Crack tip plasticity and Small scale yielding

Linear elastic analysis of the crack shows crack tip stresses and deformations of the form,

\[ \sigma \sim K \frac{1}{r} \] and \[ U \sim K \frac{1}{r} \]

where \( K = K_I, K_{II}, \) or \( K_{III}. \) Evidently, as \( r \to 0, \sigma \to \infty \) and \( U \to 0. \) Thus, crack tip stresses are singular.

In reality, however, engineering materials undergo plastic or permanent deformations when stressed beyond the "yield point" \( (\sigma_Y) \) of the material.

\[ \sigma \begin{cases} \sigma_y, & \text{linear elastic} \\ \sigma_y, & \text{linear elastic work hardening} \\ \sigma_y, & \text{elastically perfectly plastic} \\ \sigma_y, & \text{nonlinear elastic ("rubbery")} \end{cases} \]

Thus, when crack tip stresses reach \( \sigma_y, \) plastic deformation occurs at \( \varepsilon \) around the crack tip and stresses become finite, weakening the stress singularity. As the load increases plastically deformed region spreads.
Irwin's Model:

The influence of crack tip plasticity was studied by Irwin to determine the stress redistribution and the resulting effect on crack tip stress intensity factor. He assumed the material to be an "elastic—perfectly plastic material" and crack tip plasticity to cause a simple redistribution of the stresses ahead of the crack tip beyond the "plastic zone".

Recall, from the elastic analysis,

\[ \sigma_{22}(x, \theta) = \frac{K_I}{2\pi r} \cos \theta \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \]

\[ \sigma_{22}(x, \theta=0^\circ) = \sigma_{22}(x_1, x_2=0) = \frac{K_I}{\sqrt{2\pi r}} = \frac{K_I}{\sqrt{2\pi x_1}} \]

as represented by ABC in the graph.
Irwin assumed that plasticity causes a redistribution of stresses as shown by the broken line in the figure for a deformed crack. i.e., stresses are represented by DBEF (EF being a translation of BC) with crack tip stresses being \( \sigma_y \) over a length DE ahead of the crack tip.

Let \( DB = c_1 \) and \( BE = c_2 \).

For equivalence of the two cases, the shaded areas should be equal:

\[
\int_0^c \frac{k_I}{\sqrt{2\pi x}} \, dx_1 - \sigma_y c_1 = \sigma_y c_2
\]

\[
\Rightarrow \frac{2k_I}{\sqrt{2\pi}} c_1^{\frac{1}{2}} = \sigma_y (c_1 + c_2).
\]

Now, at \( B \), \( \sigma_{22} = \sigma_y = \frac{k_I}{\sqrt{2\pi c_1}} \)

\[
\sigma_y c_1 = \left( \frac{k_I^2}{\sigma_y^2} - \frac{1}{2\pi} \right)
\]

\[
\Rightarrow \left( \frac{k_I^2}{\sigma_y^2} \frac{1}{2\pi} \right) = \sigma_y \left( \frac{k_I^2}{\sigma_y^2} - \frac{1}{2\pi} \right) + \sigma_y c_2
\]

\[
\Rightarrow c_2 = \left( \frac{k_I^2}{\sigma_y^2} - \frac{k_I^2}{\sigma_y^2} \frac{1}{2\pi} \right) = \left( \frac{k_I^2}{\sigma_y^2} \frac{1}{2\pi} \right) = c_1
\]
\[ \frac{c_1}{c_2} = \frac{1}{1 + \frac{1}{2} \left( \frac{K_I}{\sigma_y} \right)^2} \]

\[ \text{plastic zone size} = c_1 + c_2 = 2\gamma \frac{K_I^2}{\sigma_y^2} \]

Irwin argued that a plastically deformed crack, being more compliant than an elastic crack, acts like a longer crack. That is, equivalent crack length, \( a_y = (a + \gamma a) \)

This linearization is also called "small scale yielding" (SSY). Hence, the modified stress intensity factor for a "plastically" deformed crack is,

\[ (K_I)_y = \sigma_0 \sqrt{\pi a_y} = \sigma_0 \sqrt{\pi (a + \gamma a)} \]

for an infinite plate with a small crack or,

\[ (K_I)_y = f \left( \frac{a}{W} \right) \sigma_0 \sqrt{\pi (a + \gamma a)} \]

\( \text{Geometric correction factor} \)

From eq. \((1)\),

\[ (K_I)_y = \sigma_0 \sqrt{\pi \left[ a + \frac{1}{2\pi} \left( \frac{K_I}{\sigma_y} \right)^2 \right]} \]

\( \text{\textbullet} \)
Note: Equation 2 can be used iteratively until convergence occurs while computing the SIF.

