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Versal Deformation and Local Bifurcation Analysis of Time-Periodic Nonlinear Systems

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Abstract. In this study a local semi-analytical method of quantitative bifurcation analysis for time-periodic nonlinear systems is presented. In the neighborhood of a local bifurcation point the system equations are simplified via *Lyapunov–Floquet transformation* which transforms the linear part of the equation into a dynamically equivalent time-invariant form. Then the *time-periodic center manifold reduction* is used to separate the 'critical' states and reduce the dimension of the system to a possible minimum. The center manifold equations can be simplified further via *time-dependent normal form theory*. For most codimension one cases these nonlinear normal forms are completely time-invariant. *Versal deformation* of these time-invariant normal forms can be found and the bifurcation phenomenon can be studied in the neighborhood of the critical point. However, in general, it is not a trivial task to find a quantitatively correct versal deformation for time-periodic systems. In order to do so, one must find a relationship between the bifurcation parameter of the original time-periodic system and the versal deformation parameter of the time-invariant normal form. Essentially one needs to find the eigenvalues of the fundamental solution matrix of the time-periodic problem in terms of the system parameters, which, in general, cannot be done due to computational difficulties. In this work two ideas are proposed to achieve this goal. The eigenvalues of the fundamental solution matrix can be related to the versal deformation parameter by sensitivity analysis and an approximation of any desired order can be obtained. This idea requires a symbolic computational procedure which can be very time consuming in some cases. An alternative method is suggested for faster results in which a second or higher order curve fitting technique is used to find the relationship. Once this relationship is established, closed form post-bifurcation steady-state solutions can be obtained for flip, symmetry breaking, transcritical and secondary Hopf bifurcations. Unlike averaging and perturbation methods, the proposed technique is applicable at any bifurcation point in the parameter space. As physical examples, a simple and a double pendulum subjected to periodic parametric excitation are considered. A simple two degrees of freedom model is also studied and the results are compared with those obtained from the traditional averaging method. All results are verified by numerical integration. It is observed that the proposed technique yields results which are very close to the numerical solutions, unlike the averaging method.

Keywords: Nonlinear, time-periodic systems, local bifurcations, versal deformation.

1. Introduction

Nonlinear time-varying equations whose coefficients are explicit periodic functions of time arise from the mathematical modeling of various engineering and physical phenomena. Equations of this kind can also be obtained from stability and bifurcation analyses of periodic motions of nonlinear systems. The study of stability and bifurcations provides a basic understanding of the dynamics of a system, which is an essential step before one can pursue the design and control aspects of the problem.

The stability of an equilibrium position (or a periodic motion) in nonlinear systems can be lost in different ways. If after the loss of stability, similar to what happens in linear systems, the local solutions diverge away from the equilibrium without any limit, it is so called a sharp loss of stability. On the other hand if there is a small attractive domain born around the equilibrium (or periodic orbit), which limits the diverging solutions from inside and attracts solutions from outside in the neighborhood, then we say that the stability is lost softly. In some practical applications such systems may still be considered stable as long as the attractive domain remains small enough, even though theoretically the stability has already been lost in the linear sense. In this paper an analytical method is proposed to study the nature of the loss of stability, compute post-bifurcation steady-state solutions and determine the size of the attractive domain.

The techniques of bifurcation analysis are well developed for autonomous systems. *Center manifold reduction*, *normal form theory* and*versal deformations* [1] have been the important tools of investigation for such systems. However, when it comes to dealing with time-periodic equations, a number of difficulties arise. The powerful tools used for autonomous systems cannot be directly applied due to the time-varying nature of the problem. One either needs to develop all new methods and theories to deal with this problem, or find ways to make the periodic system amenable to the application of the known methods, via some transformations. In time-invariant systems it is also relatively easy to determine the influence of bifurcation parameters on the critical eigenvalues of systems. In time-periodic systems, a direct relationship can only be established for commutative systems or for weakly excited systems via perturbation or averaging methods. For more general problems (including strong parametric excitation), new techniques must be developed to establish this connection, which is essential when one wants to study the dynamics quantitatively not just at bifurcation points, but also in their neighborhood.

The method proposed here is based on the *Lyapunov–Floquet transformation* technique [2, 3] that makes the linear part of periodic systems time-invariant. To a periodic system with constant linear part the *time-periodic center manifold theory* [4, 5] can be applied to reduce the dimension to that of its center manifold, separating the critical variables from the rest. This theory allows us to study the dynamics of a lower-dimensional system, describing the 'critical' behavior only. The system on the center manifold can be further simplified using the *time-dependent normal form theory* [6] to eliminate as many nonlinear terms as possible. It has been shown in [7] that for codimension one bifurcations the normal form on the center manifold may become time invariant and simple enough to be solved in a closed form. Now, to analyze the dynamics around the critical point, *versal deformation* [1] of the normal form equation can be constructed. The versal deformation equation symbolizes a small change in the bifurcation parameter as a small change in the eigenvalues of the linear part of the normal form equation. Such equations have been derived, strictly on a theoretical basis, by Chow and Wang [1, 8] even for codimension two bifurcations, but those normal forms give only qualitative information about the system behavior, and are of little use for engineering analysis and control. For practical purposes a quantitative relationship needs to be established between the versal deformation parameter and the original system parameter. Once this relation is determined, the versal deformation equation may often be simple enough to be solved in a closed form and solutions can be found in the neighborhood of the bifurcation point. The results can be transformed to the original variables to provide physically meaningful information. However, it should be noted that the Lyapunov–Floquet transformation and the center manifold relation computed at the bifurcation point are reasonably accurate only in a small neighborhood and introduce a small error into the solution in the original domain.

In the study of stability and bifurcations of time-periodic nonlinear systems traditionally averaging and perturbation methods [9] have been used. These techniques are quite limited due to the fact that they can be applied only to systems where the linear time-periodic terms as well as the nonlinearities explicitly depend on a small parameter. Also, in these methods the time-periodic linear part is treated as a perturbation, therefore, the prediction of stability and bifurcation points is not accurate since the linear part is not preserved. Point-mapping techniques [10] have also been used, but they involve elaborate numerical calculations and are not suitable for the type of analysis suggested here. The other main disadvantage of these techniques is that they provide information only in the transformed domain, and the results cannot be converted back to the original physical coordinates.

The proposed method does not require the linear periodic terms to depend on a small parameter and it is applicable at every point in the entire space of parameters. It can provide meaningful post-bifurcation solutions in the original coordinates and it is very efficient computationally. To illustrate the method, first a commutative system undergoing flip bifurcation is studied. Then a symmetry breaking bifurcation of a simple pendulum with periodic base excitation is considered. The secondary Hopf bifurcation is studied on the example of a double inverted pendulum subjected to a periodic load. Finally, a two degrees of freedom system with a small parameter is analyzed via averaging as well as the proposed method and a comparison is made. Numerical simulations are provided in each case to demonstrate the effectiveness of the method.

2. The Method of Analysis

2.1. Statement of the Problem

In this work parameter-dependent nonlinear time-periodic systems of the form

$$\dot{\mathbf{x}}(t;\boldsymbol{\alpha}) = \mathbf{A}(t;\boldsymbol{\alpha})\mathbf{x}(t;\boldsymbol{\alpha}) + \mathbf{f}_2(\mathbf{x},t;\boldsymbol{\alpha}) + \cdots + \mathbf{f}_k(\mathbf{x},t;\boldsymbol{\alpha}) + \mathbf{O}(|\mathbf{x}|^{k+1},\boldsymbol{\alpha},t) \tag{1}$$

are studied, where $\mathbf{A}(t+T;\boldsymbol{\alpha}) = \mathbf{A}(t,\boldsymbol{\alpha})$ is an $n \times n$ state matrix, $\mathbf{x}$ and $\mathbf{f}_i$ are n vectors, $\mathbf{f}_i$ contain monomials of $\mathbf{x}$ of order i and are T-periodic in time and $\boldsymbol{\alpha}$ is an m ($m \leq n$) vector containing the parameters of the system. Systems of this kind can be obtained by expanding a general nonlinear periodic system about a known equilibrium position or a periodic solution.

