Review of Theory of Elasticity

Stress at a point $Q$ in the solid is defined as,

$$\sigma = \lim_{\Delta A \to 0} \left( \frac{R}{\Delta A} \right)$$

where $n$ is a unit vector normal to the plane under consideration, $\Delta A$ is an elemental area surrounding $Q$ and $R$ is the resultant force vector at $Q$. $R$ vector in general does not coincide with $n$ and definition of stress depends on the plane under consideration.

Cartesian Stress Components:
Stress vector on the \( \eta \)-plane can be decomposed along \( \eta_i \)-directions; let the components be \( \sigma_i \), \( i = 1, 2, 3 \).

Similarly, one can also possibly choose planes whose normals are along \( \eta_i \)-axis but pass through the same point \( O \), to define three stress vectors \( \sigma \) corresponding to \( \eta_i \)-planes.

All these can be described on an elemental tetrahedron described around \( O \). Now, it can be shown that,

\[
\sigma_{ij} = \frac{\epsilon_{ij}}{\eta_j} \quad (i, j = 1, 2, 3)
\]

\[
= \epsilon_{ij} \sigma_{ij}\] where \( \epsilon_{ij} \) represent unit vectors along \( \eta_i \)-axes.

Components \( \sigma_{ij} \) are called Cartesian (or, Cauchy's) stress components.

There are 9 stress components.
\[ \overline{\sigma}_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \]

Stress vector corresponding to an arbitrary plane passing through \( Q \) can be related to the \( 9 \) Cauchy stress component by "Cauchy's Equations":

\[ \overline{n} \cdot \overline{\sigma}_{ij} \overline{n}_{ij} = 0 \quad i=j=1,2,3 \quad \overline{n}_{x}, \overline{n}_{y}, \overline{n}_{z} \]

\[ \overline{n}_{1} = \sigma_{11} \cos(\overline{e}_{1}, \overline{n}) + \sigma_{21} \cos(\overline{e}_{2}, \overline{n}) + \sigma_{31} \cos(\overline{e}_{3}, \overline{n}) \]

\[ \overline{n}_{2} = \sigma_{12} \cos(\overline{e}_{1}, \overline{n}) + \sigma_{22} \cos(\overline{e}_{2}, \overline{n}) + \sigma_{32} \cos(\overline{e}_{3}, \overline{n}) \]

\[ \overline{n}_{3} = \sigma_{13} \cos(\overline{e}_{1}, \overline{n}) + \sigma_{23} \cos(\overline{e}_{2}, \overline{n}) + \sigma_{33} \cos(\overline{e}_{3}, \overline{n}) \]

where \( \overline{e}_{1}, \overline{e}_{2}, \overline{e}_{3} \) are unit vectors along \( x_{1}, x_{2}, x_{3} \) directions.

Equality of cross-shears

By momentum balance condition, it can be shown that \[ \overline{n} \cdot \overline{\sigma}_{ij} \overline{n}_{ij} = 0 \]

\[ \overline{e} \cdot \overline{\sigma}_{ij} \overline{e}_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \Rightarrow \text{Symmetric Matrix} \]
Equilibrium Equations:

3 eqns: \( \frac{\partial \sigma_{ij}}{\partial x_j} + \int \kappa_i = F_i \) (i = 1, 2, 3)

\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = F_1 \]

\[ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = F_2 \]

\[ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = F_3 \]

Cauchy's stress representation

(components on the hidden faces are not shown)
Coordinate Transformation of Stresses:

Let $x_i$ and $x'_i$ denote two Cartesian coordinate systems at $Q$. Then, stress components in the prime coordinate system can be expressed in terms of those in the unprimed coordinates as:

$$
\sigma'_{ks} = \sigma_{ij} \cdot l_{ki} \cdot l_{sj} \quad (i, j, k, s = 1, 2, 3)
$$

(summation over repeated index is assumed)

Where $l_{ki} = \cos(x'_k, x_i)$ etc.

