Time-Varying Controller Synthesis for Nonlinear Systems Subjected to Periodic Parametric Loadings

R. PANDIYAN
Department of Aerospace Engineering, Indian Institute of Technology, Kanpur 208 016, India

S. C. SINHA
Nonlinear Systems Research Laboratory, Department of Mechanical Engineering, Auburn University, Auburn, AL 36849

(Received 26 June 1995; accepted 13 May 1997)

Abstract: Linear and nonlinear time-varying controller synthesis for systems represented by nonlinear differential equations with periodic coefficients is addressed. A recently developed technique, based on the Liapunov-Floquet (L-F) theorem, is employed so that time-varying control gains can be obtained via time-invariant techniques. Furthermore, a simple time-varying pole-placement approach for the design of linear control has also been devised for linear time-periodic systems. The robustness of the above control designs under structured perturbations of the nominal system matrices is studied. In many cases, the linear control design alone may not meet the desired performance specifications of the nonlinear periodic systems due to the time-varying nature of the problem. Therefore, to improve the controlled response of the nonlinear system, a nonlinear time-varying controller is designed and incorporated. The linear control is used to stabilize and the nonlinear controller is employed to improve the response specifications of the system. The linear control designs are based on the L-F transformation approach and the time-varying pole-placement approach, whereas the nonlinear controller is obtained using the Liapunov direct method. The responses obtained through the above approaches are compared and the advantages and disadvantages of the methods are discussed. Noticeably, the combination of linear and nonlinear controllers based on the L-F transformation approach has been found to have better performance and robustness characteristics than the other approach.

Key Words: Time-periodic systems, time-varying linear control, time-varying nonlinear control, Liapunov-Floquet (L-F) transformation

1. INTRODUCTION

Periodically time-varying systems appear naturally in various branches of science and engineering. Numerous practical applications can be found in the areas of (i) structures subjected to periodic loading (Evan-Iwanowski, 1976), (ii) helicopter rotor blades in forward flight (McKillip, 1985), (iii) asymmetric rotor-bearing systems (Roseau, 1987), (iv) robots performing repetitive tasks (Streit, Krousgrill, and Bajaj, 1989), (v) quantum mechanics (Powell and Crasemann, 1961), and (vi) electrical circuits (Richards, 1983). The equations of motion for these systems, in general, have time-varying periodic coefficients and are nonlinear. The control problems associated with linear periodic systems alone are quite

Journal of Vibration and Control, 7: 73-90, 2001

challenging owing to their time-varying nature. One of the principal reasons that could be attributed to this fact is that the time-varying eigenvalues of the periodic matrix do not determine the stability of the systems, and one must resort to Floquet analysis. Hence, it is difficult to apply classical as well as modern control techniques to these problems as opposed to the case of time-varying systems. Methods for control synthesis of linear time-varying systems have been reported in the past by several authors. Invariably, these methods are based on transforming the original system into a suitable canonical form so that some of the special properties of the canonical system can be utilized for controller designs (Calico and Wiesel, 1984; Calise, Wasikowski, and Schrage, 1992; D'Angelo, 1970). However, such transformations, if they exist, are not unique and are tedious to implement, especially for higher dimensional systems. Calico and Wiesel (1984) discussed the active control problems associated with time-periodic systems where an iterative procedure was suggested to achieve control via the pole placement techniques. However, the gains thus obtained were only a subset of possible gain selections and did not represent the most general possibilities. Calise, Wasikowski, and Schrage (1992) developed a method to design fixed-gain controllers for time-periodic systems. The approach was quite complicated and required two levels of iteration in calculating the feedback gains. In this regard, Joseph (1993) and Sinha and Joseph (1994) recently developed a new strategy in designing controllers for linear periodic systems through an application of the Liapunov-Floquet (L-F) transformation matrix, which permits the construction of an equivalent time-invariant problem. Then, the well-known time-invariant techniques can be employed to design the various controllers. Finally, the time-varying gains for the periodic systems are obtained easily owing to the invertible nature of the L-F transformation matrix. However, the robustness characteristics of all the designs stated above, especially when they are implemented in periodically time-varying nonlinear systems, are not known.

In this paper, new methods to design time-varying controllers for nonlinear time-periodic systems are presented. Both linear and nonlinear controller syntheses are presented and their robustness characteristics addressed. The linear controller designs are based on either the L-F transformation matrix approach or the simple time-varying pole placement approach. Furthermore, to improve the closed-loop system performance, nonlinear controllers are incorporated via the Liapunov direct method. In the following, the mathematical preliminaries necessary for the development of time-varying control designs are briefly discussed.

2. MATHEMATICAL PRELIMINARIES

2.1. *Stability of Nonlinear Periodic Systems*

Consider the control problem of a periodically time-varying nonlinear system represented by

$$\dot{x} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} + \mathbf{f}(x,t), \tag{1}$$

with $\mathbf{A}(t) = \mathbf{A}(t+T), \mathbf{B}(t) = \mathbf{B}(t+T), \mathbf{f}(x,t) = \mathbf{f}(\mathbf{x},t+T)$, where $\mathbf{x}$ is an $n \times 1$ state vector, $\mathbf{A}(t)$ is an $n \times n$ system matrix, $\mathbf{B}(t)$ is an $n \times m$ control matrix, and $\mathbf{f}(\mathbf{x},t)$ is the nonlinear function associated with the system, which satisfies the condition $\mathbf{f}(\mathbf{0},t) = \mathbf{0}$. Finally, $\mathbf{u}_L$ and $\mathbf{u}_N$ are the linear and nonlinear control vectors satisfying $\mathbf{u} = \mathbf{u}_L + \mathbf{u}_N$, an $m \times 1$ control vector, respectively. Here, the aim is to select the appropriate control vector $\mathbf{u}$

such that the system is asymptotically stable and meets the system performance specifications. In the following, the control and stability of equation (1) is discussed via the Liapunov direct method.

Let the linear control function $\mathbf{u}_L$ be selected as

$$\mathbf{u}_L = -\mathbf{F}(t)\,\mathbf{x} \tag{2}$$

and the Liapunov function be defined as

$$\mathbf{V}(\mathbf{x}, t) = \mathbf{x}^T \mathbf{P}(t)\,\mathbf{x}, \tag{3}$$

where matrix $\mathbf{P}(t)$ denotes an $n \times n$ symmetric and positive definite weighting matrix. For the system to be asymptotically stable in the large, the candidate Liapunov function $\mathbf{V}(\mathbf{x}, t)$ can be selected suitably as illustrated by Wiens, Lu, and Sinha (1991).

