SYMBOLIC COMputation OF FUNDAMENTAL SOLUTION MATRICES FOR LINEAR TIME-PERIODIC DYNAMICAL SYSTEMS

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A new technique which employs both Picard iteration and expansion in a set of Chebyshev polynomials is used to symbolically approximate the fundamental solution matrix for linear time-periodic dynamical systems of arbitrary dimension explicitly as a function of the system parameters and time. As in previous studies, the integration and product operations which are associated with the Chebyshev polynomials are employed. However, the need to algebraically solve for the Chebyshev coefficients of the fundamental solution matrix is completely avoided as only matrix multiplications and additions are utilized. Since these coefficients are expressed as homogeneous polynomials of the system parameters, closed form approximations to the true solutions may be obtained. Also, because this method is not based on expansion in terms of a small parameter, it can successfully be applied to periodic systems whose internal excitation is strong. Two formulations are proposed. The first is applicable to general time periodic systems while the second approach is useful when the system equations contain a constant matrix. Three different example problems, including a simple inverted pendulum subjected to a periodic follower force, are included and CPU times and convergence results are discussed.

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1. INTRODUCTION

The study of dynamical systems governed by a set of ordinary differential equations with periodic coefficients is of great theoretical and practical importance in various fields of science and engineering. These equations generally represent the perturbed dynamics about a steady state periodic motion of the system. In many situations, the linearized perturbed equations may be sufficient for the prediction of stability and therefore the problem reduces to a set of linear differential equations with periodic coefficients. The stability conditions are determined by the requirement that the eigenvalues of the fundamental solution matrix evaluated at the end of the principal period (called the Floquet multipliers) must lie within the unit circle in the complex plane. For various bifurcation studies, it is necessary to generate the non-linear equations of the perturbed dynamics. However, the local bifurcation conditions are once again determined by the nature of the Floquet multipliers of the linearized system. Further, fundamental solution matrices also play an important role in designing feedback controllers for all dynamical systems. If the fundamental solution matrix could be computed as a function of the system parameters, then it would be possible to determine the stability (bifurcation) conditions and the controller gains in a closed form. In this study an attempt has been made toward achieving this goal.

It is well known that exact solutions of periodic systems are possible only in a very limited number of cases (see section 5.1), and, in general such solutions do not exist. Two
common asymptotic methods that have been used in the past to yield an approximate fundamental solution matrix with parametric dependence in closed form include the perturbation method [1] and the averaging technique [2]. These methods are limited in application to systems with weak internal excitation since they are based on expanding the solution in terms of a small parameter that multiplies the time-periodic terms. When the excitation becomes strong, the parameter is not small and the accuracy of the solution is very poor. Moreover, increasing the order of the approximate solution is usually very difficult and does not guarantee uniform convergence to the true solution. A recent approximation technique of Guttal and Flashner [3, 4] for computing the fundamental solution matrix evaluated at the end of the principal period (called the Floquet transition matrix) by truncated point mappings avoids the restriction of a small periodic parameter but is computationally expensive in terms of CPU time. In addition, the fundamental solution matrix is not obtained in closed form as an explicit function of time.

Another approximation technique which has developed recently involves expanding periodic coefficients in terms of Chebyshev polynomials and was first introduced by Sinha and Chou [5] and Sinha et al. [6]. Although these applications were limited to second order scalar equations only, a later study by Sinha and Wu [7] outlined a scheme to obtain an approximate fundamental solution matrix in closed form for a system of second order equations. The approach was based on the idea that the state vector and the periodic system matrix can be expanded in terms of Chebyshev polynomials over the principal period. By employing the integration and product operational matrices associated with the polynomials, this expansion reduces the original problem to a set of linear algebraic equations for the Chebyshev coefficients of the state vector from which the solution in the interval of one period is obtained. In later studies [8, 9], this method was applied to single-degree-of-freedom problems to symbolically approximate the solution in terms of the system parameters. It was shown that a ten- or twelve-term Chebyshev expansion provides much better accuracy compared to a sixteenth-order perturbation series when the coefficients of the periodic terms are large. However, the Chebyshev coefficients of the solution must be obtained using Cramer’s rule, matrix inversion, etc., and therefore this approach is impractical for systems of higher dimension.

In this paper, a new technique which employs both Picard iteration and expansion in shifted Chebyshev polynomials is used to symbolically approximate the fundamental solution matrix for time-periodic dynamical systems of arbitrary dimension explicitly as a function of the system parameters and time. Since this method is not based on expansion in terms of a small parameter, it can successfully be applied to periodic systems with strong internal excitations. As in the previously mentioned studies, the integration and product operational matrices associated with the shifted Chebyshev polynomials are employed. However, the need to algebraically solve for the Chebyshev coefficients is completely avoided by employing Picard iterations (sequence of approximations). The Chebyshev coefficients of the resulting fundamental solution matrix are expressed as homogeneous polynomials of the system parameters, thus enabling a closed form approximation to the true solution to be obtained. Two formulations, one applicable to general periodic systems and one for equations which contain a constant system matrix, are outlined and applied to three different example problems including a double inverted pendulum subjected to a periodic follower force. CPU times are included and convergence results are discussed in detail.
The first non-homogeneous differential equation
\[ \frac{dx(t)}{dt} = f(x(t), t), \quad x(t) = x^0, \tag{1} \]
may be expressed in an equivalent integral form such as
\[ x(t) = x^0 + \int_0^t f(x(s), s)\, ds, \tag{2} \]
where \( \varepsilon \) is a dummy variable. Assuming an initial approximation \( x^{(0)}(t) = x^0 \) and inserting it on the right side of equation (2), an approximation \( x^{(n)}(t) \) is generated. This is inserted back into the integral to generate the second approximation \( x^{(2)}(t) \), etc. This process, called Picard iteration, satisfies the recurrence equation
\[ x^{(n+1)}(t) = x^0 + \int_0^t f(x^n(s), s)\, ds, \tag{3} \]
and results in a sequence of approximations to the true solution \( x(t) \) [10]. If \( f(x, t) \) is defined, continuous, and satisfies a Lipschitz condition with Lipschitz constant \( L \) in an interval about \( t_0 \), then equation (3) may be used to obtain the relationship
\[ |x^{(n)}(t) - x^{(n-1)}(t)| \leq ML^n|t - t_0|, \tag{4} \]
from which the neighborhood of convergence \( h = t - t_0 \) is obtained as
\[ h \leq \frac{1}{M} \ln \left( \frac{1 + LB/M}{M} \right), \tag{5} \]
where \( |x(t) - x^0| < B \) and \( |f(x, t)| < M \) [11]. In contrast to asymptotic techniques, all of the system parameters are treated equally in Picard iteration so that the convergence varies radially in the parameter space. This technique is of great theoretical importance in the theory of the existence of solutions of equation (1), but the difficulty of evaluating the integral in equation (2) has made it impractical for general numerical computations even for the special case of linear time-varying equations. It will be seen that this obstacle is circumvented here by expanding the periodic system matrix in Chebyshev polynomials. By employing the associated operational matrices defined in section 3, the successive integrations associated with the Picard iterations are replaced by simple matrix multiplications.

3. SHIFTED CHEBYSHEV POLYNOMIALS AND THE OPERATIONAL MATRICES
The shifted Chebyshev polynomials of the first kind are defined in terms of the standard Chebyshev polynomials of the first kind \( T_n(t) \) valid over the interval \([-1, 1]\) by using the change of variable
\[ t \to (t + 1)/2. \tag{6} \]
Thus, the shifted Chebyshev polynomials of the first kind are given by
\[ T^*_n(t) = T_n(2t - 1), \quad 0 \leq t \leq 1, \tag{7} \]
and are valid over the interval \([0, 1]\). All properties of \( T^*_n(t) \) can be deduced from those of \( T_n(2t - 1) \). The orthogonality and recurrence relations for these polynomials can be
found in references [12, 13]. If \( T^*(t) = (T(t_1) T^2(t_1) \cdots T_{2r-1}(t_1))^T \) is an \( m \times 1 \) column vector of the polynomials, then the outer product of two of these vectors is

\[
\begin{bmatrix}
T^2(t) & T^4(t) & T^6(t) & \cdots & T_{2r-1}(t) \\
T^2(t) & (T^2(t) + T^4(t))/2 & (T^2(t) + T^4(t))/2 & \cdots & T_{2r-1}(t)/2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_{2r-1}(t) & T_{2r-1}(t) + T_{2r-3}(t)/2 & & \cdots & \vdots \\
\end{bmatrix}.
\]

(8)

Generally, an arbitrary continuous time function \( f(t) \) can be approximated by a finite shifted Chebyshev series over the interval \([b, 1] \) where the coefficients of the polynomials can be obtained as shown in [14, 15]. If two such functions are expanded as

\[
f(t) = \sum_{i=0}^{n-1} a_i T_i(t), \quad g(t) = \sum_{i=0}^{n-1} b_i T_i(t),
\]

