Explicit Expressions for stresses, strains and displacements as Asymptotic Series

\[ \sigma_{11} = A_1 \gamma^{1/2} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{\pi}{2} \right) + 2A_2 (1) \]
\[ + A_3 \gamma^{1/2} \cos \frac{\theta}{2} \left(1 + \sin^2 \frac{\theta}{2} \right) + 2A_4 \gamma \cos \theta \]
\[ + A_5 \gamma^{3/2} \left(\cos \frac{3\theta}{2} - \frac{3}{2} \sin \theta \sin \frac{\theta}{2} \right) + 2A_6 \gamma^2 \left(1 - 3\sin^2 \theta \right) + \ldots \]

\[ \sigma_{22} = A_1 \gamma^{1/2} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{\pi}{2} \right) + A_2 (0) \]
\[ + A_3 \gamma^{1/2} \cos \frac{\theta}{2} \left(1 - \sin^2 \frac{\theta}{2} \right) + A_4 (0) \]
\[ + A_5 \gamma^{3/2} \left(\cos \frac{3\theta}{2} + \frac{3}{2} \sin \theta \sin \frac{3\theta}{2} \right) + 2A_6 \gamma^2 \sin^2 \theta + \ldots \]

\[ \sigma_{12} = A_1 \gamma^{1/2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{\pi}{2} + A_2 (0) \]
\[ - A_3 \gamma^{1/2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} + A_4 \gamma \sin \theta \]
\[ - 3A_5 \gamma^{3/2} \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} - 2A_6 \gamma^2 \sin 2\theta + \ldots \]

\[ E \varepsilon_{11} = A_1 \gamma^{1/2} \cos \frac{\theta}{2} \left[(1 - \nu)(1 - \nu) \sin \frac{\theta}{2} \sin \frac{\pi}{2} \right] + 2A_2 \]
\[ + A_3 \gamma^{1/2} \cos \frac{\theta}{2} \left[(1 + \nu)(1 + \nu) \sin^2 \frac{\theta}{2} \right] + 2A_4 \gamma \cos \theta \]
\[ + A_5 \gamma^{3/2} \left[2(1 - \nu) \cos \frac{3\theta}{2} - 3(1 + \nu) \sin \theta \sin \frac{\theta}{2} \right] \]
\[ + 2A_6 \gamma^2 \left[1 - (3 + \nu) \sin^2 \theta \right] + \ldots \]
\[ E \varepsilon_{22} = A_1 \gamma^{\frac{3}{2}} \cos \frac{\Theta}{2} \left[ (1-\gamma) + (1+\gamma) \sin \frac{\Theta}{2} \sin \frac{3\Theta}{2} \right] - 2 \gamma A_2 \]
\[ + A_3 \gamma^{\frac{3}{2}} \cos \frac{\Theta}{2} \left[ (1-\gamma) - (1+\gamma) \sin^2 \frac{\Theta}{2} \right] - 2 \gamma A_4 \gamma \cos \Theta \]
\[ + \frac{A_5}{2} \gamma^{\frac{3}{2}} \left[ 2(1-\gamma) \cos \frac{3\Theta}{2} + 3(1+\gamma) \sin \Theta \sin \frac{\Theta}{2} \right] \]
\[ + 2 \gamma A_6 \gamma^2 \left[ -\gamma + (1+3\gamma) \sin^2 \Theta \right] + \ldots \]

\[ 2E \varepsilon_{12} = \frac{A_1}{2} \gamma^{\frac{3}{2}} \sin \Theta \cos \frac{3\Theta}{2} + A_2(\Theta) \]
\[ - \frac{A_3}{2} \gamma^{\frac{3}{2}} \sin \Theta \cos \frac{\Theta}{2} - 2 \gamma A_4 \gamma \sin \Theta \]
\[ - 3A_5 \gamma^{\frac{3}{2}} \sin \Theta \cos \frac{\Theta}{2} - 2A_6 \gamma^2 \sin 2\Theta + \ldots \]