**Note:** $l_{12} \neq l_{21}$, ... , ...

Similarly, we can write the inverse transformation as:

$$
\sigma_{ij} = \sigma_{ks}' \cdot l_{ik} \cdot l_{js}
$$

**Example:** $\sigma_{12}' = \sigma_{ij} \cdot l_{1k} \cdot l_{2s} = \sigma_{ij} \cdot l_{11} \cdot l_{2j} + \sigma_{ij} \cdot l_{12} \cdot l_{2j} + \sigma_{ij} \cdot l_{13} \cdot l_{2j}

= \sigma_{11} \cdot l_{11} \cdot l_{21} + \sigma_{12} \cdot l_{11} \cdot l_{22} + \sigma_{13} \cdot l_{11} \cdot l_{23}

+ \sigma_{21} \cdot l_{12} \cdot l_{21} + \sigma_{22} \cdot l_{12} \cdot l_{22} + \sigma_{23} \cdot l_{12} \cdot l_{23}

+ \sigma_{31} \cdot l_{13} \cdot l_{21} + \sigma_{32} \cdot l_{13} \cdot l_{22} + \sigma_{33} \cdot l_{13} \cdot l_{23}$
Principal stresses:

There are special planes passing through $Q$ such that $\sigma_{ij}$'s matrix contains only normal stresses and all the shear stresses vanish. The normal stresses corresponding to these planes are called principal stresses and the planes on which they act (or, directions) are called principal planes. Principal stresses & directions can be found by determining the eigenvalues and eigen vector of

$$\begin{vmatrix}
\sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} - \sigma & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma
\end{vmatrix} = 0$$

where $\delta_{ij}$ is called Kronecker delta

$\delta_{ij} = 0$ for $i \neq j$

$= 1$ for $i = j$

$\Rightarrow \sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$

where $I_1, I_2, I_3$ are called stress invariants (unaffected by the choice of coordinates)
\[ I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \]

\[ I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} \]

\[ I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix} \]

Solution to "stress cubic" gives principal stress magnitudes: \( \sigma_1 > \sigma_2 > \sigma_3 \).

Back substitution of \( \sigma_1, \sigma_2, \sigma_3 \) into the system of linear equations below gives principal stress directions.

For \( \sigma_1 \):

\[
\begin{pmatrix}
\sigma_{11} - \sigma_1 & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} - \sigma_1 & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_1
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{pmatrix}
= 0
\]

Where \( \eta_1, \eta_2, \eta_3 \) represent direction cosines of \( \sigma_1 \). Note: The principal directions should satisfy the condition \( \eta_1^2 + \eta_2^2 + \eta_3^2 = 1 \).
Plane stress:

If thickness $t <<$ inplane dimensions and subjected to inplane loading leading stress components $\sigma_{ij}$ $(\sigma_{21}, \sigma_{22}, \sigma_{23})$ can be neglected.

$$\sigma' = \sigma'_{\theta} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

Then, stress transformation equations become:

$$\sigma'_{ks} = \delta_{ij} \delta_{kl} \sigma_{ij}$$, $i, j, k, s = 1, 2, 3$.

Long Hand:

$$\sigma'_{11} = \frac{(\sigma_{11} + \sigma_{22})}{2} + \left(\frac{(\sigma_{11} - \sigma_{22})}{2}\right) \cos 2\theta + \sigma_{12} \sin 2\theta$$

$$\sigma'_{22} = \left(\frac{(\sigma_{11} + \sigma_{22})}{2}\right) - \left(\frac{(\sigma_{11} - \sigma_{22})}{2}\right) \cos 2\theta - \sigma_{12} \sin 2\theta$$

$$\sigma'_{12} = \sigma_{12} \cos 2\theta - \left(\frac{(\sigma_{11} - \sigma_{22})}{2}\right) \sin 2\theta$$

They can also be expressed as:

$$\sigma'_{11} + \sigma'_{22} = \sigma_{11} + \sigma_{22}$$

$$\left(\sigma'_{22} - \sigma'_{11} + 2i \sigma'_{12}\right) = (\sigma_{22} - \sigma_{11} + 2i \sigma_{12}) e^{i2\theta}$$

where $e^{i2\theta} = (\cos 2\theta + i \sin 2\theta)$ and $i = \sqrt{-1}$.
Planes on which shear stresses $\tau_1$, vanish are called principal planes and the corresponding values of $\sigma_{11}' = \sigma_1$ and $\sigma_{22}' = \sigma_2$ ($\sigma_1 > \sigma_2$) are called principal stresses.