First of all, we need to find critical values of the parameter $\boldsymbol{\alpha}$, points where bifurcations occur (unless such a point is given in the actual problem at hand). One approach, of course, is to perform a numerical search and find $\boldsymbol{\alpha}_c$ by a trial and error method. In the other approach, first an analytical technique is used to compute the state transition matrix associated with the linear part of Equation (1) symbolically in terms of the system parameters. Such a technique has been developed in [11, 12], which involves *Picard iteration* and *Chebyshev polynomial expansion*. Then the bifurcation surfaces of Equation (1) can be computed in the m-dimensional space of parameter α and sets of critical values α_c can be determined for various bifurcations.

In the following the dynamics and stability of an equilibrium in the neighborhood of a bifurcation point is investigated. Further, it is desired to obtain a quantitative estimate of the post-bifurcation solution and compute the size and rate of growth of the attractive domain in case of a soft loss of stability. First, a brief discussion on the mathematical tools needed in this study is provided.

2.2. Lyapunov–Floquet Transformation

Consider the nonlinear T-periodic system given by Equation (1) and assume that for a given set of parameters $\boldsymbol{\alpha}_c$ the linear part of the system is critical, i.e., the nonlinear system is undergoing bifurcation. Substituting α_c into Equation (1), we get a parameter-independent equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}_2(\mathbf{x}, t) + \cdots + \mathbf{f}_k(\mathbf{x}, t) + \mathbf{O}(|\mathbf{x}|^{k+1}, t), \tag{2}$$

where $\mathbf{A}(t+T) = \mathbf{A}(t)$ is an $n \times n$ state matrix and $\mathbf{x}$ and $\mathbf{f}_i$ are n vectors, $\mathbf{f}_i$'s contain monomials of $\mathbf{x}$ of order i and are T-periodic in time. From assumption, matrix $\mathbf{A}(t)$ is critical, that is, n_1 of its Floquet multipliers lie on the unit circle of the complex plane and n_2 ($n_1+n_2 = n$) multipliers lie inside the unit circle. From Floquet Theory, the state transition matrix (STM) $\Phi(t)$ of the linear part of the system can be factored as

$$\Phi(t) = \mathbf{L}(t)\,\mathrm{e}^{\mathbf{C}t}, \tag{3}$$

where $\mathbf{L}(t)$ is the T-periodic complex Lyapunov–Floquet (L–F) transformation matrix and $\mathbf{C}$ is a complex constant matrix, in general. $\Phi(t)$ can also be factored as

$$\Phi(t) = \mathbf{Q}(t)\,\mathrm{e}^{\mathbf{R}t}, \tag{4}$$

where $\mathbf{Q}(t)$ is the real L–F transformation matrix, and $\mathbf{R}$ is a real constant matrix. If all the Floquet multipliers (the eigenvalues of the Floquet transition matrix, $\Phi(T)$) are in the left half of the complex plane, then there exist both the T-periodic complex and the $2T$-periodic real L–F transformations and the real one has the symmetry property $\mathbf{Q}(t+T) = -\mathbf{Q}(t)$. If some of the multipliers are in the right half plane, then the two transformations are the same and they are T-periodic and real. In this work the real L–F transformation $\mathbf{Q}(t)$ will be used. From the factorization (4) both $\mathbf{Q}(t)$ and $\mathbf{R}$ can easily be computed following the studies reported by Sinha and Pandiyan [2] and Sinha et al. [3]. Applying the transformation

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{y} \tag{5}$$

Equation (2) becomes

$$\dot{\mathbf{y}}(t) = \mathbf{R}\mathbf{y}(t) + \mathbf{Q}^{-1}(t)\mathbf{f}_2(\mathbf{y}, t) + \cdots + \mathbf{Q}^{-1}(t)\mathbf{f}_k(\mathbf{y}, t) + \mathbf{O}(|\mathbf{y}|^{k+1}, t), \tag{6}$$

where the linear part is time-invariant, making the equation amenable for the use of local methods such as *periodic center manifold reduction* and *time-dependent normal form theory.*

2.3. Center Manifold Reduction

Since in Equation (6) the linear part is time-invariant, we can transform matrix $\mathbf{R}$ into its Jordan canonical form employing a similarity transformation $\mathbf{y} = \mathbf{M}\mathbf{z}$. Then Equation (6) becomes

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{G}(\mathbf{z}, t), \tag{7}$$

where $\mathbf{J}$ denotes the Jordan form of matrix $\mathbf{R}$. According to the assumption made on the Floquet multipliers of matrix $\mathbf{A}(t)$, matrix $\mathbf{J}$ has n_1 eigenvalues with zero and n_2 with negative real parts, respectively. Equation (7) can be partitioned as

$$\begin{Bmatrix} \dot{\mathbf{z}}_c \\ \dot{\mathbf{z}}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_c \\ \mathbf{z}_s \end{Bmatrix} + \begin{Bmatrix} \mathbf{G}_c(\mathbf{z}_c, \mathbf{z}_s, t) \\ \mathbf{G}_s(\mathbf{z}_c, \mathbf{z}_s, t) \end{Bmatrix}, \tag{8}$$

where $\mathbf{G}_i = \mathbf{M}^{-1}\mathbf{Q}^{-1}(\mathbf{f}_{i2}+\cdots+\mathbf{f}_{ik}) := \mathbf{g}_{i2}+\cdots+\mathbf{g}_{ik}$, $i = c, s$. Subscripts c and s denote the critical and stable states, respectively. According to the time-periodic center manifold theory [4, 5], there exists a nonlinear transformation with time-periodic coefficients of the form

$$\mathbf{z}_s = \mathbf{h}_c(\mathbf{z}_c, t) \tag{9}$$

which decouples the critical states from the stable ones in Equation (8). Therefore, the equation on the center manifold takes the form

$$\dot{\mathbf{z}}_c = \mathbf{J}_c\mathbf{z}_c + \mathbf{G}_c(\mathbf{z}_c, t) = \mathbf{J}_c\mathbf{z}_c + \sum_{r=2}^{k} \sum_{\sum m_i = r} \mathbf{g}_{cr,(m_1,\ldots,m_{n_1})}(t) z_1^{m_1} \ldots v_{n_1}^{m_{n_1}}. \tag{10}$$

This equation is already significantly simplified, although it still may contain a large number of nonlinear terms with periodic coefficients. However, in some of the codimension one critical cases it may already be simple enough to be solved in a closed form.

2.4. Application of Time-Dependent Normal Form Theory

If the center manifold equation is not simple enough, it can be simplified further by eliminating as many nonlinear terms as possible. Since Equation (10) has a time-invariant linear part, a direct application of the method of time-dependent normal forms for equations with periodic coefficients is possible, as shown by Arnold [6]. It has been shown by Sinha et al. [7] that in most cases of codimension one bifurcations the normal form is completely time-invariant. According to the time-dependent normal form theory we can use a near-identity transformation of the form

$$\mathbf{z}_c = \mathbf{v} + \mathbf{h}_r(\mathbf{v}, t), \tag{11}$$

where $\mathbf{h}_r(\mathbf{v}, t)$ is a formal power series in $\mathbf{v}$ of degree r with unknown periodic coefficients with period $2T$. This transformation allows us to eliminate all nonresonant rth order nonlinear terms (for details, see [5, 7]). If the system is hyperbolic (stable or unstable) we can eliminate all the nonlinear terms. In case of the critical system studied here, resonant terms remain and Equation (10) takes its simplest nonlinear form

$$\dot{\mathbf{v}} = \mathbf{J}_c\mathbf{v} + \mathbf{W}(\mathbf{v}, t) = \mathbf{J}_c\mathbf{v} + \sum_{r=2}^{k} \sum_{\sum m_i = r} \mathbf{w}_{cr,(m_1,\ldots,m_{n_1})}(t) v_1^{m_1} \ldots v_{n_1}^{m_{n_1}} \tag{12}$$

which is time-dependent, in general. However, for the cases of flip, symmetry breaking, transcritical and nonresonant secondary Hopf bifurcations it becomes time-invariant, as shown in [7]. For example, if we assume that the system undergoes a flip bifurcation, then Equation (2) has a -1 Floquet multiplier which corresponds to a zero eigenvalue of matrix $\mathbf{R}$, and the normal form becomes

$$\dot{v} = \bar{w}_3 v^3, \tag{13}$$

where $\bar{w}_3$ is the average of the time-periodic coefficient function $g_{13,(3,0,\ldots,0)}(t)$ of the cubic term z_1^3 from Equation (10). A secondary Hopf bifurcation occurs when a complex pair of multipliers lie on the unit circle. Assuming that the critical eigenvalues of matrix $\mathbf{R}$ have the form $\pm\omega i$, and if ω and π/T are not integer multiples of each other then the normal form is time invariant and has the form [1]

$$\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} = \begin{bmatrix} a+i\omega & 0 \\ 0 & a-i\omega \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \begin{Bmatrix} \bar{w}_{21} v_1^2 v^2 \\ \mathrm{cc}(\bar{w}_{21}) v_1 v_2^2 \end{Bmatrix}. \tag{14}$$

The coefficient $\bar{w}_{21}$ is the average of the coefficient function of the cubic term $z_1^2 z_2$ from Equation (10). The notation 'cc' stands for complex conjugate.