Utilizing equations (1), (2), and (3), the derivative $\dot{\mathbf{V}}(\mathbf{x}, t)$ can be obtained as

$$\begin{aligned} \dot{\mathbf{V}}(\mathbf{x}, t) &= \mathbf{x}^T \left[\dot{\mathbf{P}}(t) + \mathbf{A}_c^T(t)\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{A}_c(t)\right]\mathbf{x} \\ &+ 2\mathbf{x}^T \mathbf{P}(t)\left[\mathbf{B}(t)\,\mathbf{u}_N + \mathbf{f}(\mathbf{x}, t)\right], \end{aligned} \tag{4}$$

where $\mathbf{A}_c(t) = \mathbf{A}(t) - \mathbf{B}(t)\mathbf{F}(t)$ is the closed-loop system obtained via the linear control. Thus, $\dot{\mathbf{V}}(\mathbf{x}, t) < 0$ if we require that

$$\dot{\mathbf{P}}(t) + \mathbf{A}_c^T(t)\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{A}_c(t) + \mathbf{Q}(t) = 0, \tag{5}$$

where $\mathbf{Q}(t)$ is a positive-definite matrix and

$$\mathbf{x}^T \mathbf{P}(t)\left[\mathbf{B}(t)\,\mathbf{u}_N + \mathbf{f}(\mathbf{x}, t)\right] \le -s(t)\,\|\mathbf{x}\|^2 < 0, \tag{6}$$

where $s(t)$ is a nondecreasing, continuous scalar function such that $s(0) = 0$. Equation (5) is simply the periodic Liapunov differential equation (Bittanti, Laub, and Willems, 1991) associated with equation (1), and the periodic, positive-definite matrix $\mathbf{Q}(t)$ can be obtained only by a suitable selection of the gain matrix $\mathbf{F}(t)$ such that all the multipliers of the closed-loop system $\mathbf{A}_c(t)$ are within the unit circle. However, there is no easy way to compute the gain matrices in this manner for this class of time-varying periodic systems. In this regard, it is found that the L-F transformation matrix can be very effectively used in obtaining the time-varying gain matrix $\mathbf{F}(t)$ without much effort. The condition given by equation (6) can be met by suitably selecting the nonlinear control vector $\mathbf{u}_N$. In the sequel, first, a brief description of the method to compute the L-F transformation matrix for general linear periodic systems is presented.

2.2. Computation of L-F Transformation Matrix via Chebyshev Polynomials

It has been shown by Sinha and Wu (1991), Sinha and Juneja (1991), and Joseph, Pandiyan, and Sinha (1993) that the state transition matrices (STMs) of linear periodic systems can be obtained in terms of the shifted Chebyshev polynomials of the first kind. The technique is efficient and since the STM is basically expressed in terms of powers of t, it is suitable for algebraic manipulations as well. In fact, if the dimension is small, the STM can be expressed in a closed form as an explicit function of system parameters as shown by Sinha and Juneja (1991) for the case of the Mathieu equation. In this approach, each element of the STM $\Phi(t)$ is expressed in terms of the shifted Chebyshev polynomials of the first kind.

Once the $n \times n$ $\Phi(t)$, the STM of the linear part of equation (1), has been computed using the method of Chebyshev expansions, it can be written as the product of two $n \times n$ matrices as

$$\Phi(t) = \mathbf{L}(t)\, e^{Ct}, \tag{7}$$

where $\mathbf{L}(t)$ is a T-periodic matrix and $\mathbf{C}$ is a constant matrix. Since $\Phi(0) = \mathbf{I}$, equation (7) yields $\mathbf{L}(0) = \mathbf{L}(T) = \mathbf{I}$. Hence, the FTM $\Phi(T)$ can be written as

$$\Phi(T) = e^{CT}. \tag{8}$$

By performing an eigenanalysis on the FTM, $\mathbf{C}$ can be computed easily. Then the T-periodic L-F transformation matrix is

$$L(t) = \Phi(t)\, e^{-Ct}. \tag{9}$$

To evaluate a $2T$-periodic L-F transformation matrix, $\mathbf{Q}(t)$, which yields a real constant matrix $\mathbf{R}$, first we note that (Coddington and Levinson, 1955)

$$\Phi(2T) = \Phi^2(T) = e^{CT} e^{C^*T} = e^{2RT}, \tag{10}$$

where $\mathbf{C}^*$ is the conjugate matrix of $\mathbf{C}, \mathbf{R} = (\mathbf{C} + \mathbf{C}^*)/2$ and the $2T$ L-F transformation matrix can be represented as

$$\begin{aligned} Q(t) &= \Phi(t)\, e^{-Rt} \quad ; \quad 0 \le t \le T \\ Q(\tau + T) &= \Phi(\tau)\, Q(T)\, e^{-R\tau} \quad ; \quad T \le (T+\tau) \le 2T;\ 0 \le \tau \le T. \end{aligned} \tag{11}$$

It should be noted that $\mathbf{Q}(t) = \mathbf{Q}(t + 2T)$.

If one is interested in finding $\Phi^{-1}(t)$, then there are two avenues. $\Phi(t)$ can possibly be inverted through a symbolic software like MACSYMA/MATHEMATICA/MAPLE, which is still not an easy task, or one can first find the state transition matrix $\Psi(t)$ of the adjoint system

$$\dot{\mathbf{w}}(t) = -\mathbf{A}^T(t)\, \mathbf{w}(t) \tag{12}$$

and use the following relationship (Yakubovich and Starzhinskii, 1975):

$$\Phi^{-1}(t) = \Psi^{T}(t). \tag{13}$$

The computation of $\Phi^{-1}(t)$ is essential in determining $\mathbf{L}^{-1}(t)$ or $\mathbf{Q}^{-1}(t)$. For example, the inverse T-periodic L-F transformation matrix can be evaluated utilizing the properties of the adjoint system as shown below.

$$\mathbf{L}^{-1}(t) = \left[\Phi(t)\, e^{-Ct}\right]^{-1} = e^{Ct}\,\Phi^{-1}(t) = e^{Ct}\,\Psi^{T}(t). \tag{14}$$

Such an approximation of L-F transformations has been found to be extremely convergent, and since it is periodic, the elements $\mathbf{L}_{ij}(t)$ or $\mathbf{Q}_{ij}(t)$ have the truncated Fourier representation

$$\mathbf{L}_{ij}(t) \approx \sum_{n=q}^{q} c_n \exp(i2\pi nt/T) \qquad i = \sqrt{-1} \tag{15}$$

or

$$\mathbf{Q}_{ij}(t) = \frac{a_0}{2} + \sum_{n=1}^{q} a_n \cos\frac{\pi nt}{T} + \sum_{n=1}^{q} b_n \sin\frac{\pi nt}{T}. \tag{16}$$

Since complex matrix $\mathbf{L}(t)$ (or the real matrix $\mathbf{Q}(t)$) can be computed as a function of t, all algebraic manipulations involving this matrix in equation (1) after applying the transformation can be done in symbolic form. $\mathbf{L}_{ij}^{-1}(t)$ and $\mathbf{Q}_{ij}^{-1}(t)$ have similar Fourier representations. It is important that the L-F transformation matrices and their inverses be calculated with a high degree of accuracy to guarantee a reasonably accurate system dynamics. Therefore, one must be careful in computation of the STM $\Phi(t)$. It has been shown by Sinha and Wu (1991) that a 15- to 18-term Chebyshev polynomial expansion provides extremely accurate representations of $\Phi(t)$ or $\mathbf{L}(t)$ even for relatively large systems such as 20×20. A convergence study has been reported by Joseph, Pandiyan, and Sinha (1993).