(9)

then

\[
f(t) g(t) = (a_0 a_0 \cdots a_m) (b_0 T^*(t)(b_0 b_1 \cdots b_m))^T.
\]

(10)

where \( a \) and \( b \) are Chebyshev coefficients of the functions \( f(t) \) and \( g(t) \), respectively. Using equation (8), equation (10) can be rewritten as

\[
f(t) g(t) = T^*(t) Q b
\]

(11)

where \( Q \) is the product operational matrix corresponding to \( f(t) \) given by

\[
Q_s = \begin{bmatrix}
a_0 & a_0/2 & \cdots & a_{m-1}/2 \\
a_0 & a_0 + a_2 & \cdots & (a_{m-1} + a_m) \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
a_0 & a_0 + a_2 & \cdots & a_0 + a_{m-1}/2
\end{bmatrix}
\]

(12)

and \( b = (b_0 b_1 \cdots b_m)^T \) \cite{7}. The \( a \) coefficients with \( r > m - 1 \) in equation (12) may be set to zero. The general recursive formula for integration of an \( m \times 1 \) vector of shifted Chebyshev polynomials of the first kind may be written in vector form as

\[
\int_a^b \cdots \int_a^b T^*(t_1, \ldots, t_r) dt_r \cdots dt_1 dt_0 = G T^*(t),
\]

(13)
where \( G \) is the \( m \times m \) integration operational matrix given by

\[
G = \begin{bmatrix}
1/2 & 1/2 & 0 & 0 & 0 & \cdots & 0 \\
-1/8 & 0 & 1/8 & 0 & 0 & \cdots & 0 \\
-1/6 & -1/4 & 0 & 1/12 & 0 & \cdots & 0 \\
1/16 & 0 & -1/8 & 0 & 1/16 & \cdots & \vdots \\
-1/30 & 0 & 0 & -1/12 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^{m/2}m(m-2) & 0 & 0 & 0 & \cdots & 0 & 1/(m+1)
\end{bmatrix}
\]

(14)

\( T^T \) denotes the transpose of the quantity \( T \), and all \( \tau \)'s are dummy variables [7]. Use of the \( G \) matrix results in a forward difference recurrence procedure in which the \( (m+1) \)th term is truncated in order to keep the polynomial vector the same length.

Let the \( nn \times n \) Chebyshev polynomial matrix be defined as

\[
\hat{T}(t) = I_n \otimes T(t),
\]

(15)

where \( \otimes \) signifies the Kronecker product defined in Appendix A. Then the recursive formula for integration of this matrix can be written as

\[
\int_a^b \cdots \int_a^b T(t) \, dt_n \cdots dt_1 \, dt = \hat{G} \hat{T}(t),
\]

(16)

where \( \hat{G} = I_n \otimes G \) is of dimension \( nn \times nn \). Also, the product of two periodic matrix functions which have been expanded in Chebyshev polynomials can be expressed as

\[
\Theta(t) \Psi(t) = A \hat{T}(t)B = \hat{T}(t)Q, B
\]

(17)

where

\[
A = \begin{bmatrix}
\alpha_1 & \cdots & \alpha_n \\
\alpha_1 & \cdots & \alpha_n \\
\vdots & \ddots & \vdots \\
\alpha_1 & \cdots & \alpha_n
\end{bmatrix}, \quad B = \begin{bmatrix}
\beta_1 & \cdots & \beta_n \\
\beta_1 & \cdots & \beta_n \\
\vdots & \ddots & \vdots \\
\beta_1 & \cdots & \beta_n
\end{bmatrix}
\]

(18)

are \( n \times nn \) and \( nn \times n \), respectively, and \( \alpha_1 \) and \( \beta_1 \) are the Chebyshev coefficient vectors in the expansion of \( \Theta(t) \) and \( \Psi(t) \), respectively. The \( nn \times nn \) matrix

\[
Q = \begin{bmatrix}
Q_{11} & \cdots & Q_{1n} \\
\vdots & \ddots & \vdots \\
Q_{nn} & \cdots & Q_{nn}
\end{bmatrix}
\]

(19)

consists of an \( nn \times nn \) array of product operational matrices corresponding to the elements of \( \Theta(t) \). (Note that if \( \Theta(t) \) is symmetric then \( A^T = A \)). The Chebyshev expansion of the \( n \)-dimensional identity matrix is \( I_n = T(1) \otimes I = I_n \hat{T}(t) \), where

\[
I = I_n \otimes (1 \, 0 \, \cdots \, 0)^T
\]

is the \( nn \times nn \) identity coefficient matrix.
4.1. SYSTEM UNDER CONSIDERATION

Consider a system of \( n \) linear time-periodic differential equations

\[
x(t, \alpha) = A(t, \alpha)x(t, \alpha), \quad x(0, \alpha) = x^0,
\]

(20)

where \( x(t, \alpha) \in \mathbb{R}^n \) is the state vector which depends on the time \( t \in \mathbb{R}^+ \) and a parameter vector \( \alpha \in \mathbb{R}^r \) (the derivative is with respect to time), and the \( n \times n \) matrix \( A(t, \alpha) \) can be written as \( A(t, \alpha) = A_0(t, \alpha) + A_1(t, \alpha) + \cdots + A_r(t, \alpha) \). The functions \( f_i(t) = f_i(t + \beta_i) \), \( i = 1, \ldots, r \) are periodic with period \( \beta_i \) and the \( n \times n \) constant matrices \( A_i(\alpha), i = 1, \ldots, r \) contain the coefficient of these periodic functions. Assuming that the frequencies are commensurate, the lowest positive number \( T \) such that \( q \beta_i = T \) for positive integers \( q_i \), is the principal period of the system matrix \( A(t + T, \alpha) = A(t, \alpha) \). The fundamental solution matrix \( \Phi(t, \alpha) \) of equation (20) satisfies \( \Phi(t, \alpha) = A(t, \alpha)\Phi(t, \alpha) \); \( \Phi(0, \alpha) = I \) and the solution for the given initial conditions may be expressed as \( x(t, \alpha) = \Phi(t, \alpha)x^0 \). In the following, two formulations for finding \( \Phi(t, \alpha) \) via Picard iteration and Chebyshev expansion are presented.

4.2. THE GENERAL FORMULATION

The first formulation is applicable to general systems of the form of equation (20). An equivalent integral form of equation (20) is

\[
x(t, \alpha) = x^0 + \int_0^t A(t, \alpha)x(t, \alpha) \, dt.
\]

(21)

As the zeroth approximation, let \( x^0(t, \alpha) = x(0, \alpha) = x^0 \). Use of equation (21) then leads to the first approximation

\[
x^{(1)}(t, \alpha) = x^0 + \int_0^t A(t, \alpha)x^{(0)}(t, \alpha) \, dt = x^0 + \left[ I + \int_0^t A(t, \alpha) \, dt \right] x^0,
\]

(22)

where \( t_0 \) is a dummy variable. The second approximation is obtained from equation (21) as

\[
x^{(2)}(t, \alpha) = x^0 + \int_0^t A(t, \alpha)x^{(1)}(t, \alpha) \, dt = x^0 + \left[ I + \int_0^t A(t, \alpha) \, dt \right] \left[ I + \int_0^t A(t, \alpha) \, dt \right] x^0,
\]

(23)

where \( t_1 \) is another dummy variable. Further iteration leads to the \((k + 1)\) th approximation

\[
x^{(k+1)}(t, \alpha) = x^0 + \int_0^t A(t, \alpha)x^{(k)}(t, \alpha) \, dt
\]
\[
= I + \sum_{n=1}^{\infty} A(t_n, a) \, dt_n + \sum_{n=1}^{\infty} A(t_n, a) \, dt_n \, \cdots \, dt_n \\
+ \cdots + \sum_{n=1}^{\infty} A(t_n, a) \, dt_n \, \cdots \, dt_n \, \cdots \, dt_n \, k^n, 
\]

where \( t_0, \ldots, t_n \) are all dummy variables. The series of integrals is an approximation to the fundamental matrix \( \Phi(t, a) \) since it is truncated after a finite number of terms, while the true solution is an infinite series. If \( A(t, a) = A(a) \) is a constant matrix, then this series results in the power series definition of the exponential solution of equation (20), namely

\[
x(t, a) = e^{A(a)t} = [I + A(a)t + (A(a)t)^2/2! + (A(a)t)^3/3! + \cdots]x^0. 
\]