2.5. Versal Deformation of the Normal Form Equation

Equation (12) is a general representation of the normal form of the dynamical equation at the bifurcation point. However, since in beginning of this procedure we eliminated the effect of the parameters, it does not describe the bifurcation phenomenon itself, it can only be used to study the stability of the critical point. In order to analyze the behavior in a small neighborhood of this point, we need to construct a parameter dependent matrix $\mathbf{J}(\boldsymbol{\mu})$, the versal deformation of $\mathbf{J}_c$, where $\boldsymbol{\mu}$ is an m vector of small parameters. Then the new equation can be formally written as

$$\dot{\mathbf{v}} = \mathbf{J}(\boldsymbol{\mu})\mathbf{v} + \mathbf{W}(\mathbf{v}, t). \tag{15}$$

In order to obtain a quantitatively meaningful result we need to find a relationship between the versal deformation parameter $\boldsymbol{\mu}$ and the original system parameter α.

2.5.1. *Finding the Versal Deformation Parameter via Sensitivity Analysis*

Since Equation (1) contains the bifurcation parameter $\boldsymbol{\alpha}$ while the versal deformation equation is given in terms of the parameter $\boldsymbol{\mu}$, we need to determine the relationship $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\alpha})$ satisfying the condition $\boldsymbol{\mu}(\boldsymbol{\alpha}_c) = \mathbf{0}$. We can compute the state transition matrix (STM), $\boldsymbol{\Phi}(t, \boldsymbol{\alpha})$ associated with the linear part of Equation (2) symbolically (for details, see [11]) in terms of the parameters $\boldsymbol{\alpha}$. From the relationship between the eigenvalues of the Floquet transition matrix (FTM), $\boldsymbol{\Phi}(T, \boldsymbol{\alpha})$ and matrix $\mathbf{J}(\boldsymbol{\mu})$ in Equation (15) and utilizing sensitivity analysis, we can determine an approximation for $\boldsymbol{\mu}(\boldsymbol{\alpha})$. From Equation (4), using the $2T$-periodic property of the real L–F transformation, we have the relationship

$$\boldsymbol{\Phi}(2T, \boldsymbol{\alpha}) = \boldsymbol{\Phi}^2(T, \boldsymbol{\alpha}) = \mathrm{e}^{2\mathbf{R}(\boldsymbol{\alpha})}. \tag{16}$$

Let λ_j, $j = 1, \ldots, n_1$ denote those eigenvalues of $\mathbf{R}(\boldsymbol{\mu}) = \mathbf{R}(\boldsymbol{\mu}(\boldsymbol{\alpha}))$, which are critical for $\boldsymbol{\mu} = \mathbf{0}$. These eigenvalues have the general form $\lambda_j = \mathrm{Re}(\mu_j) + (\mathrm{Im}(\mu_j) + \omega_c)i$, where $\omega_c i$ is the critical value. (Actually, ω_c is zero in the cases of flip, transcritical-symmetry-breaking and fold bifurcations. It is nonzero only for the secondary Hopf bifurcation, among the codimension one cases.) Let us assume that for some $\boldsymbol{\mu} \neq \mathbf{0}$

$$\rho_j = \mathrm{e}^{2\lambda_j(\boldsymbol{\mu})} \tag{17}$$

are the eigenvalues of $\boldsymbol{\Phi}(2T, \boldsymbol{\alpha})$, that is, they satisfy

$$D = \det[\boldsymbol{\Phi}(2T, \boldsymbol{\alpha}) - \mathbf{I}\,\mathrm{e}^{2\lambda_j(\boldsymbol{\mu})}] = 0 \tag{18}$$

for all $j = 1, \ldots, n_1$. Equation (18) gives an exact nonlinear relationship, but finding solutions of it is very impractical. Instead, we assume that $\mu_j(\boldsymbol{\alpha}) \approx a_{1j}(\alpha_j - \alpha_{cj}) + a_{2j}(\alpha_j - \alpha_{cj})^2 + \cdots$, where a_{lj}, $l = 1, 2, \ldots, p$ are unknown complex coefficients, in general. Expanding the determinant in Equation (18) into Taylor series up to the pth order terms, we obtain a set of n_1 equations as

$$\sum_{j=1}^{n_1}\left(\mu_j \left.\frac{\partial D}{\partial \mu_j}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\alpha}=\boldsymbol{\alpha}_c} + (\alpha_j-\alpha_{cj}) \left.\frac{\partial D}{\partial \alpha_j}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\alpha}=\boldsymbol{\alpha}_c}\right)$$
$$+\sum_{j+1}^{n_1}\sum_{k=1}^{n_1}\left(\mu_j\mu_k \left.\frac{\partial^2 D}{\partial \mu_j\partial\mu_k}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\alpha}=\boldsymbol{\alpha}_c} + \mu_j(\alpha_k-\alpha_{ck}) \left.\frac{\partial^2 D}{\partial \alpha_k\partial\mu_j}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\alpha}=\boldsymbol{\alpha}_c}\right.$$
$$\left.+ (\alpha_j-\alpha_{cj})(\alpha_k-\alpha_{ck}) \left.\frac{\partial^2 D}{\partial \alpha_j\partial\alpha_k}\right|_{\boldsymbol{\mu}=\mathbf{0},\boldsymbol{\alpha}=\boldsymbol{\alpha}_c}\right) + \cdots = 0. \tag{19}$$

Substituting $\mu_j(\alpha) \approx a_{1j}(\alpha_j - \alpha_{cj}) + a_{2j}(\alpha_j - \alpha_{cj})^2 + \cdots$, this becomes a set of algebraic equations for the unknowns a_{lj}, $l = 1,\ldots,p$, $j = 1,\ldots,n_1$, and the desired approximate relationship can be obtained. However, we need to note that the symbolic computation procedure of the STM is very time consuming for large systems or when more than two parameters are involved.

2.5.2. *Finding the Versal Deformation Parameter via a Curve Fitting Technique*

An alternate (simpler and faster) approach is to compute the FTM numerically, using the Chebyshev polynomial expansion method [2, 3], for several different sets of parameters α in the neighborhood of the critical α_c. This method has been shown to be much more efficient compared to Runge–Kutta type schemes. From the FTM, $\boldsymbol{\Phi}(T,\boldsymbol{\alpha})$, one can compute the matrix $\mathbf{R}(\boldsymbol{\alpha})$ based on the factorization in Equation (4). Let $\boldsymbol{\alpha}_k = \boldsymbol{\alpha}_c + \boldsymbol{\eta}_k$ denote the k different sets of parameters, where $\boldsymbol{\eta}_k$ are small, and λ_k denote the vector of eigenvalues of the matrix $\mathbf{R}(\boldsymbol{\alpha}_k)$. Again, we assume that $\lambda_k = \mathrm{Re}(\boldsymbol{\mu}_k) + (\mathrm{Im}(\boldsymbol{\mu}_k) + \omega_c)i$ and also $\mu_{kj} = a_{1j}\eta_{jk} + a_{2j}\eta_{kj}^2 + \cdots + a_{pj}\eta_{kj}^p$, $j = 1,\ldots,n_1$. The unknown coefficients a_{lj}, $l = 1,\ldots,p$ are complex, in general. Usually k can be restricted to a small number, depending on the desired order of approximation. For example, if a quadratic approximation is needed, k may be 3–5 for a reasonable accuracy. Then, by fitting a curve (quadratic or higher order) to the $\boldsymbol{\eta}_k$–$\boldsymbol{\mu}_k$ pairs, we can find the coefficients a_{lj} of the desired relationship.