Also, recall, for a Griffith crack,

\[ G = \frac{\sigma_{\infty}^2 \pi a}{E} = \frac{k_1^2}{E} \quad \text{... plane stress} \]

So for a plastically deformed crack,

\[ G = \frac{\sigma_{\infty}^2 \pi a}{E} \left[ \alpha + \frac{1}{2\pi} \left( \frac{k_1^2}{\sigma_{\infty}^2 y^2} \right) \right] \]

\[ = \frac{\sigma_{\infty}^2 \pi a}{E} \left[ 1 + \frac{1}{2\pi c} \frac{\sigma_{\infty}^2 \pi a}{\sigma_{\infty}^2 y^2} \right] \]

\[ G = \frac{\sigma_{\infty}^2 \pi a}{E} \left[ 1 + \frac{1}{2} \left( \frac{\sigma_{\infty}}{\sigma_y} \right)^2 \right] \]

plasticity correction
crack tip opening displacement \( (CTOD)_{Irwin} \)

Previously, we have shown that crack opening displacements, \( U_2^{\text{COD}} \) is given by,

\[
U_2 = \frac{2\sigma_{\infty}}{E} \sqrt{a^2 - x_1^2}
\]

In the absence of crack tip plasticity, crack-tip opening displacement \( U_2^{\text{COD}} \) at \( x_1 = a \), \( CTOD \) is zero. Let \( CTOD = \delta_t \).

Before plasticity, \( \delta_t = 2U_2^{\text{COD}} \bigg|_{x_1 = a} = 0 \).

(Note)

Now, for a plastically deformed crack, using Irwin's corrections,

\[
\delta_t = 2U_2 \bigg|_{x_1 = a} = \frac{4\sigma_{\infty}}{E} \sqrt{a^2 - x_1^2}
\]

\[
= \frac{4\sigma_{\infty}}{E} \sqrt{(a+ry)^2 - a^2}
\]

\[
= \frac{4\sigma_{\infty}}{E} \sqrt{2ary + ry^2}
\]
For small $r_y$, $r_y^2 \ll 2ar_y$

\[ \delta_t \approx \frac{4\sigma_{\infty}}{E} \sqrt{2ar_y} \]

\[ = \frac{4\sigma_{\infty}}{E} \sqrt{2a \left( \frac{1}{2\pi} \left( \frac{k_I}{\sigma_y} \right)^2 \right)} \]

\[ = \frac{4\sigma_{\infty}}{E} \sqrt{\frac{a \sigma_{\infty}^2 \pi a}{\pi \sigma_y^2}} \quad \cdots \quad K_I = \sigma_{\infty} \sqrt{\pi a} \]

\[ \delta_t = \frac{4 \sigma_{\infty}^2 a}{E \sigma_y} = \frac{4}{\pi} \left( \frac{K_I^2}{E \sigma_y} \right) \]

Recall, $G = \frac{k_I^2}{E} f(y)$

By noting that $\delta_t \propto K_I^2$, we can deduce that $\delta_t \propto G$
**Dugdale Model**

Dugdale proposed a strip yielding model in which he assumed that plastic deformations occur in a strip ahead of the crack tip. Here again, the material is assumed to be elastic-perfectly plastic. And, the plastically deformed strip has a height approximately equal to the thickness of the sheet (plane stress conditions). (These assumptions work well with low-carbon steels, polycarbonates, ...)

1. \[ \sigma \rightarrow \sigma_y \]

2. \[ \sigma_y \]

3. \[ \sigma_y \]

4. \[ \sigma_y \]
Now, consider a Griffith crack problem with an internal crack of length 2a subjected to far-field stress $\sigma_0$. Let C denote plastically yielded region. Let O denote the original crack tip and O' is the tip of the plastic strip. Now, for this problem, assuming total crack length to be $2(a+c)$,

$$ (K_I)_1 = \sigma_0 \sqrt{\pi} (a+c). \quad \text{(A)} $$

Using Green's function method, we can write SIF for configuration 2 as,

$$ (K_I)_2 = -\sigma_y \sqrt{\pi} (a+c) + \frac{2\sigma_y (a+c)}{\sqrt{\pi} (a+c)} \sin^{-1} \left( \frac{a}{a+c} \right) \quad \text{(B)} $$

where $\sigma_y$ is the yield stress of the material.