Unfortunately, the symbolic evaluation of the fundamental matrix via equation (24), in general, leads to complicated expressions for \( \Phi(t, a) \) and, in addition, is not efficient when \( r > 1 \) due to the necessary repeated integration by parts. Instead, the following approach is taken which results in a more efficient approximation of \( \Phi(t, a) \). First, the transformation \( t = T^1 \) to equation (20) normalizes the system matrix's principal period to one and, after multiplying through by \( T \), the equation is

\[
dx(t, a)/dt = \tilde{A}(t, a)x(t, a), \quad \tilde{A}(t + 1, a) = \tilde{A}(t, a), \quad x(0, a) = x^0, 
\]

where

\[
\tilde{A}(t, a) = A(a)/f(t) + \tilde{A}(a)/f(t) + \cdots + \tilde{A}(a)/f(t), \quad f(t) = f(t + 1).
\]

and

\[
\tilde{A}(a) = T\tilde{A}(a), \quad i = 1, \ldots, r.
\]

Next, the Chebyshev polynomial matrix defined in section 2 is used in expanding the normalized system matrix in \( m \) shifted Chebyshev polynomials of the first kind as

\[
D(a) = \sum_{n=1}^{2m-1} \tilde{A}(a) \tilde{d}^n, \quad \tilde{D}(a) = \sum_{n=1}^{2m-1} \tilde{A}(a) \tilde{d}^n,
\]

where \( n \times n \) (respectively \( n \times mn \)) Chebyshev coefficient matrix \( D(a) \) (respectively \( \tilde{D}(a) \)) is defined as

\[
D(a) = \sum_{i=1}^{2m-1} \tilde{A}(a) \tilde{d}^i, \quad \tilde{D}(a) = \sum_{i=1}^{2m-1} \tilde{A}(a) \tilde{d}^i.
\]

The \( m \times i \) column vectors \( \tilde{d} \) (respectively \( 1 \times m \) row vectors \( \tilde{d}' \)) contains the known coefficients in the Chebyshev system of the \( i \)-periodic functions as

\[
f(t) = \sum_{i=1}^{2m-1} \tilde{d}_i T_i(t) = \tilde{T}^1(t) \tilde{d}, \quad \tilde{d}' \tilde{T}^* (t),
\]

where \( \tilde{T}^* (t) (0 \leq t \leq 1) \) are the Chebyshev polynomials. Then, using the integration operational matrix and the identity coefficient matrix defined in section 2, equation (22) can be written as

\[
x^{\tilde{T}}(t, a) = I + \tilde{T}^1(t_0) \tilde{D}(a) \, dt_n + \tilde{T}^1(t_0) \tilde{D}(a) \, dt_n \, \cdots \, dt_n \, k^n, 
\]

where \( \tilde{T}^* (t) \).
where the superscript \((1, m)\) indicates that the first Picard iteration is approximated by \(m\) Chebyshev polynomials. Furthermore, using the product operational matrices also defined in section 2, equation (23) can be written as

\[
\begin{align*}
X^{(1)}(t, x) &= \left[ T(t)[I + G\cdot D(x)] + \int_0^t D(x)T(t - \tau)T(\tau)[I + G\cdot D(x)] d\tau \right]X^0 \\
&= T(t)[I + G\cdot D(x)] + \int_0^t T(t - \tau)Q_1(x)G\cdot D(x) d\tau X^0 \\
&= T(t)[I + G\cdot D(x)] + \sum_{n=1} Q_n(x)G\cdot D(x)^{x,n}.
\end{align*}
\]

(31)

where \(Q_n(x) = \sum_{\lambda} \lambda(x)\otimes Q_n\) is expressed in terms of the product operational matrices \(Q_n\) corresponding to the Chebyshev coefficients of the periodic functions \(f(t)\). Continuing in this way, the approximate fundamental matrix solution of equation (20) over the principal period can be written in terms of the Chebyshev polynomials as

\[
\Phi^{(n)}(t, x) = T(t)
\left[ I + \left( \sum_{n=1}^{\infty} [L(n)]^{-1} \right) P(x) \right] - T(t)B(x),
\]

(32)

where \(B(x)\) contains the Chebyshev coefficients of the elements of \(\Phi(t, x)\) and is expressed in terms of \(L(x) = G\cdot Q_1(x)\) and \(P(x) = G\cdot D(x)\), which are \(nm \times nm\) and \(nm \times n\), respectively. By selecting a value for \(p\), the number of Picard iterations, this truncated expression gives an approximate solution to any desired degree of accuracy. While this is valid only in the interval \(t \in [0, T]\) or \(t \in [0, 1]\), the solution can be easily extended for \(t > T\) for \(t > 1\) by utilizing the formula

\[
\Phi^{(n)}(t, x) = \Phi^{(n)}(t, x)[\Phi^{(n)}(1, x)]^\tau.
\]

(33)

where \(\tau = k + q, q \in [0, 1], k = 1, 2, \ldots\). The matrix \(\Phi^{(n)}(1, x)\) is the Floquet Transition Matrix (FTM) whose eigenvalues (Floquet multipliers) determine the stability characteristics of the system. While these expressions are in terms of the normalized time, the substitution \(t = \tau / T\) yields the result in real time. It should also be noted that, using the special properties of the Kronecker product operation, \(L(x)\) and \(P(x)\) can be written in a more computationally efficient form as

\[
L(x) = \sum_{\lambda} \lambda(x) \otimes [G\cdot Q_1], \quad P(x) = \sum_{\lambda} \lambda(x) \otimes [G\cdot D].
\]

(34)

in which the amount of effort spent in matrix multiplications is minimized. The integrations in equation (24) are replaced by the more computationally efficient matrix multiplications in equation (32), and an approximation to the fundamental matrix in terms of the shifted Chebyshev polynomials is made by including a finite number \(p\) of Picard iterations and an appropriate number \(m\) of Chebyshev polynomials (which determines the sizes of the various matrices). The Chebyshev expansion of the periodic matrix \(X(t, x)\) not only provides the efficient symbolic approximation of the fundamental matrix, but also results in a more compact result than if the actual integrations in equation (24) had been performed. Also, because the solution is approximated via Picard iterations, it is not pre-expanded in a Chebyshev series with unknown coefficients as was done in
earlier studies [7, 8]. This eliminates the need to solve for the Chebyshev coefficients matrix \( H(\alpha) \) via Cramer’s rule, matrix inversion, etc., and thus permits the symbolic evaluation of this matrix. Therefore an approximate expression for the fundamental solution matrix can be obtained in terms of the system parameter vector \( \alpha \) and the normalized time \( \tau \) via simple matrix multiplications and addition. Three different problems are analyzed using this formulation in section 5.

4.3. Alternate Formulation for Systems with a Constant Matrix 

Now consider a specific form of equation (26) given by
\[
\dot{x}(i, \alpha) = A(i, \alpha)x(i, \alpha), \quad x(0, \alpha) = x^0, \tag{35}
\]
where the matrix \( A(i, \alpha) \) is independent of both time and the system parameters and \( \dot{x}(i, \alpha) = A(i, \alpha)x(i, \alpha) \). The function \( f(t) \) is unity and has a one-term (in \( T^n(\tau) \) Chebyshev expansion. The associated product operational matrix is then the identity matrix, and the application of equation (32) to the entire system results in an expression which includes the power series of the exponential matrix solution for the constant matrix \( A(i, \alpha) \) of the form of equation (25). While the general formulation above may provide satisfactory convergence if the elements in \( A(i, \alpha) \) are not very large, the required number of Picard iterations to achieve accuracy may significantly increase as these matrix elements are allowed to increase in magnitude. This is due to the well-known slow convergence rate of the power series solution in equation (25). It is desired, therefore, to take advantage of the closed-form solution of the constant part of equation (35) with the anticipation that a faster convergence would be achieved. For this purpose, the time-periodic term in equation (35) is treated as a forcing term in a non-homogeneous system which has the computable generating solution \( e^{\alpha t} \). Use of the superposition integral then results in the integral equation
\[
x(i, \alpha) = e^{\alpha t} x^0 + \int_0^t e^{\alpha (\tau - t)} A(t, \alpha) x(i, \alpha) d\tau \tag{36}
\]
which may be iterated similarly to equation (24) to yield the \((k + 1)\)th approximation
\[
x^{(k+1)}(i, \alpha) = e^{\alpha t} x^0 + \int_0^t e^{\alpha (\tau - t)} A(t, \alpha) x^{(k)}(i, \alpha) d\tau + \int_0^t e^{\alpha (\tau - t)} A(t, \alpha) e^{\alpha \tau} \int_0^\tau e^{\alpha (\tau' - \tau)} A(t', \alpha) x^{(k)}(i, \alpha) d\tau' d\tau + \cdots \tag{37}
\]
from which the approximate fundamental solution matrix is obtained. However, as with equation (24), the direct symbolic evaluation of equation (37), in general, leads to complicated expressions and is not efficient for \( k > 1 \) due to repeated integrations by parts. Instead, the method of expanding in Chebyshev polynomials after normalizing the principal period to identity as in section 4.2 is utilized.