3. TIME-VARYING CONTROL DESIGN METHODS

3.1. Linear Control Design Utilizing the Liapunov-Floquet (L-F) Transformation Matrix

It is well-known from the L-F theorem that there exists a periodic, invertible transformation matrix that converts the state matrix $\mathbf{A}(t)$ in equation (1) to a time-invariant form. To determine such a transformation, one must compute the STM as an explicit function of time. Following the procedure given in Section 2.2, the real L-F transformation matrix $\mathbf{Q}(t)$ is computed such that $\mathbf{x} = \mathbf{Q}(t)\mathbf{z}$ transforms equation (1) to

$$\dot{\mathbf{z}} = \mathbf{R}\mathbf{z} + \mathbf{Q}^{-1}(t)\,\mathbf{B}(t)\,\mathbf{u} + \mathbf{Q}^{-1}(t) f(\mathbf{z}, t), \tag{17}$$

where $\mathbf{R}$ is an $n \times n$ constant matrix. Now consider the linear part of equation (17) given by

$$\dot{\mathbf{z}} = \mathbf{R}\mathbf{z} + \mathbf{Q}^{-1}(t)\,\mathbf{B}(t)\,\mathbf{u}_L. \tag{18}$$

Following the approach of Joseph (1993) and Sinha and Joseph (1994), linear control can be achieved in time-invariant domain by considering an auxiliary system of the form

$$\dot{\bar{\mathbf{z}}} = \mathbf{R}\bar{\mathbf{z}} + \tilde{\mathbf{B}}\nu, \tag{19}$$

where $\tilde{\mathbf{B}}$ is a constant matrix such that $(\mathbf{R}, \tilde{\mathbf{B}})$ is a completely controllable pair. The control law for the time-invariant equation (19) can be written as

$$\nu = -\bar{\mathbf{K}}\bar{z}. \tag{20}$$

Defining $e = z - \bar{z}$, the error dynamics from equations (18) and (19) can be represented as

$$\dot{\mathbf{e}} = \mathbf{R}e + \mathbf{Q}^{-1}(t)\,\mathbf{B}(t)\,\mathbf{u}_L - \tilde{\mathbf{B}}\nu \tag{21}$$

or

$$\dot{\mathbf{e}} = \left(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}}\right)\mathbf{e} + \mathbf{Q}^{-1}(t)\,\mathbf{B}(t)\,\mathbf{u}_L - \tilde{\mathbf{B}}\nu + \tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{e}. \tag{22}$$

Substituting ν from (20) and $e = z - \bar{z}$ in equation (22), we obtain

$$\dot{\mathbf{e}} = \left(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}}\right)\mathbf{e} + \mathbf{Q}^{-1}(t)\,\mathbf{B}(t)\,\mathbf{u}_L + \tilde{\mathbf{B}}\bar{\mathbf{K}}z. \tag{23}$$

Since $(\mathbf{R} - \tilde{\mathbf{B}}\bar{\mathbf{K}})$ is a stability matrix, the two systems described by equations (18) and (19) can be made equivalent in the least square sense if

$$\mathbf{Q}^{-1}(t)\,\mathbf{B}(t)\,\mathbf{u}_L = -\tilde{\mathbf{B}}\bar{\mathbf{K}}z; \tag{24}$$

that is,

$$\mathbf{u}_L = -\mathbf{B}^{\#}(t)\mathbf{Q}(t)\,\tilde{\mathbf{B}}\bar{\mathbf{K}}z, \tag{25}$$

where $\mathbf{B}^{\#}(t)$ is the generalized inverse (Lovass-Nagy, Miller, and Powers, 1978) of $\mathbf{B}(t)$ and satisfies

$$\mathbf{B}(t)\,\mathbf{B}^{\#}(t)\mathbf{B}(t) = \mathbf{B}(t). \tag{26}$$

Noting that $\mathbf{z} = \mathbf{Q}^{-1}(t)\mathbf{x}$, equation (25) takes the form

$$\mathbf{u}_L = -\mathbf{B}^{\#}(t)\mathbf{Q}(t)\,\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{Q}^{-1}(t)\,\mathbf{x}, \tag{27}$$

which provides the control law for the linear portion of the time-varying system (1). Furthermore, if

$$\mathbf{u}_L = -\mathbf{F}(t)\,\mathbf{x}, \tag{28}$$

then the time-varying gain matrix $\mathbf{F}(t)$ is given by

$$\mathbf{F}(t) = \mathbf{B}^{\#}(t)\mathbf{Q}(t)\,\tilde{\mathbf{B}}\bar{\mathbf{K}}\mathbf{Q}^{-1}(t)\,. \tag{29}$$

The time-invariant gain $\bar{\mathbf{K}}$ is chosen by applying the root locus, pole placement technique, and/or the optimal control theory on the auxiliary system (19) such that it is asymptotically stable.

3.2. Linear Control via Time-Varying Pole Placement

Pole placement of time-varying systems requires either special canonical transformations or iterative schemes as reported in D'Angelo (1970) and Calico and Wiesel (1984), respectively. Such procedures are complicated and time-consuming. Here, a simple alternate method is suggested. According to this procedure, a suitable periodic matrix $\mathbf{A}_c(t)$ has to be obtained whose multipliers lie within the unit circle. Most periodic systems depend on many system parameters whose values can be varied so that such a matrix can be obtained. Consider a nominal system model of a periodic system in the form

$$\dot{\mathbf{x}} = \mathbf{A}(\lambda, t)\,\mathbf{x}, \tag{30}$$

where λ is the parameter vector that may be varied to obtain the desired system characteristics. Let $\lambda = \lambda_s$ be the values of the parameter vector at which the system has all its multipliers placed inside the unit circle such that $\mathbf{A}(\lambda_s, t) = \mathbf{A}_c(t)$. Assuming a control law in the form of equation (28), the linear portion of equation (1) yields

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} + \mathbf{B}(t)\,\mathbf{u}_L = [\mathbf{A}(t) - \mathbf{B}(t)\,\mathbf{F}(t)]\,\mathbf{x} = \mathbf{A}_c(t)\,\mathbf{x}. \tag{31}$$

Then as stated above, for a given $\mathbf{A}_c(t)$, the linear control vector $\mathbf{u}_L$ can be evaluated by solving a set of linear algebraic equations given by

$$\mathbf{A}(t) - \mathbf{A}_c(t) = \mathbf{B}(t)\,\mathbf{F}(t)\,, \tag{32}$$

such that $\mathbf{u}_L = -\mathbf{F}(t)\mathbf{x}$. However, it is to be noted that this procedure depends heavily on the parameters of the nominal model and, therefore, when encountered with the large numbers of parameters, the handling will be extremely cumbersome. Since the closed-loop system $\mathbf{A}_c(t)$ thus obtained has all its multipliers within the unit circle, it also satisfies the condition given by equation (5).

3.3. Robustness of the Control Designs

A measure of the robustness of stability of the above-mentioned control designs in the presence of structured perturbations can be obtained in the following way. Consider only the linear part of equation (1),

$$\dot{\mathbf{x}} = \mathbf{A}(t)\,\mathbf{x} + \mathbf{B}(t)\,\mathbf{u}, \tag{33}$$

with $\mathbf{A}(t) = \mathbf{A}_n(t) + \mathbf{A}_p(t)$, where $\mathbf{A}_n(t)$ is the nominal system matrix and $\mathbf{A}_p(t)$ is the structured perturbation of the nominal matrix. Applying control gains given by equations (29) or (32) for the nominal matrix $\mathbf{A}_n(t)$, equation (33) takes the form

$$\dot{\mathbf{x}} = [\mathbf{A}_{nc}(t) + \mathbf{A}_p(t)]\,\mathbf{x}, \tag{34}$$

where $\mathbf{A}_{nc}(t)$ is the stable closed-loop system after the application of control laws. Stability of general time-varying systems of the type given by equation (34) has been discussed by the Dini-Hukuhara theorem II (Bellman, 1970). For time-periodic systems, the proof of the theorem is straightforward by applying the L-F transformation. The proof hinges on the fact that for periodic systems represented by equation (34), the quantity $\Phi(t)\Phi^{-1}(t)$ is bounded since the STM $\Phi(t)$ is bounded for the controlled nominal system as shown by Burton (1985). But to obtain the measure of sensitivity of the nominal control to the structured perturbations is not easy in the original coordinates $\mathbf{x}$. However, applying L-F transformation (based on $\mathbf{A}_{nc}(t)$) $\mathbf{x} = \mathbf{Q}(t)\mathbf{z}$ to equation (34), one obtains

$$\dot{\mathbf{z}} = \left[\mathbf{R}_{nc} + \mathbf{Q}^{-1}(t)\mathbf{A}_p(t)\mathbf{Q}(t)\right]\mathbf{z}. \tag{35}$$