3. Illustrative Examples

3.1. A Commutative System

For this example the L–F transformation can be computed in a closed form which makes the rest of the computations much simpler. Because of the properties of a commutative system, this case behaves just like an autonomous system, when it comes to constructing the versal deformation. Consider the commutative system

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -1+\alpha\cos^2 t & 1-\alpha\sin t\cos t \\ -1-\alpha\sin t\cos t & -1+\alpha\sin^2 t \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + b\begin{Bmatrix} x_1^3 \\ x_2^3 \end{Bmatrix}, \tag{20}$$

where $\{x_1, x_2\}$ is the state-vector, α is the bifurcation parameter and b is an arbitrary real number. This system is periodic with period $T = \pi$. The state transition matrix is given by (see [13])

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} \mathrm{e}^{(\alpha-1)t}\cos t & \mathrm{e}^{-t}\sin t \\ -\mathrm{e}^{(\alpha-1)t}\sin t & \mathrm{e}^{-t}\cos t \end{bmatrix} \tag{21}$$

which can easily be factored as

$$\mathbf{\Phi}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} \mathrm{e}^{(\alpha-1)t} & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix} = \mathbf{Q}(t)\,\mathrm{e}^{\mathbf{R}t}, \tag{22}$$

where $\mathbf{Q}(t)$ is the L–F transformation, real and $2T = 2\pi$ periodic. The Floquet transition matrix is

$$\mathbf{\Phi}(t) = \begin{bmatrix} -\mathrm{e}^{(\alpha-1)\pi} & 0 \\ 0 & -\mathrm{e}^{-\pi} \end{bmatrix} \tag{23}$$

and therefore at $\alpha = 1$ the system is critical, with one multiplier equal to -1, which indicates a flip bifurcation. Following the procedure described in Sections 2.3 and 2.4, the normal form of the equation on the center manifold can be calculated as

$$\dot{v} = \frac{3}{4} b v^3. \tag{24}$$

An exact versal deformation of this equation can be found in this case, because in the factorization of the STM, the bifurcation parameter appears only in the $\mathrm{e}^{\mathbf{R}t}$ matrix, and $\mathbf{R}(\boldsymbol{\alpha})$ can be easily computed as

$$\mathbf{R} = \begin{Bmatrix} \alpha - 1 & 0 \\ 0 & -1 \end{Bmatrix}. \tag{25}$$

Here matrix $\mathbf{R}$ is given directly in terms of the parameter α and it is also diagonal (equal to its Jordan matrix $\mathbf{J}$), which means that in this case the connection between α and the versal deformation parameter μ is given exactly, namely $\mu = \alpha - 1$. Therefore, the versal deformation equation is

$$\dot{v} = (\alpha - 1)v + \frac{3}{4} b v^3. \tag{26}$$

The solution of this equation for a given initial condition $v(0) = v_0$ is

$$v(t) = \left(\left(\frac{1}{v_0^2} + \frac{3b}{4(\alpha - 1)} \right) \mathrm{e}^{-2(\alpha-1)t} - \frac{3b}{4(\alpha - 1)} \right)^{-1/2}. \tag{27}$$

It has a non-trivial equilibrium point for $\alpha > 1$, which, after transforming back to the original states, becomes a $2T$-periodic limit cycle due to the fact that the real L–F transformation is $2T$-periodic in this case (both multipliers lie in the left half of the complex plane). It is stable for $b < 0$, with an amplitude $A = (4(1 - \alpha)/3b)^{1/2}$. This is the amplitude of the small attractive domain born around the origin after the stability is lost in the linear sense. In the neighborhood of the origin it exhibits a relatively slow quadratic growth, depending on the parameter.

In general, however, it is not possible to factor out the bifurcation parameter so nicely because even if we can compute the STM in terms of the bifurcation parameter we still cannot perform the factorization due to computational difficulties. Therefore, approximate techniques must be used to find the versal deformation parameter as shown in the following examples.

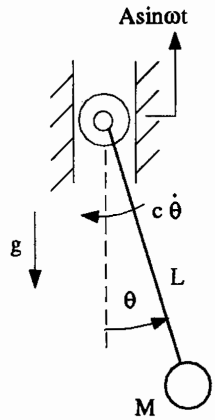


Figure 1. Parametrically excited simple pendulum.

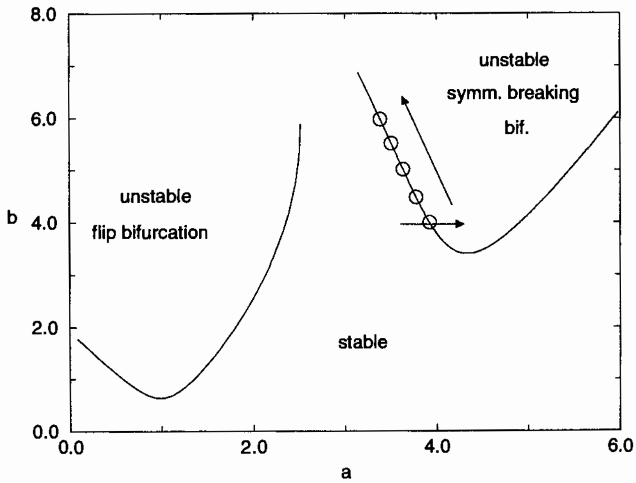


Figure 2. Stability chart of the parametrically excited pendulum.

3.2. Symmetry Breaking Bifurcation (+1 Multiplier)

Consider a parametrically excited simple pendulum (Figure 1). The equation of motion is given as

$$\ddot{\theta} + d\dot{\theta} + (a + b\sin\omega t)\sin\theta = 0, \tag{28}$$

where $a = g/L$, $b = A\omega^2/L$ and $d = c/ML^2$. Let $\{x_1 \;\; x_2\} = \{\theta \;\; \dot{\theta}\}$ and expand $\sin\theta$ in a Taylor series about the zero equilibrium point up to the cubic terms. Then Equation (28) can be written in the state-space form as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(a + b\sin\omega t) & -d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ (a + b\sin\omega t)x_1^3/6 \end{Bmatrix}. \tag{29}$$

The stability chart of this system is well known and is shown in Figure 2 in the $a \sim b$ plane for $0 \leq a \leq 6$, $0 \leq b \leq 8$ and $d = 0.31623$, $\omega = 2$. The circles in the figure show the bifurcation points investigated in this example. Observe that these points are located at large values of parameters and and the system is subjected to a strong parametric excitation. Further, these bifurcation points are relatively far away from the 1:1 resonance condition ($\omega = \sqrt{a}$). Normally in averaging and perturbation methods approximate solutions are constructed in the neighborhood of resonance points. For these reasons, averaging and perturbation methods cannot be applied in this case. It is our intent to investigate the dynamics as we fix the value of b and cross the boundary by introducing changes in a (indicated by the horizontal arrow in Figure 2). Also, we would like to compare post bifurcation dynamics for different values of b, introducing the same change in a for all five points investigated (see the vertical arrow in Figure 2). First, let parameter a be the bifurcation parameter and set the others as $b = 4$, $d = 0.31623$ and $\omega = 2$. For the critical value $a_c = 3.91778734$ the system undergoes a bifurcation (one of the Floquet multipliers of the linear part of Equation (29) is $0.999999996 \approx 1$, the other one is 0.37029). Because the multipliers lie in the right half of the complex plane, there is only one Lyapunov–Floquet transformation and it is real and T-periodic. After normalizing the time so that the period becomes 1, and applying the L–F and modal transformations at the critical point, we can apply the center manifold reduction to decouple the critical equation from the other. The center manifold equation can be simplified via normal form reduction and the normal form becomes

$$\dot{v} = -0.4306v^3, \tag{30}$$

indicating that the equilibrium of the nonlinear system is asymptotically stable at the critical point. The versal deformation of Equation (30) yields the normal form of symmetry breaking bifurcation for this system as

$$\dot{v} = \mu v - 0.4306v^3. \tag{31}$$

It is solvable in a closed from; the solution for the initial condition $v(0) = v_0$ being

$$v(t) = \left(\left(\frac{1}{v_0^2} + \frac{-0.4306}{\mu} \right) e^{-2\mu t} - \frac{-0.4306}{\mu} \right)^{-1/2}. \tag{32}$$