After normalizing, the Chebyshev expansions
\[
A(\tau, \alpha) = T^n(\tau)d(\alpha), \quad e^{\alpha \tau} = T^n(\tau)E, \quad e^{-\alpha \tau} = T^n(\tau)F \tag{38}
\]
are made where \( \tilde{A} = TA \) and \( \tilde{A}(t, \mathbf{n}) = T A(t, \mathbf{n}) \) are the normalized system matrices. The \( m \times n \) Chebyshev coefficient matrix \( \mathbf{D}(m) \) is defined as in equations (28) and (29) while the definitions of \( \mathbf{E} \) and \( \mathbf{F} \) depend on whether the eigenvalues of \( \tilde{A} \) are real, imaginary, or complex. If they occur in \( \eta/2 \) imaginary pairs \( \pm \alpha + \mathbf{t} \), then the coefficient matrices of 
\[ e^{\alpha(t)} = \sum_{i=0}^{n} \mathbf{C} \cos(\alpha(t)) + \mathbf{S} \sin(\alpha(t)) \] 
and \[ e^{\mathbf{t}} = \sum_{i=0}^{n} \mathbf{C} \cos(\mathbf{t}) - \mathbf{S} \sin(\mathbf{t}) \] 
can be expressed as
\[ \mathbf{E} = \sum_{i=0}^{n} \mathbf{C} \mathbf{b}(\alpha) + \mathbf{S} \mathbf{b}(\mathbf{t}), \quad \mathbf{F} = \sum_{i=0}^{n} \mathbf{C} \mathbf{b}(\alpha) - \mathbf{S} \mathbf{b}(\mathbf{t}), \]  
(39)

where the \( m \times 1 \) Chebyshev coefficient column vectors \( \mathbf{b}(\alpha) \) and \( \mathbf{b}(\mathbf{t}) \) may alternately be defined in terms of Bessel functions in the expansions of \( \cos(\alpha(t)) = T^\mathbf{t}(\mathbf{r}, \mathbf{b}(\alpha)) \) and \( \sin(\alpha(t)) = T^\mathbf{t}(\mathbf{r}, \mathbf{b}(\alpha)) \) [12, 17]. If the eigenvalues \( \alpha_i, \mathbf{t}_i = 1, \ldots, n \) are real, then the coefficient matrices of 
\[ e^{\alpha(t)} = \sum_{i=0}^{n} \mathbf{Z} \mathbf{e}^\alpha_i \] 
and \[ e^{\mathbf{t}} = \sum_{i=0}^{n} \mathbf{Z} \mathbf{e}^\mathbf{t}_i \] 
are
\[ \mathbf{E} = \sum_{i=0}^{n} \mathbf{Z} \mathbf{e}^\alpha_i, \quad \mathbf{F} = \sum_{i=0}^{n} \mathbf{Z} \mathbf{e}^\mathbf{t}_i, \]  
(40)

where \( \mathbf{e}^\alpha_i \) may also be defined in terms of Bessel functions in the expansion of \( e^{\alpha(t)} = T^\mathbf{t}(\mathbf{r}, \mathbf{e}^\mathbf{t}_i) \) [17]. If the eigenvalues occur in \( \eta/2 \) complex pairs \( \alpha_i \pm \mathbf{t}_i \), then the product operational matrix may be used to yield the Chebyshev coefficient column vectors as
\[ e^{\alpha(t)} = \sum_{i=0}^{n} \mathbf{C} \mathbf{e}^{\alpha_i} \cos(\alpha(t)) + \mathbf{S} \mathbf{e}^{\alpha_i} \sin(\alpha(t)) \] 
and \[ e^{\mathbf{t}} = \sum_{i=0}^{n} \mathbf{C} \mathbf{e}^{\mathbf{t}_i} \cos(\mathbf{t}) - \mathbf{S} \mathbf{e}^{\mathbf{t}_i} \sin(\mathbf{t}) \] 
(41)

from which the coefficient matrices of 
\[ e^{\alpha(t)} = \sum_{i=0}^{n} \mathbf{C} \mathbf{e}^{\alpha_i} \] 
and \[ e^{\mathbf{t}} = \sum_{i=0}^{n} \mathbf{C} \mathbf{e}^{\mathbf{t}_i} \] 
are obtained as
\[ \mathbf{E} = \sum_{i=0}^{n} \mathbf{C} \mathbf{b}(\mathbf{e}^{\alpha_i}), \quad \mathbf{F} = \sum_{i=0}^{n} \mathbf{C} \mathbf{b}(\mathbf{e}^{\mathbf{t}_i}), \]  
(42)

Combinations of these may easily be treated by summing all the appropriate terms in equations (39), (40), and (42). Substituting the expansions of equation (38) into the bracketed portion of equation (37) yields the approximate fundamental solution matrix over the principal period as
\[ \Phi_{\alpha}(\tau, \mathbf{n}) = e^{\mathbf{t}} T^\mathbf{t}(\tau) \left[ I + \left( \sum_{i=0}^{n} \mathbf{H}(\mathbf{e}^{\mathbf{t}_i}) \right)^{-1} \mathbf{K}(\mathbf{n}) \right] = e^{\alpha(t)} T^\mathbf{t}(\tau) \mathbf{R}(\mathbf{n}), \]  
(43)
where \( H(x) \) - \( \mathbf{C}_1 \mathbf{Q}_1 \mathbf{C}_1 \) and \( k(s) = \mathbf{C}_0 \mathbf{Q}_0 \mathbf{C}_1 \) are generally much less sparse than the corresponding \( L(x) \) and \( P(x) \) matrices in the general formulation. Unlike the general formulation in which all \( A(x) \) matrices may include one or more parameters, parametric dependence in the constant \( A_1 \) matrix is impractical in this formulation since the product operational matrices \( \mathbf{Q}_0 \) and \( \mathbf{Q}_1 \) in equation (43) cannot be efficiently evaluated in terms of a parameter. Given this restriction, however, the faster convergence of this formulation generally results in the use of much fewer Picard iterations to achieve a desired accuracy as opposed to use of the general formulation via equation (32). Applications of this formulation to a commutative system and in the Mathieu equation are given in section 5.

5. EXAMPLES

5.1. A COMMUTATIVE SYSTEM

Consider the commutative \( x \)-periodic system

\[
\begin{pmatrix}
\dot{x} \\
\dot{\chi}
\end{pmatrix} =
\begin{bmatrix}
1 - a \cos t & 1 - a \sin t \\
1 - a \sin t & 1 + a \cos t
\end{bmatrix}
\begin{pmatrix}
x \\
\chi
\end{pmatrix},
\]

(44)

where \( a \) is the system parameter. The exact fundamental solution matrix of the system's [16]

\[
\Phi(t, s) =
\begin{bmatrix}
e^{a(t-s)} \cos t & e^{a(t-s)} \\
-e^{a(t-s)} \sin t & e^{a(t-s)}
\end{bmatrix},
\]

(45)

It is desired to compare this exact solution with the proposed approximate one. After normalizing the period to one as in equation (36), the new periodic system matrix is

\[
\mathbf{\Lambda}(s) = \mathbf{\Lambda}_1(s) + \mathbf{\Lambda}_2(s) \cos 2\sigma t + \mathbf{\Lambda}_3(s) \sin 2\sigma t,
\]

where

\[
\mathbf{\Lambda}_1(s) =
\begin{bmatrix}
-1 + a/2 & 1 - a/2 \\
1 - a/2 & -1 + a/2
\end{bmatrix},
\]

\[
\mathbf{\Lambda}_2(s) =
\begin{bmatrix}
x/2 & 0 \\
0 & -x/2
\end{bmatrix},
\]

\[
\mathbf{\Lambda}_3(s) =
\begin{bmatrix}
0 & -x/2 \\
x/2 & 0
\end{bmatrix},
\]

(46)

so that the product operational matrices for \( f(t) = 1 \) (which is the identity matrix), \( f(t) = \cos 2\sigma t \), and \( f(t) = \sin 2\sigma t \) are utilized.