A bound on the structural perturbations $\mathbf{Q}^{-1}(t)\mathbf{A}_p(t)\mathbf{Q}(t)$ for linear uncertain systems such as equation (35) can be obtained following the procedure given by Infante (1968) as

$$\left\|\mathbf{Q}^{-1}(t)\mathbf{A}_p(t)\mathbf{Q}(t)\right\| \le \frac{\min\lambda\left(\bar{\mathbf{Q}}\right)}{\max\lambda\left(\bar{\mathbf{P}}\right)}, \tag{36}$$

where the positive-definite matrices $\bar{\mathbf{P}}$ and $\bar{\mathbf{Q}}$ satisfy the following Liapunov matrix equation

$$\mathbf{R}_{nc}^T\bar{\mathbf{P}} + \bar{\mathbf{P}}\mathbf{R}_{nc} = -2\bar{\mathbf{Q}}. \tag{37}$$

Note that the right-hand side of equation (36) depends entirely on eigenvalues of $\bar{\mathbf{P}}$ and $\bar{\mathbf{Q}}$, and, therefore, it is possible to maximize the bound. A procedure to sharpen the above bound is provided by Bhattachardya (1987). A similar measure of response bounds for the class of second-order elastic systems has been presented by Sinha and Wiens (1989), which is found to be optimal. However, it is to be noted that since the matrices $\mathbf{Q}(t)$ and $\mathbf{Q}^{-1}(t)$ are periodic, their norms $\|\mathbf{Q}(t)\|$, $\|\mathbf{Q}^{-1}(t)\|$ are bounded and, therefore, the robust bound on the structural perturbation matrix $\mathbf{A}_p(t)$ will seldom be good.

3.4. Nonlinear Control Synthesis

The linear control thus obtained by either of the above mentioned methods when implemented into the nonlinear system represented by equation (1) makes the system asymptotically stable as long as the nonlinear terms are bounded in the mean as given in Krasovskii (1963). In this way, the nonlinear terms are considered as a persistent disturbance. However, this might affect the required performance requirements. To improve the performance, the nonlinear control vector $\mathbf{u}_N$ has to be selected suitably. After implementing the linear control obtained above, equation (1) can be rewritten as

$$\dot{\mathbf{x}} = \mathbf{A}_c\,(t)\,\mathbf{x} + \mathbf{B}\,(t)\,\mathbf{u}_N + \mathbf{f}\,(\mathbf{x}, t)\,, \tag{38}$$

where $\mathbf{A}_c(t)$ is the new system matrix, which is closed-loop stable. Therefore, at least for the linear part of equation (38), there exists a Liapunov function (Krasovskii, 1963) given by $\mathbf{V}(\mathbf{x}, t) = \mathbf{x}^T\mathbf{P}(t)\mathbf{x}$, where $\mathbf{P}(t)$ is an $n \times n$ positive definite weighting matrix for $\forall t$. For the nonlinear equation (38), the derivative of $\mathbf{V}(\mathbf{x}, t)$ can be written as

$$\dot{\mathbf{V}}(\mathbf{x}, t) = -\mathbf{x}^T\mathbf{Q}_1(t)\mathbf{x} + 2\mathbf{x}^T\mathbf{P}\,(t)\,[\mathbf{B}\,(t)\,\mathbf{u}_N + \mathbf{f}\,(\mathbf{x}, t)]\,, \tag{39}$$

where $\mathbf{Q}_1(t)$ is a positive definite matrix. To make equation (38) stable, a minimum nonlinear control vector $\| \mathbf{u}_N \|$ must be chosen such that $\dot{\mathbf{V}}(\mathbf{x}, t) < 0$. Following the work of Wiens, Lu, and Sinha (1991), a time-varying nonlinear control can be selected such that

$$\mathbf{x}^T\mathbf{P}\,(t)\,[\mathbf{B}\,(t)\,\mathbf{u}_N + \mathbf{f}\,(\mathbf{x}, t)] = -\mathbf{x}^T\mathbf{P}\,(t)\,\mathbf{E}\mathbf{P}\,(t)\,\mathbf{x}, \tag{40}$$

where $\mathbf{E}$ is a time-invariant positive semidefinite matrix.

To obtain $\mathbf{u}_N$ from equation (40), a general optimization function for minimizing $\| \mathbf{u}_N \|$ can be defined as shown by Wiens, Lu, and Sinha (1991)

$$\begin{aligned} \mathbf{u}_N &= -\left[\mathbf{x}^T\mathbf{P}\,(t)\,\mathbf{B}\,(t)\right]^{\#}\left[\mathbf{x}^T\mathbf{P}\,(t)\right]\left[\mathbf{E}\mathbf{P}\,(t)\,\mathbf{x} + \mathbf{f}\,(\mathbf{x}, t)\right] \\ &+ \left\{\mathbf{I}_p - \left[\mathbf{x}^T\mathbf{P}\,(t)\,\mathbf{B}\,(t)\right]^{\#}\left(\mathbf{x}^T\mathbf{P}\,(t)\,\mathbf{B}\,(t)\right)\right\}\mathbf{d}, \end{aligned} \tag{41}$$

where $[\cdot]^{\#}$ represents the general pseudo-inverse of the quantity $[\cdot]$, $\mathbf{d}$ is a $p \times 1$ arbitrary vector, and $\mathbf{I}_p$ is an $p \times p$ identity matrix. Note that because of the orthogonal property existing between the matrices $\mathbf{x}^T\mathbf{P}(t)\mathbf{B}(t)$ and $[\mathbf{I}_p - \{\mathbf{x}^T\mathbf{P}(t)\mathbf{B}(t)\}^{\#}\{\mathbf{x}^T\mathbf{P}(t)\mathbf{B}(t)\}]$, the product of these matrices is zero. As shown by Wiens, Lu, and Sinha (1991), one obtains the desired minimum nonlinear control vector $\| \mathbf{u}_N \|$ by selecting $\mathbf{d} = \mathbf{0}$ in equation (41). Therefore,

$$\mathbf{u}_N = -\left[\mathbf{x}^T\mathbf{P}\,(t)\,\mathbf{B}\,(t)\right]^{\#}\mathbf{x}^T\mathbf{P}\,(t)\,\left[\mathbf{E}\mathbf{P}\,(t)\,\mathbf{x} + \mathbf{f}\,(\mathbf{x}, t)\right]. \tag{42}$$

Substitution of equation (42) into equation (39) yields

$$\dot{\mathbf{V}}(\mathbf{x}, t) = -\mathbf{x}^T\left[\mathbf{Q}_1(t) + 2\mathbf{P}\,(t)\,\mathbf{E}\mathbf{P}\,(t)\right]\mathbf{x}. \tag{43}$$

It is obvious that $\dot{\mathbf{V}}(\mathbf{x}, t) < 0$ holds for a given positive definite matrix $\mathbf{Q}_1(t)$ and a semipositive definite matrix $\mathbf{E}$.