Dugdale postulated that plastic zone dimension can be determined by requiring that the SIF be zero at the crack tip O since stresses are finite ($= \sigma_y$).

$$ \therefore (K_I)_1 + (K_I)_2 = 0 $$

i.e., from eqs. (A) & (B),
\[
\Rightarrow \sigma_0 \sqrt{\frac{\pi}{a+c}} - \sigma_y \sqrt{\frac{\pi}{a+c}} + \frac{2\sigma_y \sqrt{\pi(a+c)}}{a} \sin^2 \left(\frac{a}{a+c}\right) = 0
\]

\[
\Rightarrow \sigma_0 - \sigma_y + \frac{2\sigma_y}{a} \sin^2 \left(\frac{a}{a+c}\right) = 0
\]

\[
\Rightarrow \sin^2 \left(\frac{a}{a+c}\right) = \frac{\pi}{2} \left(\frac{\sigma_y - \sigma_0}{\sigma_y}\right) = \frac{\pi}{2} \left(1 - \frac{\sigma_0}{\sigma_y}\right)
\]

\[
\Rightarrow \frac{a}{a+c} = \sin \left\{\frac{\pi}{2} \left(1 - \frac{\sigma_0}{\sigma_y}\right)\right\} = \sin \frac{\pi}{2} \cos \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right) - \cos \frac{\pi}{2} \sin \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right) = 0
\]

\[
\frac{a}{a+c} = \cos \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right) = \frac{\pi}{2} \frac{\sigma_0}{\sigma_y}
\]

\[
\frac{a}{\alpha(1+e)} = \cos \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right)
\]

\[
\Rightarrow \frac{c}{a} = \sec \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right) - 1
\]

(Recall, \(\sec \theta \approx 1 + \frac{\theta^2}{2} + \cdots\))

For small values of \(\left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right)\),

\[
\sec \left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right) = 1 + \frac{1}{2}\left(\frac{\pi}{2} \frac{\sigma_0}{\sigma_y}\right)^2 = 1 + \frac{\pi^2}{8} \left(\frac{\sigma_0}{\sigma_y}\right)^2
\]
\[
\frac{c}{a} \approx \frac{\pi^2}{8} \left( \frac{\sigma_{\text{op}}}{\sigma_y} \right)^2
\]

Now, recall \( K_I = \sigma_{\text{op}} \sqrt{\pi a} \)

\[
C = \frac{\pi}{8} \left( \frac{K_I}{\sigma_y} \right)^2 \quad \frac{\pi}{8} = 0.392
\]

Note: \( C \) used in this analysis is equivalent to \( 2\gamma_y \) used in Irwin's analysis \( \Rightarrow \) plastic zone size.

\[
(2\gamma_y)_{\text{Irwin}} = \frac{1}{\pi} \left( \frac{K_I}{\sigma_y} \right)^2 = 0.318 \left( \frac{K_I}{\sigma_y} \right)^2
\]

\[
(C)_{\text{Dugdale}} = \frac{\pi}{8} \left( \frac{K_I}{\sigma_y} \right)^2 = 0.392 \left( \frac{K_I}{\sigma_y} \right)^2
\]

\[
(\alpha_y)_{\text{Dugdale}} = \alpha + C
\]

and, \( K_I = \sigma_{\text{op}} \sqrt{\pi (a+c)} \)

\[
= \sigma_{\text{op}} \sqrt{\pi \left( a + \frac{\pi}{8} \left( \frac{K_I}{\sigma_y} \right)^2 \right)}
\]
Crack Tip Opening Displacement (CTOD) Dugdale

As in Irwin's analysis, the crack tip opening displacement due to plasticity can be determined rigorously. In view of the lengthy derivation, details are avoided here. But, it can be shown that

\[ \delta_t = \left( \frac{K_I^2}{E \sigma_y} \right) \]

Where \( \sigma_y \) is the yield stress of the material.

(Note: \( \delta_t \) \text{Irwin} = \frac{\pi}{4} \left( \frac{K_I^2}{E \sigma_y} \right) \)
Crack Tip Plastic Zone Based on Von-Mises yield condition:

Consider a mode-I crack in a planar object. The stresses at the tip of an elastic crack are:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix}
1 - \sin^2 \frac{\theta}{2} & \sin \frac{3\theta}{2} \\
\sin \frac{\theta}{2} & 1 + \sin^2 \frac{3\theta}{2} \\
2 \sin \frac{\theta}{2} \cos \frac{3\theta}{2}
\end{bmatrix}
\]

Recall, principal stresses can be found using:

\[
\sigma_1, \sigma_2 = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}
\]

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2
\end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \begin{bmatrix}
1 + \sin^2 \frac{\theta}{2} \\
1 - \sin^2 \frac{\theta}{2}
\end{bmatrix}
\]

and \( \sigma_3 = 0 \) for plane stress

\[
= \gamma (\sigma_1 + \sigma_2) \quad \text{for plane strain}
\]

\[
(\sigma_2 = \gamma \frac{2K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2}, \text{ for plane strain})
\]

\( \text{(2a)} \)
According to von-Mises failure theory, material yielding occurs when

\[
\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2 = 2\sigma_y^2 - (3)
\]

Then, for plane stress

\[
\left(\sigma_1 - \sigma_2\right)^2 + \sigma_2^2 + \sigma_1^2 = 2\sigma_y^2 - (4)
\]

Let \(\gamma_y(\theta)\) denote the radial distance to the elastic-plastic boundary from the crack tip as shown in Fig.