Appendix B gives the complete first element of the approximated fundamental solution matrix using the general formulation via equation (32) where \( p = 14 \) and \( m = 7 \). The computations were performed using MATHEMATICA on a SUN SPARC 20 where the total CPU time was 5797 seconds. It is seen that the Chebyshev coefficients are homogeneous polynomials of powers of \( a \) up to the order \( \sigma^2 \) and that higher order terms possess considerably smaller coefficient magnitudes. This expression was then evaluated for three different values of the parameter \( a \) at times spanning the principal period and compared to the exact solution via equation (45). These results are shown in Table 1. To select \( p \) and \( m \), the matrix elements of the exact and approximate solutions were plotted over the principal period for all three parameter values. Figure 1 demonstrates the left-to-right convergence of the approximate solution as the number \( p \) of iterations is increased \((a = 0.1, m = 7)\), and Figure 2 demonstrates the more even convergence as the
number $m$ of polynomials is increased ($\tau = 0.1, p = 10$). In either case, however, the result is least accurate when the solution matrix becomes the FTM at $\tau = 1$. The last graph in Figure 2 demonstrates that increasing $m$ beyond a certain value cannot improve the accuracy if $p$ is too low. Since in general one does not know the exact solution, it was found that the best procedure to achieve a desired accuracy is, for a high value of $m$, to select an appropriate $p$ for a converged FTM, and then decrease $m$ until the joint just before convergence is lost. This should be done at various locations in the parameter region of interest to ensure uniform convergence before the final analytical approximation is obtained. It should be noted that this type of convergence study can be accomplished using

![Graphs](image)

Figure 1. The first element of the exact (dashed) and approximate (solid) fundamental solution matrix for the commutative system plotted over the principal period (versus normalized time) where $\tau = 0.1, m = 7$, and $p = (a) 6; (b) 8; (c) 10; (d) 12$. 

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>0.1</th>
<th>0.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00000*</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>1</td>
<td>0.34874</td>
<td>0.47754</td>
<td>1.55088</td>
</tr>
<tr>
<td>2</td>
<td>0.00000</td>
<td>0.47754</td>
<td>1.55372</td>
</tr>
<tr>
<td>3</td>
<td>0.00000</td>
<td>0.47754</td>
<td>1.55372</td>
</tr>
<tr>
<td>4</td>
<td>0.00000</td>
<td>0.47754</td>
<td>1.55372</td>
</tr>
<tr>
<td>5</td>
<td>0.00000</td>
<td>0.47754</td>
<td>1.55372</td>
</tr>
</tbody>
</table>

(a), exact solution; (b), approximate (symbolic) solution.
Figure 2. The first element of the exact (dashed) and approximate (solid) fundamental solution matrix for the commutative system plotted over the principal period (versus normalized time) when \( \alpha = 0.1 \), \( \beta = 1.0 \), and \( m = 3 \); (a) 5; (c) 7; (d) 9.

a FORTRAN program at a negligible cost compared to obtaining the parameter-dependent analytical approximation using symbolic software.

Finally, it should be noted that, although the \( A(\alpha) \) matrix includes the system parameter, the alternate formulation can still be utilized in conjunction with the generating solution \( \Phi(t, z) \) by including the parameter-dependent part of \( A(\alpha) \) (along with \( A(\beta) \) and \( \Lambda(\alpha) \)) in the \( A(t, z) \) matrix in equation (39). The term between the integrals in equation (37) simplifies to

\[
\Phi^{-1}(t, 0) A(t, z) \Phi(t, 0) = \begin{bmatrix}
\cos t & -\sin t & a & 1 + \cos 2t & -\sin 2t \\
\sin t & \cos t & b & -\sin 2t & 1 - \cos 2t \\
\end{bmatrix}
\times
\begin{bmatrix}
\cos t & -\sin t & a & 0 & 0 \\
\sin t & \cos t & b & 0 & 0 \\
\end{bmatrix}
\]

(47)

so that the actual series of integrals in brackets in that equation results in the exponential solution of this matrix, or \( \text{diag}(\exp(t), 1) \). The solution then is \( \Phi(t, \alpha) = \Phi(t, 0) \text{diag}(\exp(t), 1) \). Of course, use of the alternate formulation in conjunction with an expansion in \( m \) Chebyshev polynomials would yield a more approximate solution with \( \exp(t) \) expanded in an \( (m - 1) \)th degree power series. In this case the alternate formulation is more efficient than the general formulation only for \( \alpha < 1 \). If the commutative system matrix (with \( \alpha \) fixed) were perturbed by another time-periodic parameter-dependent matrix, however, the alternate formulation could be employed such that the complete fundamental matrix solution of equation (45) is used as the generating function. Such a procedure would be much more efficient than would be the direct application of the general formulation to a perturbed commutative system.

5.2. Mathieu Equation

The well-known Mathieu equation

\[
\ddot{y} + (a + b \cos t)y = 0
\]

(48)
is analyzed next where $a$ and $b$ are the system parameters. It should be noted that if the
time is normalized in state space form as

$$\dot{x} = [\bar{A}(a) + \bar{A}(b) \cos 2\pi \tau] x,$$  
$$\bar{A}(a) = \begin{bmatrix} 0 & 2\pi a \\ -2\pi a & 0 \end{bmatrix}, \quad \bar{A}(b) = \begin{bmatrix} 0 & 0 \\ -2\pi b & 0 \end{bmatrix}$$  
(49)

where $x' = (x, \dot{x}) - (y, \dot{y})$ and the derivatives are with respect to $\tau$, then the fundamental
solution matrix of the original second order system is given by

$$\Phi_{\xi}(\tau, a, b) = T(\tau) \Phi_{\xi}(0) = T(\tau) \begin{bmatrix} 1 & 2\pi \tau \\ 0 & 1 \end{bmatrix}$$

(50)

However, if equation (48) is first normalized and then transformed to the state space form

$$\dot{x} = [\bar{A}(a) + \bar{A}(b) \cos 2\pi \tau] x,$$  
$$\bar{A}(a) = \begin{bmatrix} 0 & 1 \\ -4\pi^2 a & 0 \end{bmatrix}, \quad \bar{A}(b) = \begin{bmatrix} 0 & 0 \\ -4\pi^2 b & 0 \end{bmatrix}$$  
(51)

then the fundamental solution matrix is given directly by equation (32). In either case, the
product operational matrices for $f(\xi) = 1$ and $f(\xi) = \cos 2\pi \tau$ are utilized in the
approximation.

Appendix C gives selected coefficients of all four elements of the approximated fundamental
solution matrix using the general formulation via equation (50) where $p = 24$
and $m = 15$. The computations were again performed using MATHEMATICA on a SUN
SPARC 20 where the total CPU time was 15 min 46 s. In Figure 3 the bold line represents the
CPU time required for each of the 24 iterations. It can be seen that higher iteration
steps require increasingly more time since the order of the polynomial coefficients
continually increases and that the first iteration requires more time than do the next two
because it includes the computation of $L(a, b)$ and $P(a, b)$. All of the Chebyshev
coefficients here are of order $\epsilon a^2 b^2$, that is they consist of various powers of $a^2 b^2$ up to

![Figure 3](image-url). The CPU time (in s) required for each of the 24 Picard iterations (not the cumulative time after each
iteration) in the Mathieu equation for the general formulation with two parameters (bold), general formulation
with one parameter (thin), and the alternate formulation with one parameter (dashed) which has equivalent
accuracy after 7 iterations.
The eigenvalues (Floquet multipliers) of (a) the "exact" Range-Kutta FTM and the approximate FTM via (d) the general formulation where \( p = 24 \), and \( m = 15 \) and (e) the constant matrix formulation where \( p = 7 \) and \( m = 17 \) for the Mackin equation with different parameter sets.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( 0 )</th>
<th>( 0.01 )</th>
<th>( 0.75 )</th>
<th>( 1.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-0.75)</td>
<td>(0.00434, 0.20, 0.754)</td>
<td>(0.00572, 0.739, 0.285)</td>
<td>(0.01816, 0.17210)</td>
<td>(0.08710, 0.17210)</td>
</tr>
<tr>
<td>(0.0)</td>
<td>(0.00493, 0.20, 0.754)</td>
<td>(0.00577, 0.739, 0.285)</td>
<td>(0.02681, 0.17212)</td>
<td>(0.17212)</td>
</tr>
<tr>
<td>(0.75)</td>
<td>(0.0001 \pm 0.00442i)</td>
<td>(-8.4771, -1.1180)</td>
<td>(-3.4734, -0.02811)</td>
<td>(-3.4734, -0.02811)</td>
</tr>
<tr>
<td>(1.5)</td>
<td>(0.0001 \pm 0.00442i)</td>
<td>(-8.4771, -1.1180)</td>
<td>(-3.4734, -0.02811)</td>
<td>(-3.4734, -0.02811)</td>
</tr>
</tbody>
</table>

