Mixed Mode Crack Initiation & Growth

models for predicting crack initiation & growth under mixed mode loading deal with:

(a) direction of crack propagation,
(b) critical value of a field quantity.

In this section, mixed-mode planar loading situations involving mode-I & mode-II are considered:

\[ k_I = \sigma_{\infty} \sqrt{\pi a} \cos^2 \alpha \]
\[ k_{II} = \sigma_{\infty} \sqrt{\pi a} \cos \alpha \sin \alpha \]

for mixed mode loading, (mode-I + - II), crack tip stress field can be expressed as,

\[ \sigma_{ij} = \frac{k_I}{\sqrt{2\pi r}} f^I_{ij}(\theta) + \frac{k_{II}}{\sqrt{2\pi r}} f^{II}_{ij}(\theta) \]

\( i, j = 1, 2 \).
Fracture Envelope: (mixed mode - I & II)

Fracture envelope is a locus of points in $K_1 - K_2$ space where combined effect of $K_1$ and $K_2$ reaches a material dependent value. For a crack in a planar object, $J$ can be expressed as,

$$f(K_1, K_2, K_1c, K_2c) = 0$$

where subscript "c" denotes the critical values. Typically, the above function is expressed in a polynomial or power law form:

$$\left( \frac{K_1}{K_1c} \right)^a + \left( \frac{K_2}{K_2c} \right)^b = 1,$$

where $a$ & $b$ are determined experimentally. (Example: for wood $a=1$, $b=2$.)

Based on energy principles, fracture envelope can be expressed as a quadratic:

$$C_{11}K_1^2 + 2C_{12}K_1K_2 + C_{22}K_2^2 = C$$

where $C_{11}, C_{12}, C_{22}$ are material dependent constants to be determined experimentally and $C$ is a real constant.
Experimental results follow an elliptical fracture envelope for many conventional materials:

\[
\left(\frac{K_I}{K_{IC}}\right)^2 + \left(\frac{K_{II}}{K_{IIc}}\right)^2 = 1
\]

**Mixed-Mode Crack Propagation Criteria**

(a) **Modified Griffith criterion**

In this, concept of energy release rate is extended to mixed-mode conditions, i.e., for mode-I + mode-II,

\[
G = G_I + G_{II}
\]

where \(G_I = \eta \frac{K_{II}^2}{E}\), \(G_{II} = \eta \frac{K_{II}^2}{E}\),

\(\eta = 1\) (plane stress)

\(= 1 - \nu^2\) (plane strain)

"Crack growth will occur in a direction in which total energy release rate is maximum and initiation will occur when total energy release rate attains a critical value."
for a cracked plate with an inclined crack as shown, for various hypothetical crack extension directions $\theta$ ($-\pi < \theta < \pi$), the associated energy release rate $G_\theta$ is evaluated and plotted as a function of $\theta$. Then, crack extension direction is $\theta_c$ corresponding to $(G_\theta)_{\text{max}}$ obtained by maximizing $G_\theta$:

$$\frac{\partial G_\theta}{\partial \theta} = 0; \quad \frac{\partial^2 G_\theta}{\partial \theta^2} < 0$$

And, crack initiation occurs when

$$(G_\theta)_{\text{max}} \geq G_c$$

where $G_c$ is a critical material dependent value obtained from pure mode-I conditions.

i.e. $G_c = \frac{\pi}{4} k_{ic} \sqrt{\frac{2}{E}}$. 

\[
\begin{align*}
\text{\textit{\textbf{(G\theta)\text{max}}}} &\geq G_c, \quad \text{where \textit{G}_c \text{ is a critical material dependent value obtained from pure mode-I conditions.}}
\end{align*}
\]
(b) **Maximum Hoop Stress Criterion** (MTS criterion)

This criterion (proposed by Erdogan & Sih) is based on a certain stress component reaching a critical condition. For a crack subjected to mode-I + II loading,

\[
\begin{align*}
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta\theta} \\
\sigma_{r\theta}
\end{bmatrix}
&= \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix}
\frac{5}{4} \cos \frac{\theta}{2} & -\frac{1}{4} \cos \frac{3\theta}{2} \\
\frac{3}{4} \cos \frac{\theta}{2} & +\frac{1}{4} \cos \frac{3\theta}{2} \\
\frac{1}{4} \sin \frac{\theta}{2} & +\frac{1}{4} \sin \frac{3\theta}{2}
\end{bmatrix} \\
&+ \frac{K_{II}}{\sqrt{2\pi r}} \begin{bmatrix}
-\frac{5}{4} \sin \frac{\theta}{2} & +\frac{3}{4} \sin \frac{3\theta}{2} \\
-\frac{3}{4} \sin \frac{\theta}{2} & -\frac{3}{4} \sin \frac{3\theta}{2} \\
\frac{1}{4} \cos \frac{\theta}{2} & +\frac{3}{4} \cos \frac{3\theta}{2}
\end{bmatrix}
\end{align*}
\]

According to this criterion, crack extension occurs when \( \sigma_{\theta\theta} \) at an infinitesimal distance \( r = r_0 \) becomes maximum and extension occurs in that direction.

\[
\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0, \quad \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0
\]

for \( r \ll \frac{a}{\gamma} \)
Now, after differentiating the expression in (1), we get,

\[ K_I \left( \sin \frac{\theta_c}{2} + \sin \frac{3\theta_c}{2} \right) + k_\Pi \left( \cos \frac{\theta_c}{2} + 3 \cos \frac{3\theta_c}{2} \right) = 0 \]

where \( \theta = \theta_c \) is the angle at which \( \sigma_{\theta\theta} \) becomes \( (\sigma_{\theta\theta})_{\text{max}} \) on \( y = y_0 \).