Note that Equation (31) has the same form as Equation (27) in the previous example, where it described a flip bifurcation. Unlike the case of autonomous systems, here the normal form does not contain all the information about the bifurcation and the L–F transformation also plays an important role. From Equation (32) we can conclude the following about the dynamics in the neighborhood of the bifurcation. For $\mu < 0$ the $v = 0$ equilibrium is stable and there are no other limit sets in the neighborhood. For $\mu > 0$ the zero equilibrium becomes unstable and there is a stable nonzero equilibrium born. This equilibrium point becomes a stable T-periodic limit cycle in the original domain because the real L–F transformation is T-periodic in this case. So we say that the stability has been lost softly. This limit cycle is the boundary of the small attractive domain around the unstable equilibrium. By computing the amplitude of the limit cycle we have estimated the size of the 'stable' region, and we can also estimate the rate of its growth as the bifurcation parameter goes further away from the critical value. The limit cycle is obtained by substituting the initial condition

$$v_0 = -\left(\frac{\mu}{-0.4306} \right) \tag{33}$$

into Equation (32) and applying the necessary transformations to yield the original state $\mathbf{x}(t)$. At this point we follow the methodology discussed in Section 2.5.1 and first establish a linear relationship between the original bifurcation parameter, a of Equation (29) and μ. (Note that

Table 1. Bifurcation parameter η *versus* versal deformation parameter μ, obtained by various techniques.

bifurcation parameter $a - ac = \eta$	critical eigenvalue μ *numerical value*	critical eigenvalue μ linear relationship *sensitivity analysis*	critical eigenvalue μ quadratic relation *sensitivity analysis*	critical eigenvalue μ quadratic relation *curve fitting*
0	0.000000009705	0	0	0
0.0001	0.000086777	0.000086800	0.000086781	0.000086484
0.001	0.00086615	0.00086800	0.00086615	0.0008553
0.005	0.0042941	0.0043400	0.0042936	0.0042484
0.01	0.0084981	0.0086800	0.0084946	0.0084234
0.02	0.016645	0.017360	0.16618	0.016554
0.03	0.024460	0.026040	0.024371	0.024391
0.04	0.031958	0.034720	0.031754	0.031934
0.05	0.039155	0.043400	0.038765	0.039185
0.06	0.046065	0.052080	0.045405	0.046142
0.08	0.059069	0.069440	0.057574	0.059178
0.1	0.071057	0.086800	0.068260	0.071040
0.16	0.10162	0.138880	0.091417	0.099590

in this case μ is nothing but the change in the critical eigenvalue of matrix **R**.) After the necessary calculations the sensitivity analysis yields

$$\mu = 0.868\eta. \tag{34}$$

We also compute a quadratic approximation by the same method and obtain

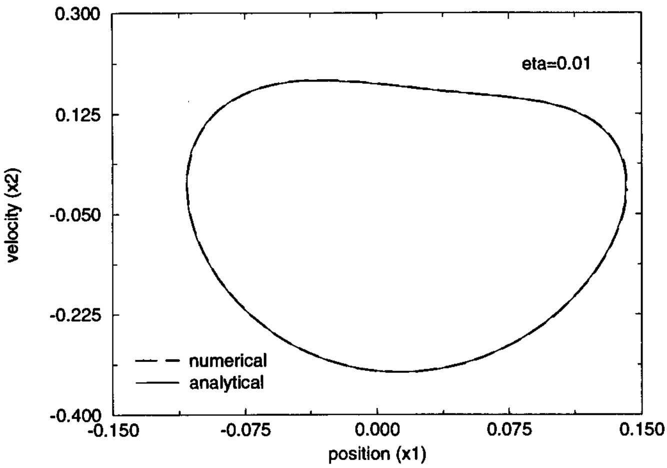
$$\mu = 0.868\eta - 1.854\eta^2, \tag{35}$$

where η is the small change in a from the critical value (i.e., $a = a_c + \eta$). To obtain Equations (34) and (35) in the symbolic computation procedure 32 Chebyshev polynomials and 30 Picard iterations were used. Note that the accuracy can be improved by increasing these numbers, however, the computation time will also greatly increase.

Similarly, following the procedure outlined in Section 2.5.2, a quadratic relationship is computed using the curve fitting technique and it is found to be

$$\mu = 0.865\eta - 1.635\eta^2. \tag{36}$$

Table 1 shows a comparison of the versal deformation parameter computed by the three different expressions given by Equations (34–36) and obtained numerically, for several values of the bifurcation parameter. From this table we can draw three conclusions. First, in this particular case, the curve fitting technique provides the best result when compared to the numerical values. Also, we can see that by increasing η up to 0.1, the critical eigenvalue has grown from order $10^{-8} \approx 0$ to the order of 10^{-2}. The noncritical eigenvalue for this problem is of order 1. Since the versal deformation of the normal form has to remain on the center manifold in order to provide correct approximation of the original dynamics, the bifurcation parameter η can only be increased until the critical eigenvalue remains at least one order smaller than the stable eigenvalues. The actual limit value of η, of course, depends on the problem at hand.



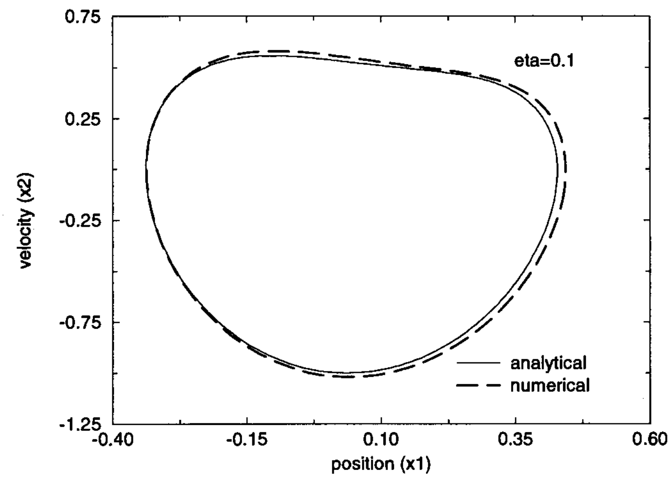


Figure 3. Comparison of the analytically and numerically obtained post-bifurcation limit cycles for $\eta = 0.01$ and $\eta = 0.1$, for the symmetry breaking bifurcation.

In this particular example, based on Table 1, it is about 0.16. Finally, we can observe that both the quadratic relationships are accurate enough compared to the numerically obtained values, there is no need to find higher order approximations. This can be expected because the problem at hand involves only cubic nonlinearities.

Figures 3a and 3b show the limit cycles in the state space for two different values of η (0.01 and 0.1) using the quadratic relationship (36). Solutions from numerical integration (IMSL subroutine) are also plotted for a comparison. We can see that the solutions agree very well for small η's but for larger values it becomes a poorer approximation. Figure 4 compares the numerical and analytical amplitudes of the steady-state periodic motion for several values of η, obtained from the three different η–μ relationships (Equations (34–36)). The results based on linear approximation diverge faster from the numerical solution, and for larger η's it does not even preserve the quadratic growth of the amplitude. Also, comparing the results yielded by the two quadratic approximations we can see, that in this case, the curve fitting method provided somewhat higher accuracy. We also observe that even though the analytical

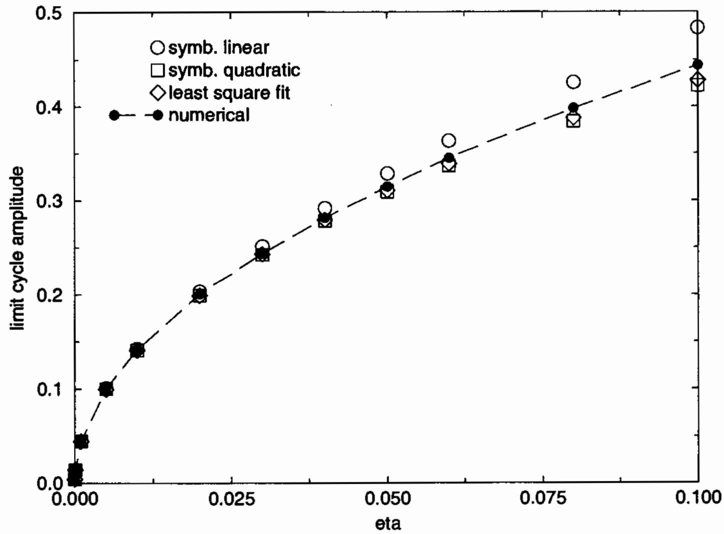


Figure 4. Comparison of the analytically and numerically obtained limit-cycle amplitudes for the symmetry breaking bifurcation.