4. APPLICATIONS

As an illustrative example, the control of a two mass inverted pendulum subjected to a nonconservative periodic load is considered. The nonlinear equations of motion of the system

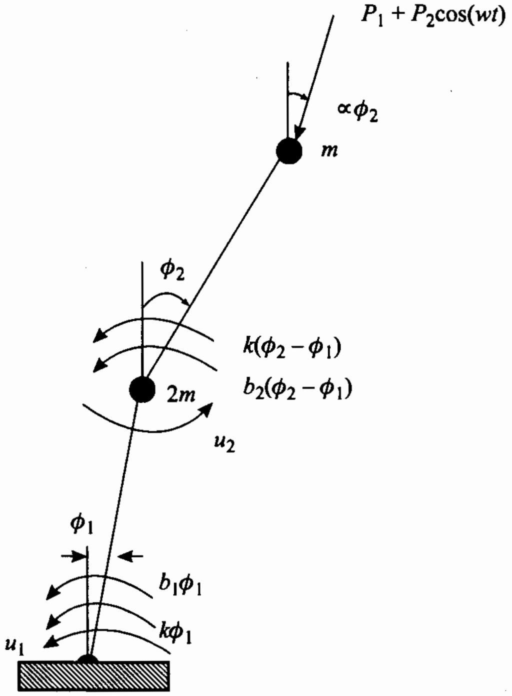

Figure 1. A double inverted pendulum subjected to a periodic loading.

shown in Figure 1 are of the form (see, e.g., Jin and Matsuzaki, 1988, for the autonomous case)

$$\begin{aligned}
\ddot{\phi}_1 &= -0.5\,(B_1 + 2B_2)\,\dot{\phi}_1 + B_2\dot{\phi}_2 + 0.5\bar{k}\,(\bar{p} - 3)\,\phi_1 + 0.5\bar{k}\,(2 - \bar{p})\,\phi_2 \\
&- 0.5\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right)(\phi_1 - \phi_2) - \left(\bar{p}\bar{k}/12\right)\left\{(\phi_1 - \alpha\phi_2)^3 - (1 - \alpha)^3\,\phi_2^3\right\} \\
&- \left((\phi_1 - \phi_2)^2/4\right)\left\{\bar{k}\,(\bar{p} - 4)\,\phi_1 + \bar{k}\,(3 + \bar{p}\,(\alpha - 2))\,\phi_2\right. \\
&- \left.(B_1 + 3B_2)\,\dot{\phi}_1 + 3B_2\dot{\phi}_2\right\} + u_1/2 - u_2/2
\end{aligned} \tag{44}$$

$$\begin{aligned}
\ddot{\phi}_2 &= 0.5\,(B_1 + 4B_2)\,\dot{\phi}_1 - 2B_2\dot{\phi}_2 + 0.5\,(5 - \bar{p})\,\bar{k}\phi_1 + \left\{(\bar{p}\,(1.5 - \alpha) - 2)\,\bar{k}\right\}\phi_2 \\
&+ 0.5\,(\phi_1 - \phi_2)\left(3\dot{\phi}_1^2 + \dot{\phi}_2^2\right) + \left(\bar{p}\bar{k}/12\right)\left\{(\phi_1 - \alpha\phi_2)^3 - 3\,(1 - \alpha)^3\,\phi_2^3\right\} \\
&+ \left((\phi_1 - \phi_2)^2/4\right)\left\{\bar{k}\,(2\bar{p} - 7)\,\phi_1 + \bar{k}\,(5 + \bar{p}\,(\alpha - 3))\,\phi_2\right. \\
&+ \left.(-2B_1 - 5B_2)\,\dot{\phi}_1 + 5B_2\dot{\phi}_2\right\} - u_1/2 + 3u_2/2,
\end{aligned} \tag{45}$$

where m is the mass, l is the length of the links of the pendulum, ϕ_1 and ϕ_2 are the displacement angles, $\dot{\phi}_1$ and $\dot{\phi}_2$ are the corresponding rates, $\alpha(0 \leq \alpha \leq 1)$ is the load-direction parameter, and $P - P_1 + P_2 \cos \omega\tau$. Other symbols appearing in equations (44) and (45) are defined as b_1 and b_2 = damping parameters, $B_1 = b_1/ml^2, B_2 = b_2/ml^2, \bar{p} = Pl/k, \bar{k} = k/ml^2$; k is the stiffness parameter, P_1 is the magnitude of static load; and P_2 is the amplitude of the dynamic periodic load. Equations (44) and (45) are rewritten in the state-space form as

$$\left\{ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{array} \right\} = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5\bar{k}\,(\bar{p}-3) & 0.5\bar{k}\,(2-\bar{p}) & -0.5\,(B_1+2B_2) & B_2 \\ 0.5\bar{k}\,(5-\bar{p}) & \bar{k}\,[\bar{p}\,(1.5-\alpha)-2] & 0.5\,(B_1+4B_2) & -2B_2 \end{array} \right]$$

$$\times \left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right\} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0.5 & -0.5 \\ -0.5 & 1.5 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}$$

$$+ \left\{ \begin{array}{c} (0) \\ \\ (0) \\ \\ \left(-0.5\,(x_3^2+x_4^2)\,(x_1-x_2) - (\bar{p}\bar{k})\right. \\ \times \left[(x_1-\alpha x_2)^3 - (1-\alpha)^3\,x_2^3\right]/12 \\ -0.25\,(x_1-x_2)^2\left[\bar{k}\,(\bar{p}-4)\,x_1 + \bar{k}\,(3+\bar{p}\,(\alpha-2))\,x_2\right. \\ \left.\left. -\,(B_1+3B_2)\,x_3 + 3B_2x_4\right]\right) \\ \\ \left(0.5\,(x_1-x_2)\,(3x_3^2+x_4^2) + \bar{p}\bar{k}\right. \\ \times \left[(x_1-\alpha x_2)^3 - 3\,(1-\alpha)^3\,x_2^3\right]/12 \\ +0.25\,(x_1-x_2)^2\left[\bar{k}\,(2\bar{p}-7)\,x_1 + \bar{k}\,(5+\bar{p}\,(\alpha-3))\,x_2\right. \\ \left.\left. -\,(2B_1+5B_2)\,x_3 + 5B_2x_4\right]\right) \end{array} \right\}, \qquad (46)$$

where $\{x_1, x_2, x_3, x_4\} = \{\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2\}$.

Approach 1

In this approach, a linear control obtained via L-F transformation method is employed in stabilizing the inverted double pendulum subjected to periodic loading. For the parameter set $\bar{k} = 2.0, B_1 = B_2 = 0.0, P_1l/k = 2.0, P_2l/k = 0.7, \alpha = 1, \omega = 2$, equation (46) yields a pair of complex Floquet multipliers with modulus one $(1.177 \times 10^{-2} \pm 0.999i)$ and two real multipliers -2.45 and -0.4082. Note that the modulus of one of the real multipliers is greater than one and therefore the system is unstable and the corresponding unstable states are shown in Figure 2. After normalizing the time with $\omega\tau = 2\pi\, t$, the L-F transformation

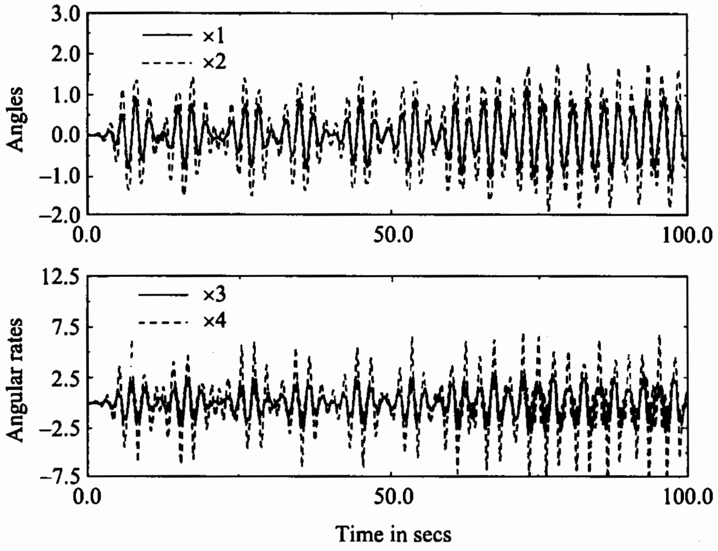

Figure 2. Unstable states of the double inverted pendulum.

