Modes of Fracture

A generic crack in a component is a line of varying curvature. Hence, state of stress varies from one point to another along the crack front. There are three basic modes of fracture, namely, mode-I, -II, and -III. Mode-I is the opening mode and the dominant displacement is normal to the surface. Mode-II is the sliding mode and the dominant displacement is in the plane of the crack along the crack orientation. In mode-III, the dominant displacement is a tangential displacement normal to the crack front in the out-of-plane direction.
A combination of two or more of these modes can be used to model state of stress of a crack located in a structural component. Mode-I usually dominates engineering investigation since it is the most dangerous.
Strain Energy & Surface Energy considerations

Whether a crack in a material is likely to grow or not for a given loading condition is the central issue in fracture mechanics. Energy methods such as the one described next addresses this issue without necessarily requiring the details regarding stresses, strains and deformation fields near the crack tip. The approach involves invoking conservation of kinetic & potential energies of a cracked body.

Griffith’s Analysis: (≈1925)

Basic idea behind Griffith’s analysis is the realization that a crack in an elastic body will not extend unless the released energy due to crack growth is equal to or greater than that needed to form (two) new surfaces. It should be recognized that, like surface tension in liquids, surfaces of all solids are associated with free surface energy. This is because atoms & molecules of a surface are in a different kind of equilibrium compared to the ones in the interior. Atoms at the surface as well as the atoms just
Under the surface readjust to achieve equilibrium, thereby straining the material close to the surface. Such surface deformations require energy referred to as "surface energy". Surface energy is a material property:

<table>
<thead>
<tr>
<th>Material</th>
<th>Surface Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond</td>
<td>5.5 J/m² or N/m</td>
</tr>
<tr>
<td>Glass</td>
<td>2.3 &quot;</td>
</tr>
<tr>
<td>Steel (mild)</td>
<td>1.2 &quot;</td>
</tr>
<tr>
<td>Copper</td>
<td>1.0 &quot;</td>
</tr>
<tr>
<td>Aluminum</td>
<td>0.6 &quot;</td>
</tr>
<tr>
<td>Ice</td>
<td>0.07 &quot;</td>
</tr>
</tbody>
</table>

Consider a thin sheet of material under uniaxial tension, held between two rigid blocks. Now, let a knife be used to cut a central crack oriented normally to the direction of the tension. If the crack length is increased gradually, at a critical value crack continues to grow without additional effort. As the crack advances, the overall stiffness of the sheet reduces. Overall strain energy stored in the solid decreases while surface energy increases as crack grows.
Finding a rigorous solution to the problem can be done only if stress and displacement fields are available near the crack tip. Hence one can only attempt an approximate analysis. Griffith assumed that most of the energy released that goes into the formation of "new" crack surfaces comes from regions adjacent to the crack faces. For simplicity one can assume this region to be a triangle of height \( \beta (2a) \) where \( \beta \) is a constant. (As the length of the crack increases, the area of the triangle grows).

Now, total released energy = \( 2 \left[ \text{vol. of the "triangle"} \right] \times \left( \text{strain energy density} \right) \)
If $E$ is the Young's modulus and $B$ is the thickness of the sheet, then,

$$
\text{released strain} \gamma = 2 \left( \frac{1}{2} (2a) \beta (2a) B \right) \left( \frac{2E}{\gamma} \right)
$$

for uniaxial state of stress.

$$
E_u = \frac{2 \beta a^2 B \gamma^2}{E} \quad \text{or} \quad E_u = \frac{\beta l^2 B \gamma^2}{2E}
$$

where $l = 2a$.

Energy consumed in the formation of new surfaces:

$$
E_p = 2 (2a) \beta \gamma = 2l \beta \gamma
$$

where $\gamma$ = surface energy ($N/m$ or $J/m^2$).

Also, it should be noted that for thin plates (plane stress), a rigorous analysis (to be presented later) shows $\beta = \pi/2$.

From the plot of $Eu$ and $Ep$ against $a$, it is clear that for an incremental increase in crack length $\Delta a$,

Initially $\Delta Eu < \Delta Ep$ for $a < a_{cr}$. 
That is, surface energy needed to advance the crack by \( \Delta a \) is more than the released energy. Hence the crack would not grow initially unless the knife cuts the material. As the crack length increases, and reaches \( a = a_{cr} \), \( \Delta E_u \) and \( \Delta E_p \) become equal for an incremental crack growth \( \Delta a \). At all values of \( a > a_{cr} \), \( \Delta E_u > \Delta E_p \) for incremental crack growth \( \Delta a \). Hence, critical condition for crack growth is,

\[
\Delta E_u \geq \Delta E_p
\]

or,

\[
\frac{dE_u}{dA} \Delta A \geq \frac{dE_p}{dA} \Delta A \quad \text{where} \quad \Delta A = (A_1 - A_2)
\]

or,

\[
\frac{1}{B} \frac{dE_u}{dl} \geq \frac{1}{B} \frac{dE_p}{dl}
\]

from \( \sigma^2 \text{E} \),

\[
\left\{ \frac{\pi (A_1)}{2} \leq \frac{2}{\sqrt{1 - \frac{2}{\pi}}} \right\} \equiv \left\{ \frac{\pi \sqrt{\sigma^2}}{E} \geq 2 \sqrt{1 - \frac{2}{\pi}} \right\}
\]

or,

\[
A_{cr} = \frac{2E\sqrt{1 - \frac{2}{\pi}}}{\pi \sigma^2}
\]