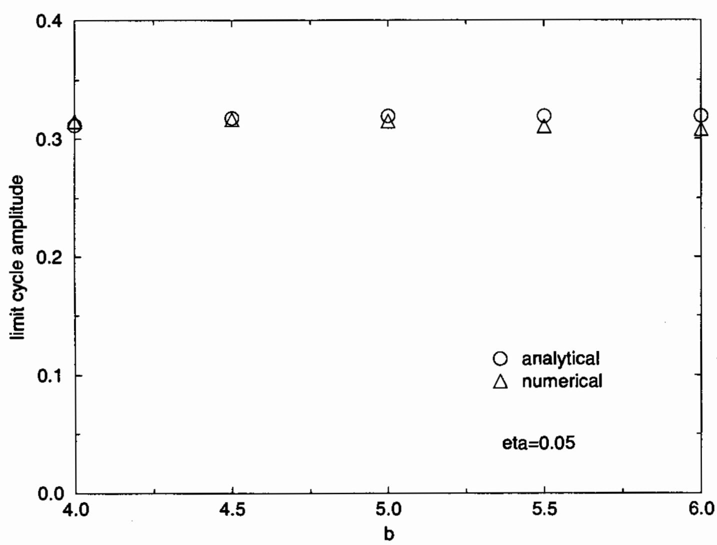


Figure 5. Comparison of the analytically and numerically obtained limit cycle amplitudes for different values of b along the stability boundary.

solutions with the quadratic approximations slowly diverge from the numerical one as η grows, both preserve the parabolic growth of the amplitude.

In the following only the quadratic relationship obtained by curve fitting will be used, since we have seen that it provides the best accuracy, and also for computational simplicity.

In order to study the dynamics as one follows the stability boundary towards larger values of b (even stronger excitations), we compute the amplitudes of steady-state solutions for four more sets of parameters, always at the same distance from the boundary (namely at $\eta = 0.05$). The values for parameter b are chosen to be 4.5, 5, 5.5 and 6 (as shown in Figure 2), keeping d and ω the same as before. The critical value of the bifurcation parameter, a_c is computed

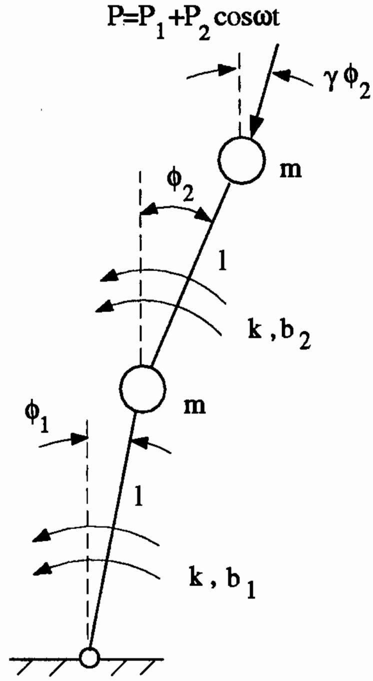


Figure 6. Double inverted pendulum with periodic load.

for each b. The post-bifurcation limit cycle amplitudes are shown in Figure 5 for all five sets of parameters computed by the proposed technique and the results are compared with those obtained from numerical integration. We observe that the accuracy of the method remains about the same as the parametric excitation becomes stronger. Interestingly enough, we also see that the amplitudes of the limit cycles at the same distance from the stability boundary are about the same in all five cases implying that the rate of growth of the attractive domain does not depend on how strong the parametric excitation is.

3.3. Secondary Hopf Bifurcation

As an example of the secondary Hopf bifurcation (two of the Floquet multipliers are complex and on the unit circle) we consider a double inverted pendulum subjected to a periodic load (Figure 6), whose equations of motion in state-space are given by (see [5])

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5\bar{k}(\bar{p}-3) & 0.5\bar{k}(2-\bar{p}) & -0.5(b_1+2b_2) & b_2 \\ 0.5\bar{k}(5-\bar{p}) & \bar{k}(\bar{p}(1.5-\gamma)-2) & 0.5(b_1+4b_2) & -2b_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \tag{37}$$

$$+ \begin{Bmatrix} 0 \\ 0 \\ \big[-0.5(x_3^2+x_4^2)(x_1-x_2)-(\bar{p}\bar{k})[(x_1-\gamma x_2)^3-(1-\gamma)^3x_2^3]/12- \\ -0.25(x_1-x_2)^2[\bar{k}(\bar{p}-4)x_1+\bar{k}(3+\bar{p}(\gamma-2))x_2-(b_1+3b_2)x_3+3b_2x_4]\big] \\ \big[0.5(x_1-x_2)(3x_3^2+x_4^2)+\bar{p}\bar{k}[(x_1-\gamma x_2)^3-3(1-\gamma)^3x_2^3]/12+ \\ +0.25(x_1-x_2)^2[\bar{k}(2\bar{p}-7)x_1+\bar{k}(5+\bar{p}(\gamma-3))x_2-(2b_1+5b_2)x_3+5b_2x_4]\big] \end{Bmatrix},$$

where $\{x_1\ x_2\ x_3\ x_4\} = \{\phi_1\ \phi_2\ \dot{\phi}_1\ \dot{\phi}_2\}$. For the parameter set $\bar{k} = k/ml^2 = 1, b_1 = b_2 = 0.01$, $p_{1\text{crit}} = -3.2570215$, $p_2 = 1$, $\bar{p} = p_1 + p_2 \cos \omega t$, $\omega = 2$ and $\gamma = 0$, the linear system matrix in Equation (37) has a pair of complex multipliers on the unit circle ($-0.9731 \pm 0.2305i$, magnitude $= 0.99999996$). This corresponds to a pair of purely imaginary eigenvalues in the transformed domain. After applying the L–F transformation, the center manifold reduction and the normal form simplifications, we find the two-dimensional versal deformation equation completely time-invariant and it is given by

$$\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} = \begin{bmatrix} \mu + 0.2326i & 0 \\ 0 & \mu - 0.2326i \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} + \begin{Bmatrix} (-0.1367 - 0.2586i)v_1^2 v_2 \\ (-0.1367 + 0.2586i)v_1 v_2^2 \end{Bmatrix}, \tag{38}$$

where μ is the small change in the eigenvalues of the normal form due to a change in the bifurcation parameter, $p_1 = p_{1\text{crit}} + \eta = -3.2570215 + \eta$. Once again following Section 2.5.2, the relationship between the bifurcation parameter and the eigenvalues of (38) is obtained as $\mu = (0.2270 - 0.8047i)\eta + (1.600 - 4.7940i)\eta^2)$ in the quadratic approximation. Utilizing the complex change of variables, $v_1 = u_1 - iu_2$ and $v_2 = u_1 + iu_2$, we can transform Equation (38) to a real form, and then by introducing polar coordinates, $u_1 = R\cos\theta$ and $u_2 = R\sin\theta$, the versal deformation equation

$$\begin{aligned} \dot{R} &= \text{Re}(\mu)R - 0.1367R^3, \\ \dot{\theta} &= -(0.2326 - 0.8047) - 0.2586R^2 \end{aligned} \tag{39}$$

becomes simple enough to be solved in a closed form. It is easy to obtain the steady-state solution of Equation (39) which yields a limit cycle amplitude $R_0 = \sqrt{\text{Re}(\mu)/0.1367}$. Then transforming it back to the complex domain, the steady-state solution of Equation (38) can be written as

$$\begin{Bmatrix} v_1(t) \\ v_2(t) \end{Bmatrix} = \begin{Bmatrix} \left(\frac{\mu}{0.1367}\right)^{1/2} \exp\left[\left(-(0.2326 - 0.8047) + \frac{0.2586}{0.36032}\mu\right) it\right] \\ -\left(\frac{\mu}{0.1367}\right)^{1/2} \exp\left[\left(-(0.2326 + 0.8047) + \frac{0.2586}{0.36032}\mu\right) it\right] \end{Bmatrix}. \tag{40}$$