is computed and applied to equation (46), leading to the following dynamically equivalent form:

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{Bmatrix} = \begin{bmatrix} 0.0 & 0 & -0.2506 & 0.2109 \\ 0 & 0 & -0.5255 & -0.5185\times 10^{-2} \\ -3.211 & -0.9894 & 0 & 0 \\ 13.9700 & 1.1170 & 0 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix}$$

$$+ \quad \mathbf{Q}^{-1}(t)\,\mathbf{B}(t)\,\mathbf{u} + \mathbf{Q}^{-1}(t)\,\mathbf{f}(z,t)\,, \tag{47}$$

where $\mathbf{B}(t)$ and $\mathbf{f}(z,t)$ are the control matrix and the nonlinear vector as defined in equation (46).

Using the procedure described in Section 3.1, a linear control based on the linear time-invariant matrix $\tilde{\mathbf{B}}\bar{\mathbf{K}}$ for this parametric set is provided as given below.

$$\tilde{\mathbf{B}}\bar{\mathbf{K}} = \begin{bmatrix} 30 & -10 & -0.2506 & 0.2109 \\ 10 & 30 & -0.5255 & -0.0052 \\ -3.2110 & -0.9894 & 35 & -12 \\ -13.9700 & 1.1170 & 12 & 35 \end{bmatrix}.$$

Although there are 16 gains in the above matrix, due to the nature of the pseudo-inverse of the control matrix $\mathbf{B}(t)$, it is to be noted that in effect the last two rows of the gain matrix only are used for control. When implemented in the nonlinear system model, even though the system

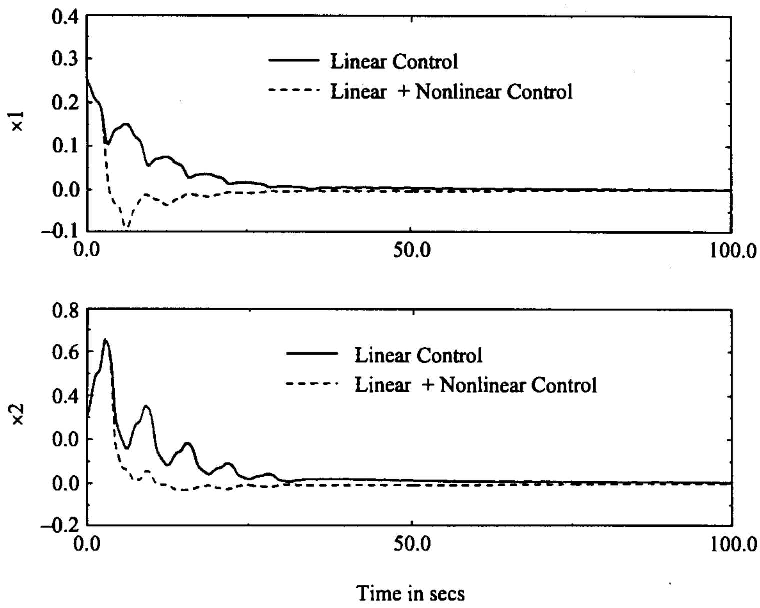

Figure 3. Comparison of controlled angles6 Liapunov-Floquet transformation approach.

is stable, the nonlinear persistent disturbance alters the overshoot or undershoot and settling time characteristics. Therefore, the nonlinear control based on the procedure described in Section 3.4 is also computed and implemented. It can be noticed that by suitably varying the magnitude of the positive-definite matrix $\mathbf{P}(t)$, the performance of the nonlinear system steady-state response characteristics can be altered to the desired specifications. This is clearly seen from Figures 3 and 4 wherein the response characteristics of the pendulum when implemented with linear controller as well as linear and nonlinear controllers are compared. The superb robustness nature of this design can be visualized from Figure 5 wherein the following structured perturbations in the stiffness and damping parameters $\Delta\bar{k} = 0.8$, and $\Delta B_1 = -0.01$ and $\Delta B_2 = -0.01$ are considered, respectively.

Approach 2

Here, a linear control obtained via time-varying pole placement is employed to stabilize the double pendulum. It can be easily verified from the stability diagram given in Figure 6 that for the parameter set $\bar{k} = 1.0, B_1 = B_2 = 0.01, P_1 l/k = 1.0, P_2 l/k = 0.7, \alpha = 1, \omega = 2$, equation (46) without the control matrix and nonlinear vector yields two pair of complex Floquet multipliers with modulus equal to 0.966 and 0.9789. Since the modulus of the Floquet multipliers obtained for the parameter set is found to be within the unit circle, the system will have stable response characteristics. A similar diagram is used when $B = B_2 = 0.0$ has simply stable and unstable boundaries and, therefore, does not provide an asymptotically stable system.

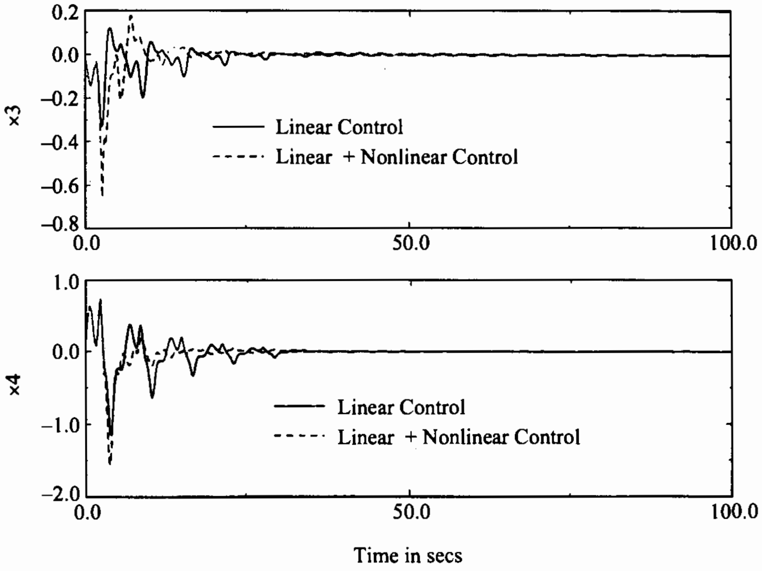

Figure 4. Comparison of controlled angular rates6 Liapunov-Floquet transformation approach.