order \( p/2 = 12 \) such that \( i + j \leq 12 \), \( i = 0, \ldots, 12 \), \( j = 0, \ldots, 12 \). However, only the coefficients of \( T^2(T^3) \), \( T^2(T^3) \), and \( T^2(T^3) \) such that \( i + j \leq 2 \) are shown in Appendix C for brevity. It should be noted that some lower order powers of \( a \) and \( b \) do not appear in the coefficients of higher polynomials such as \( T^2(T^3) \) and that higher order terms possess considerably smaller coefficients. This expression was then evaluated for different values of the parameters at the end of the principal period to obtain different Floquet Transition Matrices. The eigenvalues of Floquet multipliers were then computed and compared to those obtained via a fourth order Range-Kutta (DVPBK) routine in the INSL library, considered here as the "exact" solution, where a tolerance of \( 10^{-4} \) was used. These results are shown in Table 2. It should be noted that, as explained in section 2, the accuracy of Picard iteration is sensitive to this increase in coefficients unlike asymptotic techniques which depend on small values of the periodic parameter alone. Hence, convergence in the \( (a, b) \) plane varies radially from the origin. Convergence in both \( a \) and \( b \) was studied by plotting the FTM elements versus \( p \) while varying \( m \) for various parameter sets in the region of interest as described in section 5.1. Convergence plots for two of the parameter sets in Table 2 are shown in Figure 4 (one of the \( a = 0 \) cases to which small-parameter techniques cannot be applied) and Figure 5 (at the edge of the converged region in parameter space). It can be seen that the FTM near the origin in parameter space converges faster than does the FTM farther from the origin. Appendix D contains the trace of the FTM which is used to compute the stability boundaries via the relation

\[
tr(\hat{M}(\hat{H}_0)^{-1}(a, b)) = \pm 2, 
\]

where \( 2 \) and \(-1 \) correspond to destabilization via tangent and period doubling routes, respectively, in this Hamiltonian system [17]. Substitution of various values for \( b \) into equation (52) yields 12th-degree polynomial equations for which may be solved to find points on the stability boundaries once the complex roots and those outside the converged region have been eliminated. Figure 6 shows the familiar stability curves computed in this way in the converged parameter region. The curves coincide with those obtained using numerical integration of the FTM. The thin line in Figure 3 represents the CPU time...
Figure 4. All four FTM elements versus number \( p \) of Picard iterations for the Mathieu equation via the general formulation where \( a = 0.0, b = 0.75 \), and \( m = 8 \). The two diagonal elements (solid line) remain equal.

required for each of the 24 iterations in directly approximating the \( b \)-dependent fundamental solution matrix for \( z = 1.5 \). This approach, which yields the same \( c(b') \) coefficients also obtained from substituting \( a = 1.5 \) into the expression in Appendix C, requires a total CPU time of 3 min 26 s, a savings of 12 min 20 s over the time required for the 2-parameter approximation.

Equation (48) may also be analyzed using the alternate formulation outlined in section 4.3. As was discussed there, however, \( \alpha \) must be fixed at a given value and the fundamental solution matrix obtained in terms of only the periodic parameter \( b \). Since the constant part of the system has the generating solution

\[
e^{4t} = I \cos \omega t + S \sin \omega t,
\]

\[
S = \begin{bmatrix} 0 & 1/\omega \\ -1/\omega & 0 \end{bmatrix}, \quad \omega = 2\pi \sqrt{a},
\]

(53)

using the state space version in equation (53), \( H(b) \) and \( K(b) \) in equation (43) are expressed as

\[
H(b) = (I \otimes G')(I \otimes Q_b - S \otimes Q_b)(\tilde{A}(b) \otimes Q_b)(I \otimes Q_b + S \otimes Q_b)
\]

Figure 5. All four FTM elements versus number \( p \) of Picard iterations for the Mathieu equation via the general formulation where \( a = 1.5, b = 1.5 \), and \( m = 15 \). The two diagonal elements (solid line) remain equal.
Figure 6. Stability boundaries for the Mathieu equation in the \( a, b \) parameter plane which result from using the expression in Appendix D, generalized by application of the general formulation:

\[
\tilde{\alpha}(b) \circ (G(Q_1, Q_2) \circ Q_3) + (\tilde{\alpha}(b) S - S \tilde{\alpha}(b)) \circ (G(Q_1, Q_2) \circ Q_3) \\
- (S \tilde{\alpha}(b) S \circ (G(Q_1, Q_2) \circ Q_3), \\
K(b) = (G(Q_1, Q_2) \circ Q_3) \circ (\tilde{\alpha}(b) \circ Q_3) + (\tilde{\alpha}(b) S - S \tilde{\alpha}(b)) \circ (G(Q_1, Q_2) \circ Q_3, \\
- (S \tilde{\alpha}(b) S \circ (G(Q_1, Q_2) \circ Q_3), \\
(54)
\]

where \( \tilde{\alpha} \) contains the Chebyshev coefficients in the expansion of \( f_r(z) = \cos \Delta z \). Because a must be fixed beforehand, the value of \( a = 1.5 \) is chosen and the \( b \)-dependent fundamental solution matrix is approximated via equation (43) where \( p = 7 \) and \( m = 17 \) (see Appendix E). The computations required just 34 s, a savings of 2 min 52 s over the time required for the \( b \)-dependent approximation using the general formulation where both results are accurate to within \( 10^{-2} \) at \( a = b = 1.5 \). The dashed line in Figure 3 represents the CPU time required for each of the 7 iterations and is higher at first from the note used in computing \( H(b) \) and \( K(b) \). An additional power in \( b \) is gained at each iteration in this formulation so that all of the Chebyshev coefficients are of order \( \epsilon(b^2) = \epsilon(b^3) \) and, in general, are more accurate than the \( \epsilon(b^1) = \epsilon(b^2) \) coefficients of the general formulation.

Figure 7. All four TM elements versus number \( n \) of Picard iterations for the Mathieu equation via the alternate formulation where \( a = 1.5, b = 1.5 \), and \( m = 17 \). The two diagonal elements (solid line) remain equal.
as can be seen from the corresponding Floquet multipliers in Table 2. The faster convergence is also shown in Figure 7 for the same parameter set as in Figure 5. Thus, an improvement over the general formulation in accuracy as well as CPU time is achieved via the alternate formulation provided that $a$'s selected beforehand.

5.3. DOUBLE INVERTED PENDULUM WITH PERIODIC FOLLOWER FORCE

As an example of a higher order system, consider the double inverted pendulum of Figure 8 subjected to a follower force with both constant and periodically varying components. The time-periodic equations of motion for this system (the time-invariant form of which has been treated by Leipholz [18] and Hermann [19]), may be expressed in the linearized state space form as

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix}
= 

\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
(−3\bar{K} + \beta(t))/2 & \bar{K} − \beta(t)/2 & −B_1/2 − B_2 & B_2 \\
(2\bar{K} − \beta(t))/2 & −2\bar{K} + (3/2 − \gamma)\beta(t) & B_1/2 + 2B_2 & −2B_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
$$

(55)

where $(x_1, x_2, x_3, x_4) = (\phi, \dot{\phi}, \phi', \dot{\phi}')$ is the state vector, $\bar{K} = k/|m|^2$ is the normalized stiffness, $B_1 = b_1/|m|^2$ and $B_2 = b_2/|m|^2$ are the normalized damping constants, $\beta(t) = (\bar{P}_1 + \bar{P}_2 \cos \omega t)/|m| = \bar{P}_1 + \bar{P}_2 \cos \omega t$ is the normalized applied load, $\gamma$ is the load direction parameter, and $\omega$ is the internal driving frequency. As with the Mathieu equation, the fundamental solution matrix may be computed after normalizing in either second order or state space form. If the latter is done, then the solution matrix is given by

$$
\Phi(x; a) = T(x; \bar{a}) = T(x) \begin{bmatrix}
\cos(2\pi B_2) & 0 & 0 & 0 \\
0 & I_{2\times 2} & B_2(2\pi B_2)I_2 & 0
\end{bmatrix}
$$

(56)

where $x$ includes all the system parameters. In either case, the product operational matrices for $f_1(t) = 1$ and $f_2(t) = \cos 2\pi t$ are utilized after normalizing the period to one.
For the fixed parameter set $\kappa = 1.0, B_1 = 0.01, B_2 = 0.01, w = 2.0,$ and $\gamma = 1.0,$ the $P_i = P_j$ dependent solution may be approximated. Appendix F gives selected coefficients of the leading four elements of the resulting fundamental solution matrix using the general formulation via equation (56) where $p = 24$ and $m = 19.$ The computations were again performed using MATHEMATICA on a SUN SPARC 20 where the total CPU time to obtain these coefficients was $6$ h $6$ min $38$ s and the hold line in Figure 9 represents the time required for each of the $24$ iterations. All of the Chebyshev coefficients here are of order $(P_i^j P_i^j),$ that is they consist of various powers of $P_i^j P_i^j$ up to order $\gamma/2 = 12$ such that $i = j < 12, i = 0, \ldots, 12, j = 0, \ldots, 12.$ However, only the coefficients of $T_{P_i}^j (\tau)$ and $\bar{T}_{P_i}^j (\tau)$ such that $i = j < 2$ are shown in Appendix F for brevity. The thin line in Figure 9 represents the CPU time required for each of the $24$ iterations in directly approximating the $P_i$-dependent fundamental solution matrix for $P_i = 1.0.$ This approach, which yields the same $C(P_i^j)$ result also obtained from substituting $P_i = 1.0$ into the expression in Appendix F, requires a total CPU time of $1$ h $41$ min $50$ s, a savings of $4$ h $24$ min $48$ s over the time required for the $2$-parameter approximation. It should be noted that the

![Figure 9](image-url)

Figure 9. The CPU time (in s) required for each of the $24$ fixed iterations (not the cumulative time after each iteration) in the double inverted pendulum problem for the general formulation with two parameters (solid) and the general formulation with one parameter (thin).