The solution for the state variable $x(t)$ of the original time-periodic Equation (38) is obtained by transforming Equation (40) via the normal form reduction, center manifold relations and, finally, the L–F transformation. From Equations (39) and (40) we can conclude the following. For $\mu < 0$ the zero equilibrium is stable, and locally there are no other limit sets (the steady-state amplitude $R_0 = \sqrt{\mu/0.1367}$ is not real). For the case $\mu > 0$ the trivial solution becomes unstable and there is a stable limit cycle born with amplitude R_0. This limit cycle corresponds to a quasi-periodic attractor in the original domain, due to the fact that the $2T$-periodic L–F transformation is applied to the solution (40), whose periodicity is obviously not an integer multiple of 2. (Note that time is normalized such that $T = 1$, which allows the use of shifted Chebyshev polynomials in the computation procedure.) Figure 7a shows the Poincaré map of a transient solution converging to the attractor while Figure 7b shows the attractor itself as the state-space trajectory and its Poincaré map for a given value of η (only one pair of states is shown out of the six possible pairs of all four states). It seems to be natural to characterize the motion with its simple Poincaré map instead of the fairly complicated state-space trajectory. Poincaré sections of the attractor for two different values of η (0.001 and 0.003) are compared to results obtained from numerical integration in Figures 8a and 8b (all four states are shown in two pairs, positions *versus* velocities). From these figures we can conclude that the magnitude of the solutions is fairly accurate even relatively far from

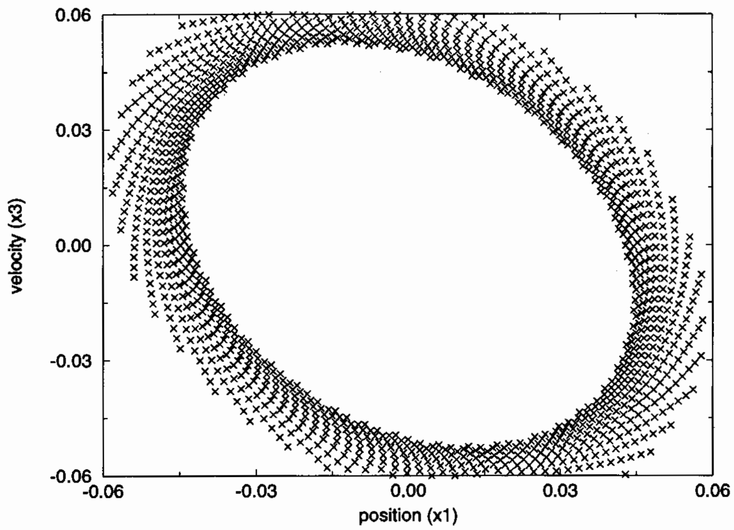


Figure 7a. Poincaré map of a transient solution converging to the attractor for the secondary Hopf bifurcation.

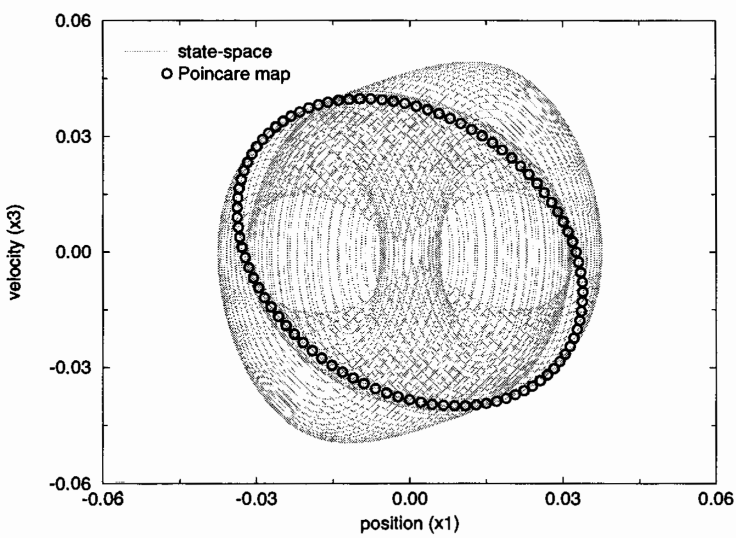
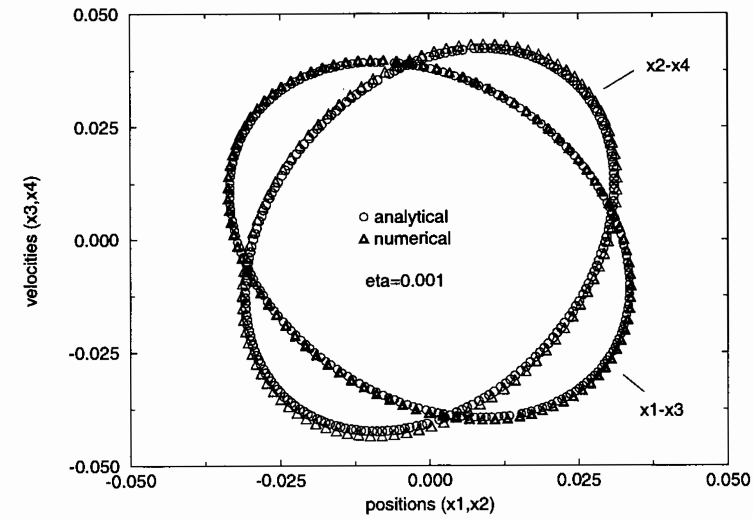


Figure 7b. State-space trajectory and Poincaré map of the attractor for the secondary Hopf bifurcation.

the critical point. However, in Figure 8b it is clearly seen that the numerical solution runs faster around the ellipse than the analytical one which implies that the error in 'frequency' is larger than the error in the 'amplitude'. The 'amplitude', the size of the attractive domain, is significant because it shows how softly the stability vanishes in this case. Since here we have a quasi-periodic four-dimensional attractor, it is not as obvious to determine what we mean by the size of the attractive domain, as in the previous example. A physically meaningful way is to consider the maximum values of the two position coordinates, $x_1 = \phi_1$ and $x_2 = \phi_2$, as a measure of the size of the domain. For brevity a comparison of numerical and analytical results is given for only the amplitude (max. value) of the first position coordinate x_1 in Figure 9. This



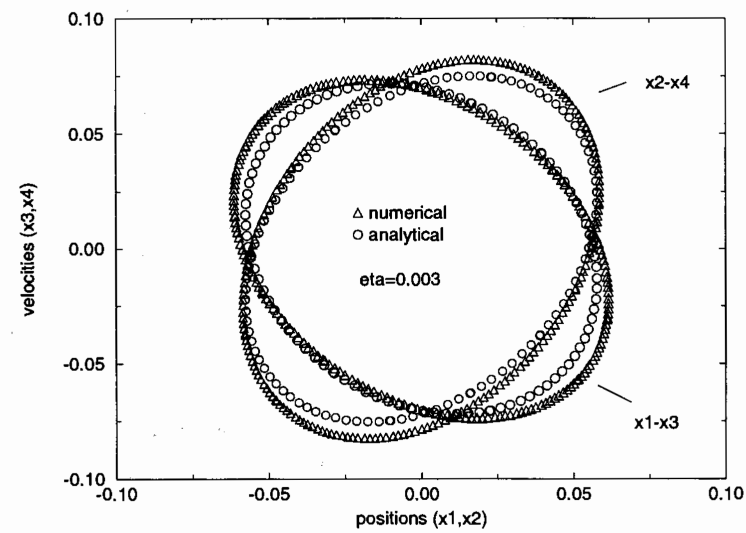


Figure 8. Comparisons of Poincaré maps of the analytically and numerically obtained post-bifurcation attractors for $\eta = 0.001$, and $\eta = 0.003$, for the secondary Hopf bifurcation.

figure shows the quadratic growth of the attractive domain as a function of the change in the bifurcation parameter.