Eq. 4 can also be used to determine the stress level required to advance a given crack.
By rearranging eq. (4),
\[ \sigma_{cr} = \sqrt{\frac{2E}{\pi a}} \quad \text{.. plane stress} \quad (5) \]

For plane strain conditions ("thick" sheet), replace \( E = E/(1-\nu^2) \) to obtain \( \sigma_{cr} \),
where \( \nu \) is the Poisson's ratio.
\[ \sigma_{cr} = \sqrt{\frac{2E}{(1-\nu^2)\pi a}} \quad \text{.. plane strain} \quad (6) \]

[Note: In Eu, Ep vs. a plot, why does the crack grow in this experiment even though Eu < Ep? This is because the difference in the two energies has already been supplied by the knife. This is very useful quantity because many small cracks in a body are not able to grow and they remain dormant.

From equations (4) and (5), we can recognize that, \( \sigma_{cr} \propto E \) and \( \nu \) and \( \frac{1}{\sqrt{2}} \) while \( \sigma_{cr} \propto \sqrt{E} \), and \( \sqrt{\nu} \) and \( \frac{1}{\sqrt{a}} \).

Also, \( \sigma \sqrt{a} = \sqrt{\frac{2E\nu}{\pi}} \) (Material constant).]
Energy Balance in Cracked Bodies

Consider a closed boundary $S$ of a cracked 2-D body. Let the thickness be unity and crack length $2a$.

Also let $E_w$ denote the energy input into the system, $E_u$ denote the strain energy, $E_k$ the kinetic energy, $E_p$ is irreversibly dissipated energy.

In the context of fracture mechanics, $E_w$ represents the work done by the external forces on the body. $E_u$ is the recoverable elastic strain energy, $E_k$ is the kinetic energy associated with crack growth. $E_p$ is typically the dissipated energy through plastic (or, permanent) deformations, intermolecular friction and formation of new surfaces.

Hence, for energy balance,

$$E_w = E_p + E_u + E_k$$
If the crack extends by a length $\delta a$

$\delta E_W = \delta E_p + \delta E_u + \delta E_k$

Then, for small energy changes, we have

$$\frac{dE_w}{da} = \frac{dE_p}{da} \delta a + \frac{dE_u}{da} \delta a + \frac{dE_k}{da} \delta a$$

or,

$$\frac{dE_w}{da} = \frac{dE_p}{da} + \frac{dE_u}{da} + \frac{dE_k}{da}$$

For an elastic body, $(\frac{dE_p}{da})$ means the rate of dissipation of energy during the formation of new crack faces. This is referred to as "fracture resistance" $R$.

i.e.,

$$R = \frac{dE_p}{da}$$

Also,

$$\left( \frac{dE_w}{da} - \frac{dE_u}{da} \right) \equiv \frac{d}{da} (E_w - E_u)$$

represents changes in the external work and internal energy (or, strain energy) due to crack growth.

(Also, $-(E_w - E_u)$ denotes potential energy drop $\equiv E'$). This is represented by $G$, (for "Griffith"),

i.e.,

$$G = \frac{d}{da} (E_w - E_u)$$

where $G = "energy\,\,release\,\,rate".$

$$G = \frac{d(E_w - E_u)}{dA}$$

Thus, $A = B \delta a$
Thus, energy balance eq. becomes,

\[ G = R + \frac{dE_k}{da} \]

If changes in the kinetic energy are relatively negligible (due to slow crack growth), then the energy balance eq. becomes,

\[ G = R \]

which represents the critical crack growth condition. Thus, the condition for crack initiation and growth is \((G - R) \geq 0\).

(Note: In the example of knife cutting through a stretched sheet, \(\frac{dE_N}{da} = 0\).

\[ -\frac{dE_u}{da} = G \] (negative represents drop in pot, energy)

For a given \(G\) for a given \(G\):

\[ G = \frac{\pi \sigma^2 a}{E} \]

\[ R = 2y \]

\(R\) \(G\)

For a cracked panel example with central crack (constant load)
Griffith's Results can be represented graphically as:

S, R

For growth to occur, the critical point $P$ must be exceeded. Beyond $P$, excess energy is available to initiate crack growth.

For all values of stress below $\tau_s$, the stress that results in crack growth, one could also pose the same problem alternately: For a known crack length $a$, what is the stress that results in crack growth for different stress levels of applied stress and same crack growth resistance $\Gamma$. Evidently, for different crack lengths of applied stress and same crack growth resistance $\Gamma$, different $\sigma_s$ values exist.