\[ EU_1 = 2A_1 \gamma^{\frac{3}{2}} \left[ (1-\gamma) \cos \frac{\Theta}{2} + (1+\gamma) \sin \Theta \sin \frac{\Theta}{2} \right] + 2A_2 \gamma \cos \Theta \]
\[ + \frac{2}{3} A_3 \gamma^{\frac{3}{2}} \left[ (1-\gamma) \cos \frac{3\Theta}{2} - (1+\gamma) \frac{3}{2} \sin \Theta \sin \frac{\Theta}{2} \right] \]
\[ + \frac{1}{2} A_4 \gamma^2 \left[ -2 \cos 2\Theta + 2(1+\gamma) \sin \Theta \sin 2\Theta \right] + \ldots \]

\[ EU_2 = 2A_1 \gamma^{\frac{3}{2}} \left[ 2 \sin \frac{\Theta}{2} - (1+\gamma) \sin \frac{\Theta}{2} \cos \frac{\Theta}{2} \right] - 2 \gamma A_2 \gamma \gamma \sin \Theta \]
\[ + \frac{2}{3} A_3 \gamma^{\frac{3}{2}} \left[ 2 \sin \frac{3\Theta}{2} - (1+\gamma) \frac{3}{2} \sin \Theta \cos \frac{\Theta}{2} \right] \]
\[ - 2 \gamma A_4 \gamma^2 \sin 2\Theta + \ldots \]
Methods of Determining $K_I$, $K_{II}$:

Finite Element Methods:

- Typically finite element method involves:
  - discretizing the continuum into elements
  - assuming "shape functions" to describe the degree of variation of displacements within an element (linear, quadratic, ...)
  - choosing the degrees of freedom at each node that connects one element to the next
  - formulating "elemental stiffness matrix" $[k_e]$ and mass matrix $[m_e]$
  - assembling elemental stiffness matrices to get "global stiffness matrix" $[K]$ and "global mass matrix" $[M]$
  - Applying displacement and force boundary conditions
  - Establishing a set of linear equations of the form $[M] \ddot{\xi} \{u\} + [K] \dot{\xi} \{u\} = \{f\}$ where $\dot{\xi} \{u\}$, $\{f\}$ denote global displacement and nodal force vectors and $\ddot{u} = \frac{\partial^2 u}{\partial t^2}$.

- In the absence of inertial forces, $[M] \ddot{\xi} \{u\}$ is negligible.
The quantities determined from a typical finite element analysis (FEA) include nodal displacements, strains, and stresses. E.g. in a 2-D analysis, nodal values of \((u_1, u_2)\), \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22})\) and \((\sigma_{11}, \sigma_{12}, \sigma_{22})\) can be obtained.

(a) Finding \(K_I, K_{II}\) from stresses

For a mode-I crack, we have \(K\)-dominant expression

\[
\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \cos \theta \left[ 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right]
\]

Along \(\theta = 0^\circ\) (\(x_1\)-axis), \(\sigma_{22} (r, \theta = 0^\circ) = \frac{K_I}{\sqrt{2\pi r}} = \frac{K_I}{\sqrt{2\pi x_1}}\)

\[
\log \sigma_{22} (r, \theta = 0^\circ) = \left[ \log \left( \frac{K_I}{2\pi} \right) \right] - \frac{1}{2} \left[ \log (2r) \right]
\]

\[
\log \left( \frac{K_I}{\sqrt{2\pi}} \right) = \text{intercept}
\]

\[
\log \sigma_{22} \text{ computed values} \quad \text{slope} \left( -\frac{1}{2} \right) \quad \text{X-X} \rightarrow \log x_1
\]
This is of the form, \( y = A + m x \) where \( m \) is the slope and \( A \) is the intercept. If the intercept is determined by extrapolating the computed values to the crack tip, one can calculate \( K_I \).

One can also obtain \( K_I \) by plotting \( \sigma_{12}(r, \theta = 0) \) against \( \frac{1}{\sqrt{x_1}} \). The "slope" of the curve can then be related to \( \frac{K_I}{\sqrt{2\pi}} \).

**Note:** The region over which one can obtain the "straight-line" behavior is affected by the geometry of the cracked body and the discretization.