![Figure 10](image-url)

Figure 10. The response $\phi_i (\tau)$ (solid line) and $\phi_i (\tau)$ (dashed line) versus normalized time for the double inverted pendulum with parameter set $\kappa = 1.0, B_1 = 0.01, B_2 = 0.01, w = 2.0, \gamma = 1.0, P_i = 1.0, P_j = 0.7$ and initial conditions $(\phi_0, \phi_0, \phi_0) = (1.0, 0.0).$
alternate formulation of section 4.3 could also be used to obtain a solution in terms of $P_i$ only, although that is not done here. For $P_i = 1/0$ and $P_i = 0/7$, the response to initial conditions $(x_1, x_2, x_3) = (0 0 0 0)$ is plotted in Figure 10 versus the normalized time $t$ over fifty principal periods directly via equation (33). Here it is seen that, in contrast to the truncated point mapping method of Guttafu and Flashner [3,4], the proposed technique allows the solution to be obtained as an explicit function of time. The Flocquet multipliers for this parameter set were computed to be $0 + 0.9418 * i$ and $0 - 0.9501 * i$ which have absolute values of $0.9789$ and $0.9669$, respectively. Since their moduli are less than one, the multipliers lie within the unit circle of the complex plane thus indicating asymptotic stability. This can be seen from the decay of the response in Figure 10, which is an interesting feature of the output obtained via numerical integration.

6. CONCLUSIONS

A technique for symbolic computation of the fundamental solution matrix for linear time-periodic dynamical systems of arbitrary dimension explicitly as a function of the system parameters and time has been presented. Both Picard iteration and expansion in shifted Chebyshev polynomials with their associated integration and product operational matrices are employed in this technique. It was shown that only matrix multiplications and additions are utilized in the computation of the Chebyshev coefficients of the fundamental solution matrix, which are expressed as homogeneous polynomials of the system parameters. This procedure permits one to obtain a closed form approximation of the fundamental solution matrix for general periodic systems. It was also shown that, unlike traditional perturbation or averaging, the technique is not based on expansion in terms of a small parameter and can therefore be successfully applied to periodic systems whose internal excitation is strong. Two formulations were proposed: one applicable to general systems and the other for equations which contain a constant system matrix. The latter formulation restricts the set of parameters that can be used in the solution but converges much faster than the former formulation. Three different example problems, including a double inverted pendulum subjected to a periodic follower force, were included, and results indicated that the proposed method is efficient and the convergence is excellent. It was also shown that the stability boundaries for the Mathieu equation may be obtained from the trace of the parameter-dependent FTM. Possibilities for future work include using the formulations presented here to determine stability and bifurcation conditions as well as controller gains in closed form as a function of the system parameters for higher dimensional systems.

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REFERENCES

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APPENDIX A: THE KRONECKER PRODUCT

Consider a 2 × 2 square matrix A and an m × m matrix B. The Kronecker product is

A ⊗ B = 
| a₁₁ B  a₁₂ B |
| a₂₁ B  a₂₂ B |

(A.1)

The resulting 2m × 2m matrix has dimension dim (A ⊗ B) = (2m, 2m). If dim (A) = dim (C) and dim (B) = dim (D), then the relation (A ⊗ B)(C ⊗ D) = (AC ⊗ BD) holds.

APPENDIX B: APPROXIMATE SOLUTION OF COMMUTATIVE SYSTEM

The following is the complete approximation of the first element of the fundamental solution matrix for the commutative system in section 5.1 obtained using MATHEMATICA. Five significant figures are included.
\[ \phi(t, y) = \\
(0.24553 + 0.0051070 \cdot x - 0.02630 \cdot x^2 - 0.028741 \cdot x^3 - 0.040813 \cdot x^4 \\
- 0.037130 \cdot x^5 + 0.0013716 \cdot x^6 + 0.00033286 \cdot x^7 - 0.018309 \cdot x^8 \\
+ 0.0036181 \cdot x^9 + 0.00023367 \cdot x^{10} - 0.00016598 \cdot x^{11} \\
+ 0.000012657 \cdot x^{12} + 0.000066197 \cdot x^{13} - 0.00001919 \cdot x^{14} \cdot \tau^2(t) \\
+ (0.00000128 \cdot x^{15} - 0.0136367 \cdot x^{16} - 0.008702 \cdot x^{17}) \\
- 0.00123 \cdot x^{18} + 0.00009378 \cdot x^{19} + 0.00004909 \cdot x^{20} - 0.03367 \cdot x^{21} \\
+ 0.0007491 \cdot x^{22} - 0.00038622 \cdot x^{23} - 0.0003029 \cdot x^{24} + 0.000026449 \cdot x^{25} \\
+ 0.000011909 \cdot x^{26} + 0.0000025182 \cdot x^{27} \cdot \tau(t) + 0.0002442 - 0.037125 \cdot x^{28} \\
- 0.05656 \cdot x^{29} - 0.048153 \cdot x^{30} - 0.03241 \cdot x^{31} - 0.067243 \cdot x^{32} \\
+ 0.018349 \cdot x^{33} - 0.0007886 \cdot x^{34} - 0.024197 \cdot x^{35} + 0.0069384 \cdot x^{36} \\
+ 0.002846 \cdot x^{37} - 0.00021739 \cdot x^{38} + 0.000025315 \cdot x^{39} + 0.0001181 \cdot x^{40} \\
+ 0.0006964 \cdot x^{41} - 0.000018914 \cdot x^{14} \cdot \tau^2(t) + (-0.045825 \\
+ 0.11466 \cdot x + 0.054535 \cdot x^2 + 0.03091 \cdot x^3 - 0.027666 \cdot x^4 \\
- 0.02986 \cdot x^5 + 0.020138 \cdot x^6 - 0.0023430 \cdot x^7 - 0.013850 \cdot x^8 \\
+ 0.0051657 \cdot x^9 + 0.00076101 \cdot x^{10} - 0.000012716 \cdot x^{11} \\
+ 0.0001961 \cdot x^{12} + 0.0000053216 \cdot x^{13} - 0.00001232 \cdot x^{14} \cdot \tau^2(t) \\
+ (-0.00062382 - 0.040524 \cdot x + 0.00086368 \cdot x^2 + 0.027509 \cdot x^3 \\
+ 0.01038 \cdot x^4 - 0.01069 \cdot x^5 + 0.013582 \cdot x^6 - 0.001923 \cdot x^7 \\
- 0.00064293 \cdot x^8 + 0.00036461 \cdot x^9 + 0.00000754 \cdot x^{10} \\
- 0.00063279 \cdot x^{11} + 0.00013092 \cdot x^{12} - 0.00001416 \cdot x^{13} \\
- 7.1176 \cdot 10^{-5} - 6.14 \cdot \tau(t) + 0.00050816 + 0.0002050 \cdot x^{-4} \\
- 0.016524 \cdot x^2 + 0.001662 \cdot x^3 + 0.0001551 \cdot x^4 \\
- 0.0045295 \cdot x^5 + 0.00084135 \cdot x^6 - 0.00009975 \cdot x^7 - 0.002510 \cdot x^8 \\
- 0.0017482 \cdot x^9 - 0.00013697 \cdot x^{10} - 0.00002361 \cdot x^{11} \\
+ 0.000066212 \cdot x^{12} + 2.4179 \cdot 10^{-10} - 6 \cdot x^{13} - 2.8263 \cdot 10^{-10} \\
- 6 \cdot x^{14} \cdot \tau(t) + (-0.00001099 + 0.00025677 + 0.0005407 \cdot x^2 \\
+ 0.00004076 \cdot x^3 - 0.0021004 \cdot x^4 - 0.0027421 \cdot x^5 + 0.0003448 \cdot x^6 \\
- 0.00073006 \cdot x^7 - 0.000078287 \cdot x^8 + 0.00007168 \cdot x^9 \\
+ 0.00010833 \cdot x^{10} - 0.00005833 \cdot x^{11} + 0.000022008 \cdot x^{12} \\
+ 2.742 \cdot 10^{-10} - 7 \cdot x^{13} - 8.9391 \cdot 10^{-10} - 7 \cdot x^{14} \cdot \tau(t). \)
APPENDIX C: APPROXIMATE SOLUTION OF MATHEU EQUATION VIA THE GENERAL FORMULATION