$$
\sigma_1, \sigma_2 = \frac{1}{2} \left( \sigma_{11} + \sigma_{22} \pm \sqrt{\left( \sigma_{11} - \sigma_{22} \right)^2 + 4 \tau_1^2} \right)
$$

and principal directions $\theta_p$ are given by:

$$
\tan 2\theta_p = \frac{2\tau_1}{\sigma_{11} - \sigma_{22}} \quad (\ll 2 \text{ angles})
$$

Maximum in-plane shear stress $\tau_{\text{max}}$ is:

$$
\tau_{\text{max}} = \frac{\sigma_1 - \sigma_2}{2}
$$

and maximum in-plane shear stress direction $\theta_s$,

$$
\tan 2\theta_s = -\frac{\sigma_{11} - \sigma_{22}}{2\tau_1}
$$

Mohr's circle representation ($\sigma_1 > \sigma_2 > 0$)
\[ \sigma_1 > 0, \sigma_2 < 0 \]

\[ \sigma_{\text{max}} = \frac{1}{2} (\sigma_{11} + \sigma_{22}) \]

**Note:**

For plane stress, \( \sigma_3 = 0 \).

When \( \sigma_1 \) and \( \sigma_2 \) are of the same sign, the max. shear stress at a point (not the max. in-plane shear stress) is determined by:

\[ (\tau_{\text{max}})^{3D} = \left| \frac{\sigma_1}{2} \right| \text{ or } \left| \frac{\sigma_2}{2} \right| \]

If \( \sigma_1 \) and \( \sigma_2 \) are of the opposite sign:

\[ (\tau_{\text{max}})^{3D} = (\tau_{\text{max}})^{\text{in-plane}} = \left| \frac{\sigma_1 - \sigma_2}{2} \right| \]

**Equilibrium Eqs. in 2-D plane stress condition**

**Cartesian coordinates:**

\[ \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = F_1 \]

\[ \frac{\partial \sigma_{22}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = F_2 \]

**In polar coordinates:**

\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \left( \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \right) = F_r \]

\[ \frac{\partial \sigma_{\theta\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{rr}}{\partial \theta} + \frac{2 \sigma_{r\theta}}{r} = F_\theta \]
Strain - Displacement Relations

If $u_1$, $u_2$ and $u_3$ represent displacement components in $x_1$, $x_2$ and $x_3$ directions, then,

Normal Strains

$$
\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \\
\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \\
\varepsilon_{33} = \frac{\partial u_3}{\partial x_3}
$$

Shear Strains

$$
\varepsilon_{12} = \frac{1}{2} \frac{\partial u_1}{\partial x_2} + \frac{1}{2} \frac{\partial u_2}{\partial x_1} \\
\varepsilon_{23} = \frac{1}{2} \frac{\partial u_2}{\partial x_3} + \frac{1}{2} \frac{\partial u_3}{\partial x_2} \\
\varepsilon_{31} = \frac{1}{2} \frac{\partial u_3}{\partial x_1} + \frac{1}{2} \frac{\partial u_1}{\partial x_3}
$$

Cylindrical coordinates ($r, \theta, z(\equiv x_3)$)

Normal Strains

$$
\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
\varepsilon_{zz} = \frac{\partial u_z}{\partial z}
$$

Shear Strains

$$
\varepsilon_{r\theta} = \frac{1}{2} \frac{\partial u_r}{\partial \theta} \equiv \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \right) \\
\varepsilon_{\theta z} = \frac{1}{2} \frac{\partial u_\theta}{\partial z} = \frac{1}{2} \left( \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \\
\varepsilon_{z r} = \frac{1}{2} \frac{\partial u_z}{\partial r} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)
$$

Strain Transformation Eqs.

$$
\varepsilon_{ks} = \varepsilon_{ij} l_{ki} l_{sj},
$$

$i$, $j$, $k$, $s = 1, 2, 3,$

$($or, $x, y, z$)$
Principal Strains:

As in the case of stresses, there are three mutually orthogonal planes at \( \theta \) which experience only extension or contraction but do not experience any shear strain. These are called principal strain planes and they coincide with those of principal stresses.