Using Eq. (2) in (4), on \(\gamma = \gamma_y\) ⇒

\[
\frac{K_I^2}{2\pi\gamma} \left[ \sin^2 \theta + \cos^2 \theta \left( \frac{1}{2} \sin^2 \theta - \frac{1}{2} \cos \theta \right) \left( \frac{1}{2} \sin^2 \theta + \frac{2}{2} \sin^2 \theta - \frac{2}{2} \cos \theta \right) \right] \bigg|_{\gamma = \gamma_y} = 2\sigma_y^2
\]

\[
= \frac{K_I^2}{2\pi\gamma} \left[ \sin^2 \theta + 2\cos^2 \theta + 2\sin^2 \theta \cos^2 \theta \cos \theta \sin \theta \right] \bigg|_{\gamma = \gamma_y} = 2\sigma_y^2
\]

\[
= \frac{K_I^2}{2\pi\gamma} \left[ \frac{1}{2} \sin^2 \theta \left( \frac{1}{2} \sin^2 \theta \right) + (1 + \cos \theta) \right] \bigg|_{\gamma = \gamma_y} = 2\sigma_y^2
\]
or, \[ \frac{K_I^2}{2\pi \gamma_y} \left[ \frac{3}{2} \sin^2 \theta + (1+\cos \theta) \right] = 2\sigma_y^2 \]

or, \[ \gamma_y = \frac{1}{2\pi} \left( \frac{K_I}{\sigma_y} \right)^2 \left[ \frac{3}{4} \sin^2 \theta + \left( \frac{1+\cos \theta}{2} \right)^2 \right] f^0(\theta) \]

or, \[ \gamma_y = \frac{1}{2\pi} \left( \frac{K_I}{\sigma_y} \right)^2 f^0(\theta). \]

Along, \( \theta = 0^\circ \), \[ \gamma_y = \frac{1}{2\pi} \left( \frac{K_I}{\sigma_y} \right)^2 = (\gamma_y)_{\text{Irwin}} \]

ie., the extent of the plane-stress plastic zone ahead of the crack tip matches with the plastic zone size estimated by Irwin.

Now, for plane strain using eqs. (2), (2a) and (3), we can get,

\[ \frac{K_I^2}{2\pi \gamma_y} \left[ \frac{3}{2} \sin^2 \theta + (1-2\nu)^2 (1+\cos \theta) \right] = 2\sigma_y^2 \]

Solving for \( \gamma_y \) \[ \gamma_y = \frac{1}{2\pi} \left( \frac{K_I}{\sigma_y} \right)^2 \left[ \frac{3}{4} \sin^2 \theta + \left( \frac{1-2\nu}{2} \right)^2 (1+\cos \theta) \right] \]

\( \gamma_y = \frac{1}{2\pi} \left( \frac{K_I}{\sigma_y} \right)^2 f^0(\theta) \)
Along, \( \theta = 0^\circ \), eq. 6 reduces to,

\[
\gamma_y = \left[ \frac{1}{2\pi} \left( \frac{K_I}{\sigma_y} \right)^2 \right] \left( \frac{1}{y} \right) \quad \text{for} \quad y = \frac{1}{3}.
\]

\( \gamma_y \), plastic zone extent is only 11% of the plane-stress counterpart!

\( \gamma_y \), spread of plastic zone is highly limited under plane-strain conditions when compared to plane-stress condition.

In practice, fracture tests are conducted on finite thickness specimens. Hence, the planes adjacent to the surfaces are close to plane-stress while the interior is close to plane-strain condition.
The elastic-plastic boundary (EPB) is a 3-D envelop as shown above. The material inside is plastically deformed and the size of the EPB shrinks as one approaches the mid-plane from the surface. This is due to the "constraint" experienced by the mid-plane unlike the free-surfaces. Thus, the spread of plasticity occurs easily on the surface (and hence a great deal of plastic energy absorption/dissipation) when compared to the interior.
Figure 3.8. Plastic zones, appearing as dark regions, in a cracked plate at (a) the surface of the specimen, (b) the section halfway between the surface and the midsection and (c) the midsection. (Photograph by P. N. Mincer, Battelle Memorial Institute.)

Figure 3.9. Plastic zones, appearing as light regions, in a cracked plate at (a) the front and (c) the back surfaces of the plate and (b) a section normal to the crack plane. (Photograph by P. N. Mincer, Battelle Memorial Institute.)