The following contains selected coefficients of all four elements of the approximated fundamental solution matrix from the general formulation via equation (50) for the Mathieu equation in section 5.2 obtained using MATLABICA. All of the Chebyshev coefficients are of order $6(a+b)^2$; however, only the coefficients of order $6(a+b)^2$ in $T^n(t)$, $T^7(t)$, and $T^n(t)$ are shown. Five significant figures are included.

$$\Phi^{(2)}(t, a, b) = (1 - 7.4022^6 a - 0.6853^6 b + 17.756^6 a^2 - 0.6194^6 a^4 b$$
$$- 3.0628^6 b^2 + \cdots) T^n(t) + \left(-9.896^6 a + 4.3651^6 b - 12.96 + 26.410^6 a^2 - 1.5087^6 b^2 - 4.9348^6 b^2 + 2 + \cdots\right) T^7(t) + \cdots + \left(1.690^6 b^2 - 8.6^6 b^2 + 6.1690^6 b^2 - 7.2^6 b^2 + 0.00012710^6 b^2 + \cdots\right) T^n(t)$$

$$\Phi^{(3)}(t, a, b) = (0.5 - 2.0561^6 a + 0.6521^6 b + 3.1962^6 a^2 - 1.5662^6 a^4 b$$
$$- 0.049695^6 b^2 + \cdots) T^n(t) + (0.5 - 0.0842^6 a + 1.0760^6 b + 5.3270^6 a^2 - 2.5872^6 b^2 - 0.096485^6 b^2 + \cdots) T^7(t) + \cdots + (5.3489^6 a - 9.6^6 b + 1.604^6 b^2 - 6.3447^6 a^10 - 6.6^6 b^2 + 2) T^n(t)$$

$$\Phi^{(4)}(t, a, b) = (19.759^6 a + 8.7302^6 b - 12.96 + 81.174^6 a^2 + 8.0949^6 a^4 b + 9.696^6 b^2 + 3.5765^6 b^2 + 121.76^6 a^2 + 16.676^2 a^6 b$$
$$- 12.778^6 b^2 + 7.7^6 b^2 + \cdots + (-2.6698^6 a^10 - 6.6^6 b^2 + 1.1808^6 b^2 - 18.6^6 b^2 + 2) T^n(t)$$

APPENDIX D: TRACE OF FTM FOR MATHEU EQUATION

The following is the complete trace of order $(6(a+b)^2)$ of the approximated Floquet Transition Matrix (FTM) for the Mathieu equation in section 5.2 obtained using the expression in Appendix C. The stability boundary is obtained by setting the following expression equal to ±2. Five significant figures are included.

$$(1 - 7.4022^6 a + 0.6853^6 b + 17.756^6 a^2 - 0.6194^6 a^4 b - 3.0628^6 b^2 + \cdots) T^n(t) + \left(-9.896^6 a + 4.3651^6 b - 12.96 + 26.410^6 a^2 - 1.5087^6 b^2 - 4.9348^6 b^2 + 2 + \cdots\right) T^7(t) + \cdots + \left(1.690^6 b^2 - 8.6^6 b^2 + 6.1690^6 b^2 - 7.2^6 b^2 + 0.00012710^6 b^2 + \cdots\right) T^n(t)$$
\[ \text{tr}(\phi^{(m)}(a, b)) = 2 \cdot 0 - 90^* \cdot 7^* - 128^* \cdot 8^* - 170^* 9^* 3^* + 120^* 48^* a^* 4 - 52^* 8^* a^* 5 + 15^* 30^* 7^* 6 - 34227^* a^* 7 + 5640^* 8^* 0 - 07275^* a^* 9 + 0007556^* a^* 10 - 000004593^* a^* 11 + 000000613^* a^* 12 + 35792^* 10^* - 9^* 5^* 3^* 410^* 10^* - 7^* a^* 5^* + 3^* 47710^* - 6^* a^* 2^* b^* + 6^* 9151^* 10^* - 6^* a^* 3^* 5^* 8^* 1218^* 1^* - 7^* a^* 4^* - 0000001377^* a^* 5^* - 5^* 0 - 00017882^* a^* 6^* + 0000003443^* a^* 7^* 8^* 8^* 4924^* 10^* - 6^* a^* 8^* 9^* 6^* - 0000017321^* a^* 9^* b^* + 0000018431^* a^* 10^* 8^* 1420^* 10^* - 6^* a^* 11^* 1^* 6^* 1973^* b^* 2^* + 50^* 921^* a^* b^* 2^* - 52^* 682^* a^* 2^* b^* 2 + 30^* 247^* a^* 3^* 2^* - 11^* 137^* a^* 4^* b^* 2 + 2^* 8625^* a^* 5^* b^* 2 - 0045473^* a^* 6^* b^* 2 + 0080236^* a^* 7^* b^* 2 - 0100905^* a^* 8^* b^* 2 + 00012691^* a^* 9^* b^* 2 - 000018231^* a^* 10^* b^* 2 + 000025892^* b^* 3 - 000041025^* a^* b^* 3 + 00016618^* a^* 2^* b^* 3 + 000046893^* a^* 3^* b^* 3 + 000093555^* a^* 4^* b^* 3 - 0000123^* a^* 5^* b^* 3 + 0000043437^* a^* 6^* b^* 3 + 0000001001401^* a^* 7^* b^* 3 - 000006680^* a^* 8^* b^* 3 + 0000004835^* a^* 9^* b^* 3 + 1^* 6271^* b^* 4 - 2^* 2923^* a^* 2^* b^* 4 + 1 4233^* a^* 2^* b^* 4 - 025286^* a^* 3^* b^* 4 + 001312^* a^* 4^* b^* 4 - 0023634^* a^* 5^* b^* 4 + 00030291^* a^* 6^* b^* 4 - 00009092^* a^* 7^* b^* 4 - 000001620^* a^* 8^* b^* 4 - 000001446^* b^* 5 + 00009426^* a^* 6^* 5 + 00002065^* a^* 2^* b^* 5 + 00002076^* a^* 3^* b^* 5 - 00000846^* a^* 4^* b^* 5 - 00002340^* a^* 5^* b^* 5 + 00000511^* a^* 6^* b^* 5 - 0000033510^* a^* 7^* b^* 5 - 001846^* b^* 6 + 001840^* a^* b^* 6 - 00008020^* a^* 2^* b^* 6 + 00002087^* a^* 3^* b^* 6 - 000017100^* a^* 4^* b^* 6 - 000020011^* a^* 5^* b^* 6 + 0000017612^* a^* 6^* b^* 6 + 0000047915^* b^* 7 - 000001469^* a^* 7^* b^* 7 + 0000005106^* a^* 2^* b^* 7 + 000004105^* a^* 3^* b^* 7 + 0000005052^* a^* 4^* b^* 7 + 0000005327^* a^* 5^* b^* 7 + 0000006073^* b^* 8 - 0000004384^* a^* 6^* b^* 8 - 1^* 3738^* 10^* - 6^* a^* 2^* b^* 8 + 0000002981^* a^* 3^* b^* 8 - 0000000324^* a^* 4^* b^* 8 - 2^* 5103^* 10^* - 6^* a^* 2^* b^* 9 + 3^* 4760^* 10^* - 6^* a^* 2^* 9^* 6^* - 9^* 6^* 8^* 6^* 2^* 83^* 10^* - 7^* a^* 2^* b^* 9^* 1^* 1^* 1777^* 10^* - 6^* a^* 3^* 2^* b^* 9^* 3^* 3^* 1946^* 10^* - 5^* 842^* 10^* - 7^* a^* 2^* b^* 10^* 1 1^* 3108^* 10^* - 6^* a^* 2^* b^* 10^* 5^* 728^* 10^* - 9^* 6^* 11^* - 7^* 3996^* 10^* - 9^* a^* b^* 11^* 7^* 4707^* 10^* - 9^* b^* 12^*.

**APPENDIX E: APPROXIMATE SOLUTION OF MATHEIU EQUATION VIA THE ALTERNATE FORMULATION**

The following contains selected coefficients of the first element of the approximated fundamental solution matrix from the alternate formulation via equation (43) for the Mathieu equation in section 5.2 with \( a = -1.5 \) obtained using MATHEMATICA. All of the Chebyshev coefficients are of order \( \theta^6 \); however, only the coefficients in \( T_7^*(\theta) \), \( T_7^*(\theta) \), and \( T_7^*(\theta) \) are shown. Five significant figures are included.
\[
\Phi(\gamma(t), \theta) = (1 + \frac{b}{19130956 - 0.15369902 - 0.11196784 - 0.005856984})^4
+ 0.0014198^4 \cdot 0.000046639^4 \cdot 0.00000005769 \cdot 7^r (t)
+ 0.00005897^4 \cdot 0.0000543^4 \cdot 0.000000075^4 \cdot 6^r (t)(t) + \ldots + (0.0000000109)^6
- 0.00102663^2 \cdot 0.0000000030639^2 \cdot 0.000000006964\]