Principal strains: \( \varepsilon_1 > \varepsilon_2 > \varepsilon_3 \)

\( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are solutions of a "strain-cubic" equation:

\[
\varepsilon^3 - J_1 \varepsilon^2 + J_2 \varepsilon - J_3 = 0
\]

where \( J_1, J_2, J_3 \) are called "strain invariants":

\[
J_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}
\]

\[
J_2 = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{vmatrix} + \begin{vmatrix} \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{23} & \varepsilon_{33} \end{vmatrix}
\]

\[
J_3 = \text{det} (\varepsilon_{ij}) \text{ where } (\varepsilon_{ij}) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}
\]

is the 3-D strain matrix.
Strain compatibility conditions

Recall, \[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

In the above 6 independent strains one related to 3 displacement components. Therefore, determining \( \varepsilon_{ij}(x_1, x_2, x_3) \) by knowing \( u_i(x_1, x_2, x_3) \) is unique. However, the reverse process is not. Hence, one needs additional restrictions to be imposed so that "continuum mechanics" assumptions are not violated. These restrictions are called compatibility conditions:

\[ \frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_i} - \frac{\partial^2 \varepsilon_{kl}}{\partial x_l \partial x_j} = 0 \]

\[ \Rightarrow \] represents a set of 81 eqs. But, only a set of 6 equations are independent.
Plane strain: This special case occurs when a body has thickness, say, in the $x_3$-direction, is very large compared to its dimensions in the $x_1$, $x_2$ directions (constrained in $x_3$-direction).

This leads to $\varepsilon_{3i} = 0$ ($i = 1, 2, 3$) $|\varepsilon_{3i}| = \varepsilon_{12}$

'\varepsilon_{ij}\', only non-zero strains are:

$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$, $\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$ and $\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$

Also, $\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 0$, leads to $u_3 = 0$ or const.,

and $\varepsilon_{13} = \varepsilon_{23} = 0$ leads to $U_3 = 0$ or const.

Strain transformation equations are identical to those for plane stress case if $\sigma$ is replaced by $\varepsilon$.

\[ E_1, E_2 = \left(\frac{E_{11} + E_{22}}{2}\right) + \sqrt{\left(\frac{E_{11} - E_{22}}{2}\right)^2 + E_{12}^2} \]

and $\tan 2\theta_p = \left(\frac{2E_2}{E_{11} - E_{22}}\right)$
Also, strain compatibility eq. for plane strain is,

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_2^2} = 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

**Stress-strain Relations (Hooke's Law)**

For a linear elastic material, stresses and strains are related as,

\[\text{Strain-strain Relations}\]

\[\left( \varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \delta_{ik} \varepsilon_{kj} \right)\]

\[\text{Long Hand:}\]

\[\varepsilon_{11} = \frac{1}{E} \sigma_{11} - \frac{\nu}{E} (\sigma_{22} + \sigma_{33})\]

\[\varepsilon_{22} = \frac{1}{E} \sigma_{22} - \frac{\nu}{E} (\sigma_{11} + \sigma_{33})\]

\[\varepsilon_{33} = \frac{1}{E} \sigma_{33} - \frac{\nu}{E} (\sigma_{11} + \sigma_{22})\]

\[\varepsilon_{12} = \frac{1}{2\mu} \sigma_{12}\]

\[\varepsilon_{23} = \frac{1}{2\mu} \sigma_{23}\]

\[\varepsilon_{13} = \frac{1}{2\mu} \sigma_{13}\]