On the same lines, for mode \(-II\),

\[
\sigma_{12}(r, \theta) = \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[ 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right]
\]

Again, for \( \theta = 0^\circ \),

\[
\sigma_{12}(r, \theta = 0^\circ) = \frac{K_{II}}{\sqrt{2\pi r}}
\]

As in mode \(-I\), \( \log \sigma_{12}(r, \theta = 0^\circ) \) vs. \( \log(x_1) \) plot can be used to get \( \frac{K_{II}}{\sqrt{2\pi}} \).
(b) $K_I, K_{II}$ from Displacements:

A procedure similar to the one used for stresses can also be used with displacements to determine $K_I, K_{II}$.

Consider mode-I displacement expression,

$$U_2(r, \theta) = \frac{K_I}{2 \mu \sqrt{2\pi}} \sqrt{r^2} \sin \frac{\theta}{2} \left[ (\eta+1) - 2 \cos \frac{\theta}{2} \right]$$

where $\eta = \frac{3-v}{1+v}$ ... plane stress $\mu = $ Shear Mod.

$\eta = 3-4v$ ... plane strain $\nu = $ Poisson's Ratio

Now, for $(r, \theta = \pm \pi)$,

$$U_2(r, \theta = \pi) = \frac{K_I}{2 \mu \sqrt{2\pi}} \sqrt{r^2} \left[ (\eta+1) \right]$$

by plotting $\log U_2(r, \theta = \pi)$ vs. $\log x_1$ we get,

$$\log \left[ \frac{K_I (\eta+1)}{2 \mu \sqrt{2\pi}} \right]$$

data.

slope $= +\frac{1}{2}$
Similarly for mode $-\Pi$, use crack tip displacement expression,

$$U_1(\gamma, \theta) = \frac{K_{\Pi}}{2\mu \sqrt{2\pi}} \gamma^{\frac{1}{2}} \sin \frac{\theta}{2} \left[ (\eta+1) + 2 \cos^2 \frac{\theta}{2} \right]$$

Again, along $\theta = \pm \pi$,

$$U_1(\gamma, \theta = \pi) = \frac{K_{\Pi}}{2\mu \sqrt{2\pi}} \gamma^{\frac{1}{2}} (\eta+1)$$

which can be used to determine $K_{\Pi}$.

(c) Energy Release Rate calculations

By definition, $G = \frac{d}{da} (E_W - E_u)$ ... const. load

$$G = -\frac{d}{da} E_u \ldots$$ fixed grip

Computing the rate of change of potential energy with respect to crack extension involves two numerical simulations with crack lengths ($a$) and ($a + da$). In each case by knowing global potential energy, $G$ can be computed. Post-processing of the data is rather limited since most FE packages compute strain energies. Here, one needs to pay attention to the errors due to finite difference approximation, $G \approx \frac{\Delta E_u}{\Delta a}$ for small $\Delta a$. 
(d) Virtual crack extension method

In this method, change of stiffness matrix due to crack extension is used to compute $G$. Consider crack tip mesh shown as solid lines for crack length $a$ and broken lines for $(a + da)$.

Then, recall, $G = \frac{d}{da} (E_w - E_u)$ where

$$E_w - E_u = \frac{1}{2} U^T [K] U - \frac{1}{2} U^T F$$

where $U, F, K$ denote displacement vector, force vector and stiffness matrix.

Now, $\frac{d}{da} (E_w - E_u) = \frac{1}{2} \frac{dU^T}{da} K U + \frac{1}{2} U^T \frac{dK}{da} U$

$$+ \frac{1}{2} U^T K \frac{du}{da} - \frac{dU^T}{da} F - U^T \frac{df}{da}$$

$$= \frac{dU^T}{da} K U - \frac{1}{2} \frac{dU^T}{da} K U + \frac{1}{2} U^T \frac{dK}{da} U$$

$$+ \frac{1}{2} U^T K \frac{du}{da} - \frac{dU^T}{da} F - U^T \frac{df}{da}$$
\[
\frac{d(E_N \cdot E_u)}{da} = \frac{du^T}{da} (k_u - F) - \frac{1}{2} \frac{du^T}{da} k u + \frac{1}{2} u^T R \frac{du}{da} \\
+ \frac{1}{2} u^T \frac{dk}{da} u - u^T \frac{df}{da}
\]

\[G = \frac{1}{2} u^T \frac{dk}{da} u - u^T \frac{df}{da}\]