3.4. A Two Degrees of Freedom System with Combined Internal and Parametric Resonance (Small Parameter Case – Comparison with Averaging Method)

Consider a two degrees of freedom system,

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 + \varepsilon(e_{11}\dot{x}_1 + e_{12}\dot{x}_2 + \cos\Omega t(b_{11}x_1 + b_{12}x_2) + a_{11}x_1^3 + a_{13}x_1x_2^2) &= 0, \\ \ddot{x}_2 + \omega_2^2 x_2 + \varepsilon(e_{21}\dot{x}_1 + e_{22}\dot{x}_2 + \cos\Omega t(b_{21}x_1 + b_{22}x_2) + a_{21}x_1^3 + a_{23}x_2x_1^2) &= 0, \end{aligned} \qquad (41)$$

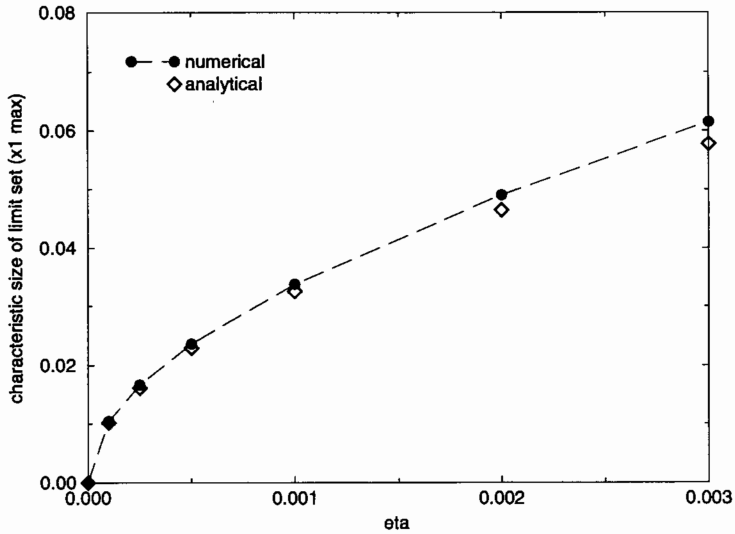


Figure 9. Comparison of the numerically and analytically obtained sizes of the attractive domain for the secondary Hopf bifurcation.

where ε is a small parameter. This system has been studied by Tso and Asmis [14] using averaging method in the cases of parametric, internal and combined parametric and internal resonances. Here we study only the case of combined resonance via the proposed technique as well as via averaging. Let $\Omega_0 = 2\omega_1$, $\omega_2 = 3\omega_1$, and $\Omega = \Omega_0(1 - \lambda)$, where λ is a detuning parameter, and let us choose $\Omega/\Omega_0 = (1 - \lambda)$ to be the bifurcation parameter. For the parameter set $\varepsilon = 0.1$, $\omega_1 = 0.45$, $e_{11} = 0.1$, $e_{22} = 0.15$, $e_{12} = e_{21} = 0$, $b_{11} = 0.8$, $b_{12} = -1.2$, $b_{21} = 0.6$, $b_{22} = 0.1$ $a_{11} = 0.4$, $a_{13} = 0.3$, $a_{21} = 0.5$ and $a_{23} = 0.27$ the averaging method predicts that the loss of stability occurs at $\Omega/\Omega_0 = 0.91625$. (In [14] this value is reported to be 0.918, but that is incorrect.) From the analysis of the state transition matrix computed either through the IMSL numerical subroutines or in terms of Chebyshev polynomials, we get $\Omega/\Omega_0 = 0.91985$ for the bifurcation point, which differs from the averaging method result only in the third digit. At this point a flip bifurcation occurs and the center manifold is one-dimensional. The equation on the center manifold has the same form (cf., Equation (31)) as in the second example, and it can be solved in a closed form to obtain the post-bifurcation limit cycles for both the coordinates x_1 and x_2. The amplitudes of the limit cycles thus obtained are compared with those obtained from averaging method. Figure 10 shows a comparison for the amplitudes of the two coordinates as a function of the bifurcation parameter Ω/Ω_0. The growth of the amplitude exhibits a quadratic characteristic in both results, and the rate of the growth is about the same. However, because the location of the bifurcation point was predicted incorrectly by the averaging method, the two results are significantly different. Considering that at the real bifurcation point averaging method predicts an amplitude of about 0.22 instead of zero, a very small error in the location of the stability boundary leads to relatively serious inaccuracy of the post-bifurcation dynamics. The proposed method follows the numerical results closely near the bifurcation point and it provides much better approximations than averaging.

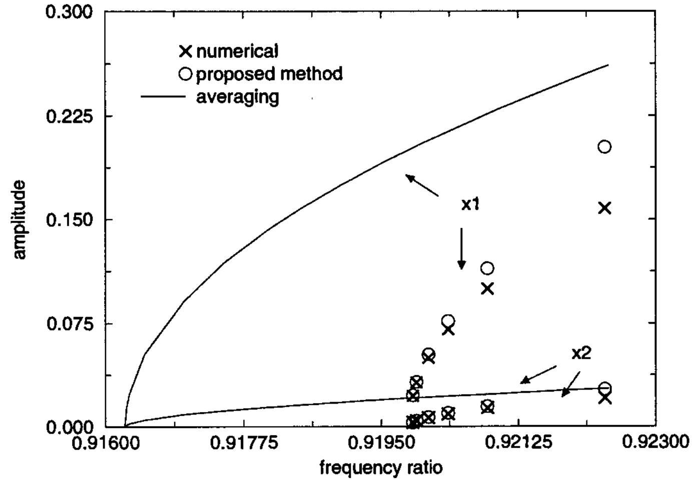


Figure 10. Comparison of the averaging method to the proposed technique.

4. Conclusions

In this paper, a technique to construct semi-analytical solutions in the neighborhood of bifurcation points of nonlinear time-periodic systems is presented. The method is based on the application of Lyapunov–Floquet transformation, time-periodic center manifold reduction, time-dependent normal form theory and versal deformation of the normal form equation. The Lyapunov–Floquet transformation converts the original system to a dynamically equivalent problem with a time-invariant linear part. Then the 'time-periodic center manifold theory' is used to separate the 'critical' dynamics, and reduce the dimension of the system to the dimension of its center manifold. 'Time-dependent normal form theory' is applied to further simplify it by reducing the number of nonlinear terms to a minimum. For most of the simple codimension one bifurcations the normal form is completely time-invariant and solvable in a closed form. Versal deformations of the normal forms are employed to study the dynamics in the neighborhood of the critical point. The versal deformation equation on the center manifold is solved to obtain post-bifurcation steady-state solutions and transformed to the original state variables by substituting all the transformations in the reversed order. However, it is observed that transforming the solution to the original coordinates introduces a small error close to the bifurcation point but it becomes significant as we move farther from it. This error is due to the fact that the Lyapunov–Floquet transformation as well as the center manifold relations are computed at the critical point and are reasonably accurate only in a small neigborhood of this point. Therefore the method is useful for local analysis only. The comparisons with numerical integration in the illustrative examples clearly show this phenomenon. But they also demonstrate that as long as we are close enough to the critical point, we obtain excellent results regardless where the critical point lies in the parameter space and how strong the periodic excitation is.

Post-bifurcation steady-state solutions of periodic nonlinear systems have been traditionally obtained by averaging and perturbation methods. However, those techniques do not preserve the correct dynamics, because they treat the periodic part of linear term as a perturbation. This leads to incorrect prediction of the bifurcation point in most cases and results in crude approximations. This is clearly illustrated by the last example that even in the case of a small parameter, the proposed method provides better approximations than the averaging method.

It is anticipated that this method would be a useful tool in local bifurcation analysis of time periodic systems by providing a better understanding and a quantitative estimate of the post-bifurcation dynamics. This could also be useful in the design and control of nonlinear systems with time-periodic coefficients.

Acknowledgment

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