Where \(E\): Young's modulus and \(\mu = \frac{E}{2(1+\nu)}\) = shear modulus

\(\nu\): Poisson's ratio

\[\text{Stress-strain relations}\]

\[\left( \sigma_{ij} = \frac{E}{1+\nu} \left( \varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{kk} \delta_{ij} \right) \right)\]

\[\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)} \left( (1-\nu) \varepsilon_{11} + \nu (\varepsilon_{22} + \varepsilon_{33}) \right)\]

\[\sigma_{22} = \ldots \]

\[\sigma_{33} = \ldots \]
Stress-Strain Relations for plane stress and plane strain

**Plane Stress**:

\[ \sigma_{3i} = 0 \quad i = 1, 2, 3. \]

\[ \varepsilon_{11} = \frac{1}{E} \left( \sigma_{11} - \nu \sigma_{22} \right) \]

\[ \varepsilon_{22} = \frac{1}{E} \left( \sigma_{22} - \nu \sigma_{11} \right) \]

\[ \varepsilon_{12} = \frac{1 + \nu}{E} \sigma_{12} \]

and,

\[ \varepsilon_{33} = -\frac{\nu}{E} \left( \sigma_{11} + \sigma_{22} \right) \]

**Plane Strain**:

\[ \varepsilon_{3i} = 0 \]

\[ \sigma_{33} = 0 \quad \text{yields} \quad \sigma_{33} = \nu (\sigma_{11} + \sigma_{22}) \]

\[ \varepsilon_{11} = \frac{1 + \nu}{E} \left[ (1 - \nu) \sigma_{11} - \nu \sigma_{22} \right] \]

\[ \varepsilon_{22} = \frac{1 + \nu}{E} \left[ (1 - \nu) \sigma_{22} - \nu \sigma_{11} \right] \]

\[ \varepsilon_{12} = \frac{1 + \nu}{E} \sigma_{12} \]

**Note**: Replace \( E \) by \( \frac{E}{1 - \nu^2} \) and \( \nu \) by \( \frac{\nu}{1 - \nu^2} \) in plane stress equations to get plane strain \( \varepsilon - \sigma \) relations.
Stress Deviator: A state of stress at a generic point \( Q \) can be decomposed as:

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{pmatrix} = \begin{pmatrix}
\sigma_{11} - \rho & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} - \rho & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33} - \rho
\end{pmatrix} + \rho \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where \( \rho = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \).

In matrix \( M_1 \), \( I_1 = 0 \) and in \( M_2 \), \( I_1 = 3 \rho \).

\( M_1 \) represents shape changes (distortion) while \( M_2 \) represents volume changes (dilatation).

Strain Energy: In an ideal elastic solid, work done by external forces is stored as strain energy. For no dissipation, strain energy is completely recoverable upon unloading.

In a 1-D case, strain energy \( E_u \) is,

\[
E_u = \frac{1}{2} \int_{V} \sigma_{11} \varepsilon_{11} \, dv.
\]

If \( E'_u \) represents strain energy density, then

\[
E'_u = \int \sigma_{11} \, d\varepsilon_{11} = \frac{1}{2} E \varepsilon_{11}^2 = \frac{1}{2} \sigma_{11} \varepsilon_{11} = \frac{1}{2E} \sigma_{11}^2.
\]

Similarly, elastic strain energy due to pure shear is

\[
E'_u = \frac{1}{2} \mu \left(2 \varepsilon_{12}\right)^2 = \frac{1}{2} \mu_1 \varepsilon_{12}^2 = \frac{\sigma_{12}^2}{2\mu}.
\]
In a general 3-D case,

\[ E_u' = \int \sigma_{ij} \, d\varepsilon_{ij} \quad i,j = 1,2,3. \]

Stress or strain can be eliminated using stress-strain relations to get,

\[ E_u' = \frac{1}{2E} \left[ \left( \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 \right) \right. \\
-2\nu \left( \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11} \right) \\
\left. + 2(1+\nu) \left( \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 \right) \right] \]

OR

\[ E_u' = \frac{E}{2} \left[ -\frac{\nu}{(1+\nu)(1-2\nu)} \left( \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \right)^2 \\
+ \frac{2}{1+\nu} \left( \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 \right) \\
+ \frac{2}{1+\nu} \left( \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2 \right) \right] \]

Then \( E_u = \int_V E_u' \, dV \) where \( V = \) Volume.