If \[\frac{df}{da} = 0\], \[G = \frac{1}{2} u^T \frac{dk}{da} u\]

(e) \textbf{J-integral or Contour Integration}

It will be shown later that one can evaluate energy release rate \(G\) using a path-independent integral called the J-integral. For elastic cases, the J-integral on a closed path around the crack tip is equal to the energy release rate.

Knowing that \(G = \frac{k_i^2}{E}\) for mode-I, \(k_i\) can be computed. Specifically,

\[J = \int_{\Gamma} W dx_2 - T_i \frac{\partial u_i}{\partial x_1} ds\]

Where \(W = \int_{\Omega} \sigma_{ij} \epsilon_{ij}\)

and, \(T_i = \sigma_{ii} n_i\), \(i, j = 1, 2\).

\[J = \int_{\Gamma} W dx_2 - [T_1 \frac{\partial u_1}{\partial x_1} + T_2 \frac{\partial u_2}{\partial x_1}] ds\]

\(T_1 = \sigma_{11} n_1 + \sigma_{12} n_2\)
\(T_2 = \sigma_{21} n_1 + \sigma_{22} n_2\)
Alternative method for evaluating $K_I$:

Stress Method: For Mode I, recall from the Williams' expansion field,

$$
\sigma_{22}(r, \theta) = \left( \frac{KI}{2\pi} \right)^{\frac{1}{2}} f_1(\theta) + A_2 \sqrt{2} f_2(\theta) + A_3 \sqrt{2} f_3(\theta) + \ldots
$$

where $f_1(\theta), f_2(\theta) \equiv 0$, $f_3(\theta)$ are defined earlier. If only the first term is used to evaluate $K_I$ using data points obtained at finite $(r, \theta)$ locations, then $K_I = (K_I)_{\text{apparent}}$.

Along $\theta = 0^\circ$, $f_1(\theta) = 1$, $f_3(\theta) = 1$, and $r = x_1$.

Then,

$$(K_I)_{\text{apparent}} = \left[ \sigma_{22}(r, \theta = 0^\circ) \right] \sqrt{2\pi x_1}
$$

From Eq. (1) $\Rightarrow$ (for the first 3 terms of the expansion and $\theta = 0^\circ$)

$$
\sigma_{22}(r, \theta = 0^\circ) \sqrt{2\pi x_1} = K_I + A_3 \sqrt{2\pi x_1}
$$

$$(K_I)_{\text{apparent}} = K_I + A_3 \sqrt{2\pi x_1}
$$

As, $x_1 \to 0$, $(K_I)_{\text{apparent}} = (K_I)$.
Displacement Method:

Using the first three terms of the Williams' displacement field for $U_2$, we can write,

$$E U_2 (r; \theta) = 2 \frac{k_{I}}{\sqrt{2\pi}} r^2 g_1 (\theta; r) - 2 A_2 g_2 (\theta; r) r^2 + \frac{2}{3} A_3 g_3 (\theta; r) r^{3/2}$$

Along $\theta = \pm \pi$, $g_1 (\theta; r) = 2$, $g_2 (\theta; r) = 0$, $g_3 (\theta; r) = -2$

$$\therefore (k_{I})_{\text{apparent}} = \frac{E \sqrt{2\pi} U_2 (r; \theta = \pi)}{4 \sqrt{r}}$$

Eq. (3) for $(r; \theta = \pi)$ can be written as,

$$(k_{I})_{\text{apparent}} = k_{I} + (\text{constant}) (r)$$

= linear eq. for $(k_{I})_{\text{app}}$ vs. $r$

As, $r \to 0$, $(k_{I})_{\text{apparent}} = k_{I}$.