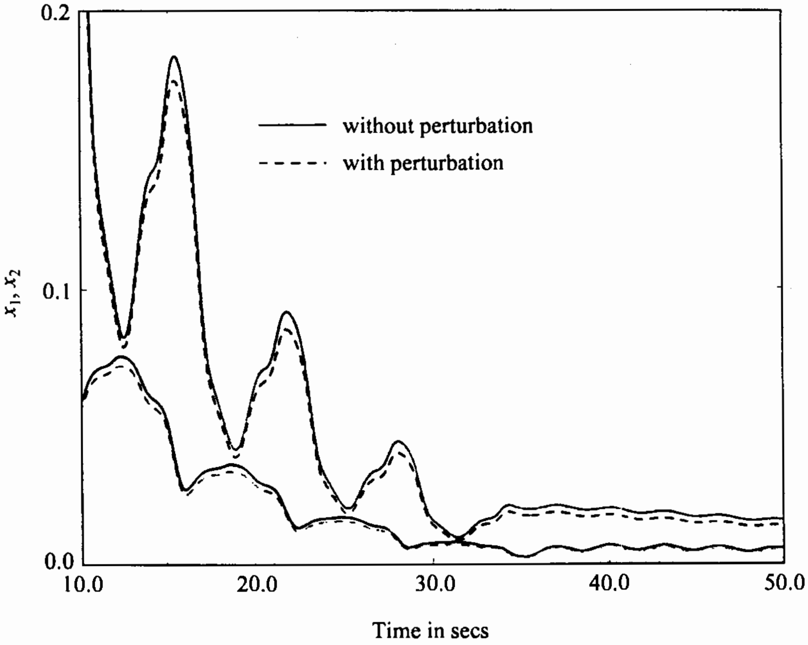

Figure 5. Response with structural perturbations6 Liapunov-Floquet transformation approach.

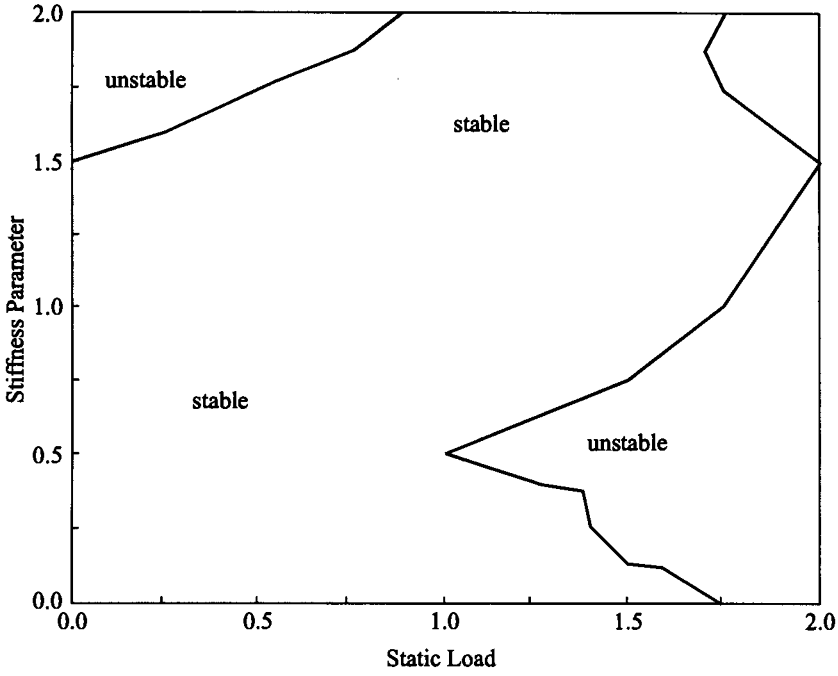

Figure 6. Stability diagram for double inverted pendulum for time-varying pole placement approach.

Since the matrix $\mathbf{A}(t)$ corresponding to the parametric set $\bar{k} = 2.0, B_1 = B_2 = 0.0, P_1 l/k = 2.0, P_2 l/k = 0.7, a = 1, \omega = 2$, the matrix $\mathbf{A}_c(t)$ corresponding to the parametric set $\bar{k} = 1.0, B_1 = B_2 = 0.01, P_1 l/k = 1.0, P_2 l/k = 0.7, a = 1, \omega = 2$ and the control matrix $\mathbf{B}(t)$ are all known quantities, the control gain vector $\mathbf{F}(t)$ can be easily computed from equation (32).

The control gain matrix $\mathbf{F}(t)$ thus obtained is shown as follows:

$$\mathbf{F}(t) = \begin{bmatrix} (1 + 0.7\cos 2t) & (2 + 0.7\cos 2t) & -0.025 & 0.01 \\ -1 & 1 & 0.01 & -0.01 \end{bmatrix}.$$

Later, the nonlinear controller has also been designed and implemented to obtain the desired response characteristics. The comparison of the response of the pendulum with linear as well as linear and nonlinear controllers is shown in Figures 7 and 8. By comparing Figures 3 and 4 to Figures 7 and 8, one can very well conclude that the response of the pendulum obtained through the control via L-F transformation is well behaved and more highly damped than the response obtained by control via the time-varying pole placement approach. Nonetheless, the control design obtained via the time-varying pole placement approach has been found to possess moderate robustness in stability as seen from Figure 9, when structured perturbations in the stiffness and damping parameters $\Delta\bar{k} = 0.2$ and $\Delta B_1 = -0.005$ and $\Delta B_2 = -0.005$ are considered, respectively.

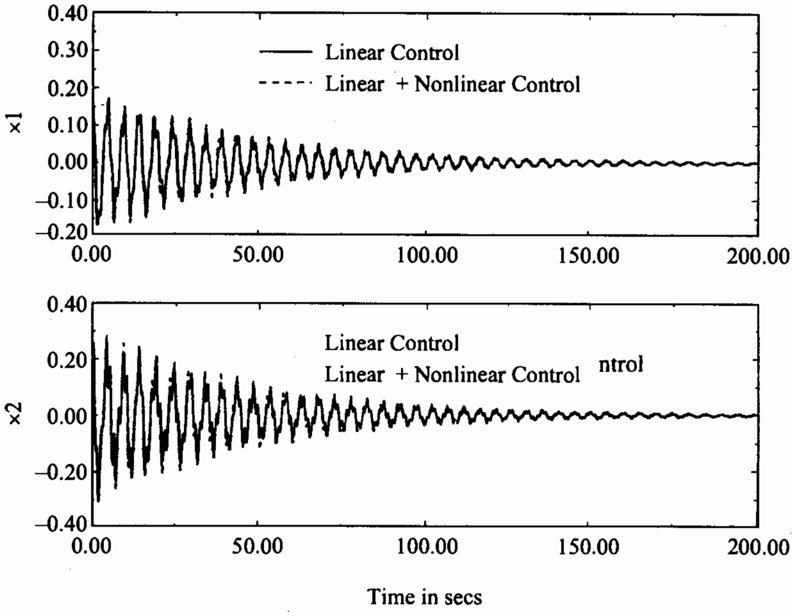

Figure 7. Comparison of controlled anglesó time-varying pole placement approach.

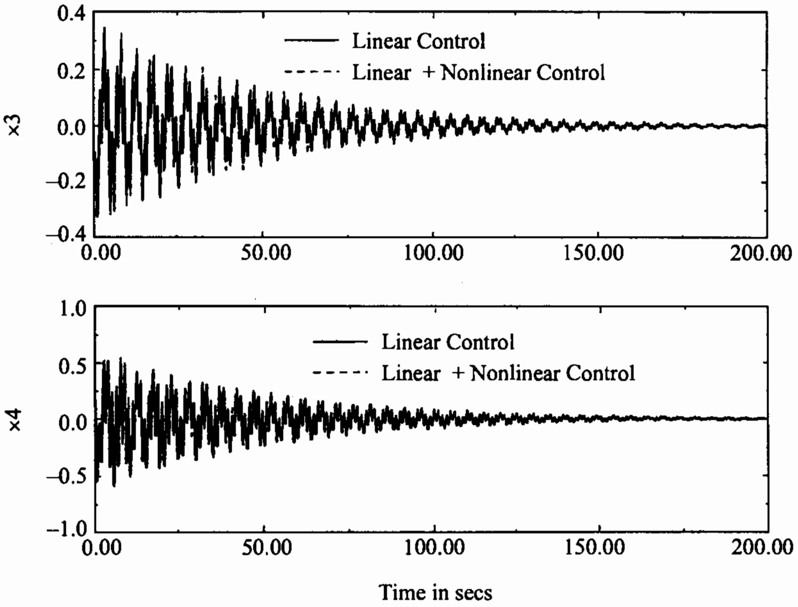

Figure 8. Comparison of controlled angular ratesó time-varying pole placement approach.