In terms of principal stresses,

\[ E_u' = \frac{1}{2E} \left[ \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - 2\nu \left( \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1 \right) \right] \]

Total strain energy density can be decomposed into two parts (a) the one due to volume changes \( (E_u') \), (b) the one due to distortion or shape change \( (E_u) \).
1. \[ E_u' = (E_u')_v + (E_u')_d \]

It can be shown that

\[ (E_u')_d = \frac{1+\nu^2}{3E} \left[ \frac{1}{a_1^2 + a_2^2 + a_3^2} - \left( 1 + \frac{a_1^2 + a_2^2 + a_3^2}{a_1^2 + a_2^2 + a_3^2} \right) \right]. \]

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**Energy Stored in Simple Configurations**

1. **Axial Loading:** \[ E_u = \int \frac{P^2}{2AE} dx \]

   where \( P \) is the load (function of \( x \)), \( x \) is the axis along which load is acting, and \( A \) is the c.s. area normal to the load.

2. **Direct Shear:** \[ E_u = \int \frac{F^2}{2AM} dx \]

   where \( F \) is the shear force on area \( A \) and \( M \) shear modulus.

3. **Bending:** \[ E_u = \int \frac{M^2}{2EI} dx \]

   where \( M \) is the moment (function of \( x \)), \( I \) : Moment of Inertia about moment axis.

4. **Torsion:** \[ E_u = \int \frac{T^2}{2\mu J} dx \]

   where \( T \) is the torque (function of \( x \)), \( J \) : Polar moment of inertia of the c.s.
Airy's Stress Function (no body forces)

For 2-D problems, it is possible to find a function \( \chi(x_1, x_2) \) such that:
(a) it satisfies stress equilibrium eqs.
(b) strain compatibility conditions
(c) boundary conditions

Let \( \chi(x_1, x_2) \) be chosen such that,

\[
\sigma_{11} = \frac{\partial^2 \chi}{\partial x_2^2}, \quad \sigma_{12} = -\frac{\partial^2 \chi}{\partial x_1 \partial x_2}, \quad \sigma_{22} = \frac{\partial^2 \chi}{\partial x_1^2}
\]

which satisfy equilibrium eqs.

Recall, strain compatibility in 2-D case:

\[
\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}
\]

Using strain-stress relations, (say, for plane stress) \( \varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E} \sigma_{22} \), \( \varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E} \sigma_{11} \), \( \varepsilon_{12} = \frac{\sigma_{12}}{2\mu} \)

we get,

\[
\frac{\partial^4 \chi}{\partial x_1^4} + 2 \frac{\partial^4 \chi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \chi}{\partial x_2^4} = 0
\]

\[ \nabla^2 (\nabla^2 \chi) = 0 \] where \( \nabla^2 = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \)

\( \Rightarrow " \)biharmonic eqn."

From \( \sigma - \chi \) relations above, \( \nabla^2 \chi = \sigma_{11} + \sigma_{22} \)

\( \nabla^2 (\sigma_{11} + \sigma_{22}) \) is harmonic,
\( \nabla^2 (\nabla^2 \chi) = 0 \) satisfies all the necessary equations of plane elasticity except the boundary conditions. \( \Rightarrow \) it offers a method of finding solution to 2-D elasticity problems.

\textbf{Bi-harmonic equation in polar coordinates}

\[
\nabla^2 \chi (r, \theta) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \chi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}
\]

\( \Rightarrow \)

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \chi}{\partial r} \right) + \frac{1}{r} \frac{\partial \chi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \right) = 0
\]

\textbf{Stress components in terms of } \chi (r, \theta) : 

\[
\sigma_{rr} = \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} \quad \text{(radial stress)}
\]

\[
\sigma_{\theta \theta} = \frac{\partial^2 \chi}{\partial r^2} \quad \text{(hoop stress)}
\]

\[
\sigma_{r \theta} = \frac{1}{r^2} \frac{\partial \chi}{\partial \theta} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \chi}{\partial \theta} \right) \quad \text{(shear stress)}
\]