This equation can be manipulated to yield,

\[ K_I \sin \theta_c + k_\Pi (3 \cos \theta_c - 1) = 0 \quad (2) \]

\[ \text{Ex: for pure mode-I conditions, } (k_\Pi = 0), \]
\[ \theta_c = 0. \]

and for pure mode-II conditions, \( (K_I = 0) \),
\[ \theta_c = \cos^{-1} \left( \frac{1}{3} \right) \Rightarrow \theta_c = \pm 70.5^\circ \]
\[ \theta_c = -70.5^\circ \text{ is valid because it satisfies, } \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0 \text{ inequality.} \]
To obtain the critical value of $(\sigma_{00})_{\text{max}}$, one needs to substitute $\theta = \theta_c$ in the expression for $\sigma_{00}$ in (1). Trigonometric manipulation of $\sigma_{00}$ in (1) leads to,

$$(\sigma_{00})_{\text{max}} = \frac{k_i}{\sqrt{2\pi r_0}} \frac{\cos^3 \theta_c}{2}$$

$$- \frac{3}{2} \frac{k_\Pi}{\sqrt{2\pi r_0}} \cos \frac{\theta_c}{2} \sin \theta_c - 3$$

Crack extension occurs when $(\sigma_{00})_{\text{max}}$ becomes equal to $\sigma_c$, a material dependent constant. $\sigma_c$ is usually determined from pure mode-I conditions, i.e., $\theta_c = 0^\circ$ and $k_i = (k_i)_c$.

$\therefore \quad \sigma_c = \frac{(k_i)_c}{\sqrt{2\pi r_0}}$. - 4

By eliminating $r_0$ in (3) & (4), one can get,

$$\frac{k_i \cos^3 \theta_c}{2} - \frac{3}{2} \frac{k_\Pi \cos \theta_c}{2} \sin \theta_c = (k_i)_c$$

[Diagram showing the failure surface with $k_\Pi / (k_i)_c$ on the y-axis and $k_i / (k_i)_c$ on the x-axis.]
for pure mode $-I$ conditions, ($\theta_c = 0^\circ$),

$$K_I = K_{Ic}$$

for pure mode $-II$, ($K_I = 0$),

$$K_{II} = -\frac{2}{3} \left( \frac{K_{Ic}}{\cos \frac{\theta_c}{2} \sin \theta_c} \right) = \frac{\sqrt{3}}{2} \frac{(K_I)_c}{\theta = -70.5^\circ}$$
Strain Energy Density Criterion (SED)

Based on energy principles, G. C. Sih has proposed the so-called SED criterion.

For a crack subjected to mixed mode-I + mode-II loadings, the strain energy is,

\[ E_u = \int_\Omega \left[ \sigma_{ij} \varepsilon_{ij} \right] \, dv, \quad V = \text{volume.} \]

Then, strain energy density \( E'_u \) is

\[ E'_u = \frac{dE_u}{dv} = \int_\Omega \sigma_{ij} \varepsilon_{ij} \]

For plane elasticity problems,

\[ E'_u = \frac{1}{2} \frac{\eta+1}{E} \left[ \frac{\eta+1}{4} (\sigma_{11}^2 + \sigma_{22}^2) - 2(\sigma_{11} \sigma_{22} - \sigma_{12}^2) \right] \]

where \( \eta = \frac{3-4\nu}{1+\nu} \quad \text{... plane strain} \)

\( = \frac{3-\nu}{1+\nu} \quad \text{... plane stress} \)

Also, crack tip stresses are:

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{pmatrix} = \begin{pmatrix}
\frac{\cos \theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \\
\frac{\cos \theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) + \frac{K_I}{\sqrt{2\pi r}} \\
-\sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}
\end{pmatrix}
\begin{pmatrix}
-\sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) \\
\cos \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) \\
\cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right)
\end{pmatrix}
\]
Using these stress expressions \( \hat{\sigma}_{ij} \) in eq. (1),

\[
E' = \frac{1}{\pi r} \left[ g_{11} k_1^2 + 2 g_{12} k_1 k_2 + g_{22} k_2^2 \right] - \(3\)
\]

where \( g_{11} = \frac{1}{16\mu} (1 + \cos\theta)(\eta - \cos\theta) \)

\( g_{12} = \frac{1}{16\mu} (2\cos\theta - (\eta - 1)) \)

\( g_{22} = \frac{1}{16\mu} [(\eta + 1)(1 - \cos\theta) + (1 + \cos\theta)(3\cos\theta - 1)] \)

\( \mu = \frac{E}{2(1+v)} \)  shear modulus.

