac{dT}{dt} = -V_x \frac{dT}{dx} + D \frac{d^2 T}{dx^2} - k_T T
\]
\[
R_D \frac{dD}{dt} = -V_x \frac{dD}{dx} + D \frac{d^2 D}{dx^2} + k_T T - k_D D
\]
\[
R_V \frac{dV}{dt} = -V_x \frac{dV}{dx} + D \frac{d^2 V}{dx^2} + k_D D - k_V V
\]
\[
R_E \frac{dE}{dt} = -V_x \frac{dE}{dx} + D \frac{d^2 E}{dx^2} + k_V V
\]

Non-Aqueous Phase

Liquids

(NAPLs)

- LNAPL (dense)
- LNAPL (light)

\[1000 \text{ mg/l}
\]
\[1000 \text{ mg/l}
\]
\[1,000, 000 \text{ mg/l}
\]

HEALTH RISK \(\rightarrow 5 - 10 \text{ mg/l}\)

\[\text{compute mass of TCE required to contaminate superfine water}\]
\[> 5 \text{ mg/l}\]

\[\text{PCE} \ ?\]

\[\text{teaspoons} \ ?\]

\[\frac{\text{mg}}{L} = \frac{\text{mg}}{1000 \text{ml}} = \frac{\text{mg}}{1000 \text{cm}^3} = \frac{10^{-6} \text{g}}{1000 \text{cm}^3} = 10^{-9} \text{g/cm}^3\]

\[V = 3.5 \times 10^6 \text{ m}^3 \times \left( \frac{10^6 \text{cm}^3}{	ext{m}^3} \right)\]

\[V = 3.5 \times 10^6 \text{ cm}^3 \times \left( \frac{5 \times 10^{-9} \text{g/cm}^3}{1 \text{ g/L}} \right)\]
Transport (Saltwater Intrusion)

\[ \frac{\partial c}{\partial t} = -V_x \frac{\partial c}{\partial x} + D_x \frac{\partial^2 c}{\partial x^2} \quad \text{(TRACER)} \]

\[ V_x = \frac{q}{n} \quad D_x = \alpha_c V_x \]

Adsorption

\[ R \frac{\partial c}{\partial t} = \text{RHS} \quad R = \left(1 + \frac{\rho_{Kd}}{n}\right) \]

Decay Reactions

\[ R \frac{\partial c}{\partial t} = -V_x \frac{\partial c}{\partial x} + D_x \frac{\partial^2 c}{\partial x^2} - R k_c \]

→ Why do we consider density when looking at salt water?

<table>
<thead>
<tr>
<th>Fresh water</th>
<th>Salt water</th>
<th>Density ( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000 kg/l</td>
<td>1.02 - 1.03 g/cm³</td>
<td></td>
</tr>
<tr>
<td>1.0 g/cm³</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Large quantities make a difference

Saltwater Intrusion Presentation

→ Where is this wedge?
at equilibrium

\[\gamma_f (h_f + h_s) = \gamma_s h_s \ \iff \ \text{hydrostatics}\]

\[\gamma_f h_f + \gamma_f h_s = \gamma_s h_s\]

\[\gamma_f h_s - \gamma_f h_s = -\gamma_f h_f\]

\[\gamma_s h_s - \gamma_f h_s = \gamma_f h_f\]

\[h_s (\gamma_s - \gamma_f) = \gamma_f h_f\]

\[h_s = \left(\frac{\gamma_f}{\gamma_s - \gamma_f}\right) h_f = \delta h_f = 40 h_f\]

**Gyben - Herzberg Principle**

- \(\gamma_f = 1\)
- \(\gamma_s = 1.025\)
- \(\delta = \left[\frac{1}{1.025-1}\right] = 40\)

If \(h_f = 2 m\), \(h_s = 80 m\)