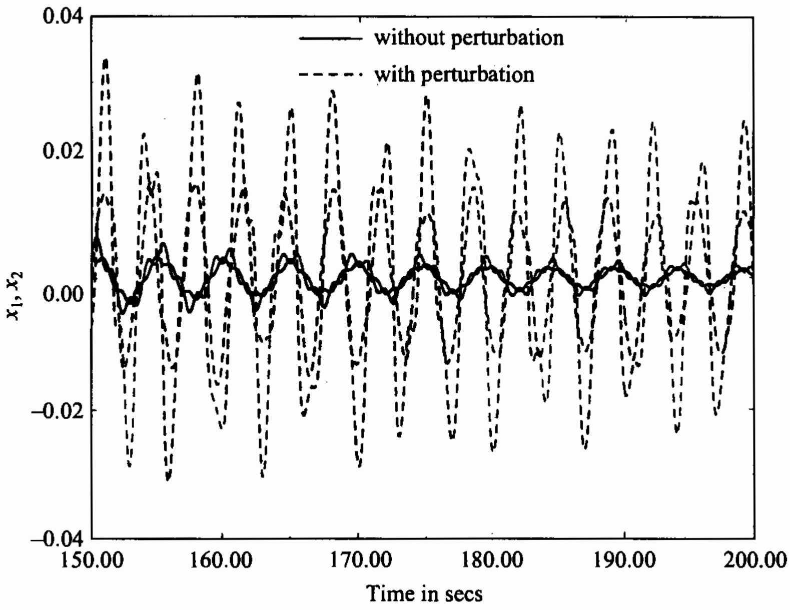

Figure 9. Response with structural perturbations6 time-varying pole placement approach.

5. DISCUSSION AND CONCLUSIONS

As seen from Figures 3 through 9, the double inverted pendulum subjected to a periodic nonconservative load can be controlled using the various designs elaborated in Section 3. In the first approach, a linear controller is designed through an application of the L-F transformation matrix. To improve the system performance, the linear controller thus obtained is supplemented by a nonlinear design based on a Liapunov-type methodology. The results are shown in Figures 3 and 4. In the second approach, the linear controller design is obtained through the time-varying pole placement method and the system performance enhancement is once again achieved via a nonlinear controller. The results for this case are provided in Figures 7 and 8. The robustness of the above control designs in the presence of structured perturbations of the nominal system parameters has been shown in Figures 5 and 9, respectively. The results indicate that the control design based on the L-F transformation approach has better robustness character than the time-varying pole-placement approach. Thus, it can be concluded that the L-F transformation approach forms a viable method in the design of controllers for general, periodic, nonlinear systems, and it is hoped that such a design approach will serve as a fundamental tool in the design of robust controllers for this class of systems.

REFERENCES

Bellman, R., 1970, *Methods of Nonlinear Analysis*, Academic Press, New York.

Bhattachardya, S. P., 1987, *Robust Stabilization Against Structured Perturbations*, Springer-Verlag, Heidelberg, Germany.

Bittanti, S., Laub, A. J., and Willems, J. C., 1991, *The Riccati Equation*, Springer-Verlag, New York.

Burton, T. A., 1985, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, New York.

Calico, R. A. and Wiesel, W. E., 1984, "Control of time-periodic systems," *Journal of Guidance, Control and Dynamics* 7(6), 671-676.

Calise, A. J., Wasikowski, M. E., and Schrage, D. P., 1992, "Optimal output feedback for linear time-periodic systems," *Journal of Guidance, Control and Dynamics* **15**(2), 416-423.

Coddington, E. A. and Levinson, N., 1955, *Theory of Ordinary Differential Equations*, McGraw Hill, New York.

D'Angelo, H., 1970, *Linear Time-Varying Systems: Analysis and Synthesis*, Allyn and Bacon, Boston, MA.

Evan-Iwanowski, R. M., 1976, *Resonance Oscillations in Mechanical Systems*, Elsevier, Amsterdam.

Infante, E. F., 1968, "On the stability of some linear nonautonomous random systems," *Journal of Applied Mechanics* **35**, 7-12.

Jin, J.-D. and Matsuzaki, Y., 1988, "Bifurcations in a two-degree-of-freedom elastic system with follower forces," *Journal of Sound and Vibration* **126**(2), 265-277.

Joseph, P., 1993, *New Strategies in the Control of Linear Dynamic Systems With Periodically Varying Coefficients*, Ph.D. dissertation, Auburn University, Auburn, AL.

Joseph, P., Pandiyan, R., and Sinha, S. C., 1993, "Optimal control of mechanical systems subjected to periodic loading via Chebyshev polynomials," *Optimal Control Applications and Methods* **14**, 75-90.

Krasovskii, N. N., 1963, *Stability of Motion*, Stanford University Press, Stanford, CA.

Lovass-Nagy, V., Miller, R. J., and Powers, D. L., 1978, "An introduction to the application of the simplest matrix-generalized inverse in system science," *IEEE Transactions: Circuits System* **CAS-25**, 766-771.

McKillip Jr., R. M., 1985, "Periodic control of individual-blade-control helicopter rotor," *Vertica* **9**, 199-225.

Powell, J. L. and Crasemann, B., 1961, *Quantum Mechanics*, Addison-Wesley, Boston, MA.

Richards, J. A., 1983, *Analysis of Periodically Time-Varying Systems*, Springer-Verlag, Berlin.

Roseau, M., 1987, *Vibrations in Mechanical Systems: Analytical Methods and Applications*, Springer-Verlag, New York.

Sinha, S. C. and Joseph, P., 1994, "Control of general dynamic systems with periodically varying parameters via Liapunov-Floquet transformation," *Journal of Dynamic Systems, Measurement, and Control* **116**, 650-658.

Sinha, S. C. and Juneja, V., 1991, "An approximate analytical solution for systems with periodic coefficients via symbolic computation," in *AIAA/ASME/ASCE/AHS/ASC 32nd Structures*, Structural Dynamics and Materials Conference, A Collection of Papers, Part 1, April 8-10, pp. 790-797.

Sinha, S. C. and Wiens, G. J., 1989, "Response bounds for lumped vibrational systems with time-dependent parameters," *Journal of Sound and Vibration* **132**(1), 1-9.

Sinha, S. C. and Wu, D.-H., 1991, "An efficient computational scheme for the analysis of periodic systems," *Journal of Sound and Vibration* **151**, 91-117.

Streit, D. A., Krousgrill, C. M., and Bajaj, A. K., 1989, "Nonlinear response of flexible robotic manipulators performing repetitive tasks," *Journal of Dynamic Systems, Measurement, and Control* **111**(83), 470-480.

Wiens, G. J., Lu, M.-S., and Sinha, S. C., 1991, "Controller design for nonlinear mechanical systems via Liapunov's direct method," in *AIAA/ASME/ASCE/AHS/ASC 32nd Structures*, Structural Dynamics and Materials Conference, Part 3, Baltimore, April 8-10, pp. 2239-2246.

Yakubovich, V. A. and Starzhinskii, V. M., 1975, *Linear Differential Equations With Periodic Coefficients*, Part I and Part II, John Wiley, New York.