\( E' \) in eq. (3) has a \( \frac{1}{r} \) singularity. Shi proposed that SED function \( S \) be of the form,

\[
S(\theta) = (g_{11} k_1^2 + 2 g_{12} k_1 k_2 + g_{22} k_2^2)^{-\frac{1}{2}}
\]

and crack extension occurs in a direction of minimum \( S(\theta) \) and extends when minimum \( S(\theta) \) reaches a critical value \( S_c \), a material dependent parameter.

\[ \Rightarrow \text{crack extension direction is given by,} \]

\[
\frac{\partial S}{\partial \theta} = 0 \text{ and } \frac{\partial^2 S}{\partial \theta^2} > 0
\] - \(5\)
These lead to:

\[ 2 \cos \theta - (\eta-1) \sin \theta \ K_I^2 \]
\[ + 2 \left[ 2 \cos 2\theta - (\eta-1) \cos \theta \right] K_I K_{II} \]
\[ + \left[ (\eta-1) - 6 \cos \theta \right] \sin \theta \ K_{II}^2 = 0 \]

and,

\[ 2 \cos 2\theta - (\eta-1) \cos \theta \ K_I^2 \]
\[ + 2 \left[ (\eta-1) \sin \theta - 4 \sin 2\theta \right] K_I K_{II} \]
\[ + \left[ (\eta-1) \cos \theta - 6 \cos 2\theta \right] K_{II}^2 > 0 \]

Using 6a, b, direction of crack growth can be determined.

Ex: for pure mode I, (\( K_{II} = 0 \))

6a \Rightarrow \left[ 2 \cos \theta - (\eta-1) \right] \sin \theta \ K_I^2 = 0

which leads to \( \theta = \theta_c = 0^\circ \)

and \( \theta = \theta_c = \cos^{-1} \left( \frac{\eta-1}{2} \right) \) for \( \eta = 0.3 \)

for \( \nu = 0.3 \), \( \eta = 1.8 \) plane stress
\( = 2.08 \) plane strain

6b \Rightarrow 2 \cos 2\theta - (\eta-1) \cos \theta > 0

is satisfied by only \( \theta_c = 0^\circ \)
Similarly for pure mode-II loading,

\[ \text{Eq. 6a} \Rightarrow [(\eta-1) - 6\cos \theta_c] \sin \theta_c \ k_{II}^2 = 0 \]

\[ \Rightarrow \theta_c = 0^\circ \]

\[ = \cos^{-1} \left( \frac{\eta-1}{6} \right) \approx 82.3^\circ \]

\[ \approx 79.6^\circ \]

\[ \text{Eq. 6b} \Rightarrow [(\eta-1) \cos \theta_c - 6 \cos 2\theta_c] k_{II}^2 > 0 \]

is satisfied by \( \theta_c = -82.3^\circ \) plane stress

or, \(-79.6^\circ \) plane strain

\[ \Rightarrow \text{Now, crack initiation occurs when } S_{\text{min}} \geq S_c \ldots \text{a material constant} \]

For pure mode-I,

\[ S_{\text{min}} = g_{11} k_{I}^2 = \frac{(\eta-1)}{8\pi MU} k_{I}^2 \quad \ldots \text{from eq. 4} \]

\[ S_{\text{min}} = \frac{\eta-1}{8\pi MU} k_{IC} \]

\( \Rightarrow \) failure occurs when \( S_{\text{min}} \geq S_c \)

or, \([ k_I \geq k_{IC} ] \)

For pure mode-II,

\[ S_{\text{min}} = g_{22} k_{II} \quad \text{where } g_{22} \text{ is given in eq. 3.} \]
for $\theta_c = -82.3^\circ$,

$$S_{\text{min}} = \frac{1}{16\mu} \left[ \frac{(\eta+1)(0.866) + (1.134)(-0.6)}{-0.678} \right] \frac{K_{\Pi}^2}{K}$$

$$= \frac{1}{16\pi\mu} \left[ (\eta+1)(0.866) - 0.678 \right] K_{\Pi}^2$$

As before, $S_c = \frac{(\eta-1)}{8\pi\mu} \frac{K_{\Pi}^2}{K_{IC}}$ for $K_{IC}$ lower than $K_{\Pi}$

For $S_{\text{min}} = S_c$, we get

$$\frac{K_{\Pi}^2}{2} \left[ (\eta+1)(0.866) - 0.678 \right] = \frac{(\eta-1)}{8\pi\mu} \frac{K_{\Pi}^2}{K_{IC}}$$

$$\Rightarrow K_{\Pi} = \sqrt{\frac{2(\eta-1)}{(\eta+1)0.866 - 0.678}} K_{IC} \approx 0.96 K_{IC} \text{ for plane stress}$$

$$\approx 1.09 K_{IC} \text{ for pl. strain}$$

Note: MTS criterion is same for both plane stress and plane strain. However, SED criteria differs for plane stress and plane strain because of the presence of $E, v$ in "S".