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Stability and Control of a Parametrically Excited Rotating System. Part I: Stability Analysis

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Abstract. This paper analyses the stability of a parametrically excited double pendulum rotating in the horizontal plane. The equations of motion for such a system contain time varying periodic coefficients. Floquet theory and the method of Hill's determinant are used to evaluate the stability of the linearized system. Stability charts are obtained for various sets of damping, parametric excitation, and rotation parameters. Several resonance conditions are found, and it is shown that the system stability can be significantly altered due to the rotation. Such systems can be used as preliminary models for studying the lag dynamics and control of helicopter blades and other gyroscopic systems.

Keywords: time-varying periodic systems, Floquet theory, State Transition Matrix, helicopter blades

1. Introduction

Systems with time-varying periodic coefficients appear in almost all branches of science and engineering. In the field of mechanical engineering, systems with periodic coefficients appear in modeling of structures subjected to periodic loading, helicopter rotor blades in forward flight, asymmetric rotor-bearing systems, robots performing repetitive tasks and walking machines, among others.

The stability problem associated with these systems involves the analysis of a set of linear ordinary differential equations with periodic coefficients. The same mathematical problem arises in the study of nonlinear autonomous systems when the stability of a particular periodic solution needs to be investigated. Beside the stability issues, the linear control problems associated with such systems lead to the same type of equations. For the case of linear systems with constant coefficients $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the eigenvalues of $\mathbf{A}$ determines the stability of its solution. This is not the case for the time-dependent system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. Hill's method, Floquet theory, perturbation and averaging methods are some of the most commonly used mathematical methods in the analysis of linear periodic systems. Using Hill's approach one can determine the stability boundaries and the perturbation and averaging methods can only be applied to systems where the periodic coefficients can be expressed in terms of a small parameter. Therefore, Floquet analysis coupled with a numerical code has served as the main tool in various applications. Floquet theory requires the computation of the fundamental matrix (also called the *State Transition Matrix, STM*) at the end of the principal period. Instead of using a numerical code, one can also use the Chebyshev polynomial expansion technique to construct solutions in numerical or symbolic form as shown by

Sinha and his associates [1]. The Floquet theory provides a quantitative measure of the stability level of the system, in terms of the eigenvalues of the *STM* at the end of the principal period.

One of the commonly investigated mechanical system is an inverted pendulum with a periodic base excitation or a periodic load at the other end.

Dugundji and Chhaptar [2] made a complete analysis of a parametrically excited one degree of freedom problem. In their work a single pendulum was investigated in detail, both theoretically and experimentally. Both linear and nonlinear cases were considered. The linear theory showed the presence of various instability regions and the presence of *beats* and *waviness* in the transient response. The nonlinear theory showed the existence of steady-state limit cycles arising from the instability regions and also from subharmonic resonance.

Anderson and Tadjbakhsh [3] applied the averaging method to a double inverted pendulum with a parametrically excited base. Using the averaging method the stability conditions for the pendulum were determined. It was found that if the pendulum support undergoes a small amplitude with high frequency oscillatory motion in the vertical direction, then the system becomes stable, performing small oscillations about the vertical position. However, this result can only be valid for a small part of the parameter space due to the limitations of the method. In the absence of a small parameter, the averaged dynamics fails to retain the bifurcation characteristics of the original system. A detailed critical comparison between the results given by the Floquet theory and averaging method for a specific case has been done by Pandiyan and Sinha [4].

Guttalu and Flashner [5, 6] derived analytical expressions for stability and bifurcation conditions of periodic systems using a point mapping technique. By converting the original equations of motion into a discrete time-invariant representation, stability and bifurcation conditions can be found in terms of algebraic equations.

In the first part of this paper the stability of a planar double pendulum with base excitation and rotation is analyzed. The base has a known uniaxial periodic motion and the entire systems rotates in the horizontal plane with a constant angular velocity. The stability analysis is carried out using the Floquet theory and the method of Hill's determinant. Stability charts for different combinations of parameters are presented.

2. The System Model

The mechanical system considered is shown in figure 1. The massless base has a known uniaxial sinusoidal movement, in the horizontal plane, of amplitude L_1 and frequency ω. Two link arms are connected to the moving base. The whole system can rotate in the horizontal plane (no gravity forces) with a constant angular speed Ω. Each link has length L and carries a concentrated mass m at the end point. On each joint there are restoring spring hinges and dampers, with the stiffness constant k and the damping constant c, respectively.

Applying Lagrange's method, the equations of motion for the system (assuming small

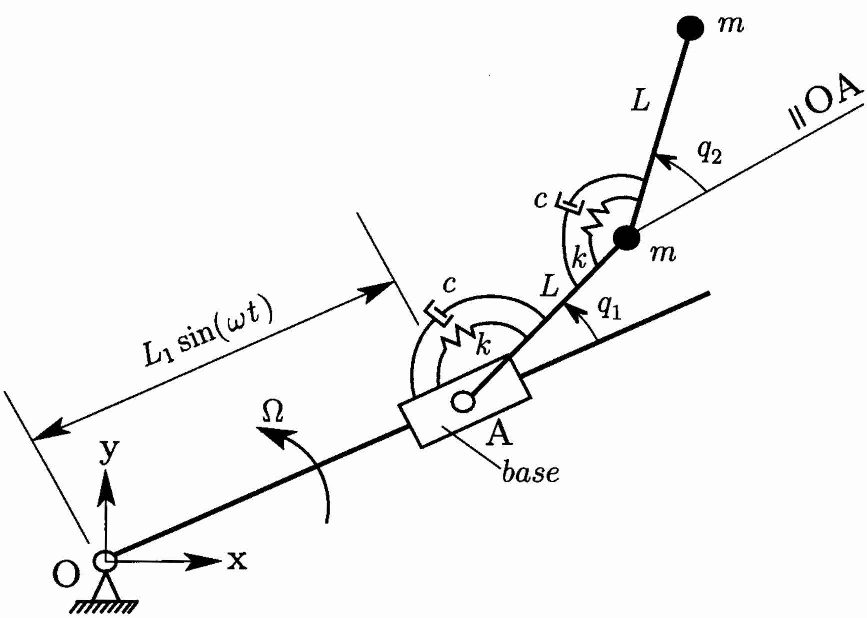

Figure 1. Planar double pendulum with a vibrating base.

values for the angles q_1 and q_2 of the links in the local reference frame) are given by

$$
\begin{aligned}
& mL^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + c \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \\
& + \left(\begin{bmatrix} 2k + mL^2\Omega^2 & -k - mL^2\Omega^2 \\ -k - mL^2\Omega^2 & k + mL^2\Omega^2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} mLL_1(\Omega^2 + \omega^2) \sin \omega t \right) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\
& = \begin{bmatrix} -4mLL_1\Omega\omega \cos \omega t \\ -2mLL_1\Omega\omega \cos \omega t \end{bmatrix}. \qquad (1)
\end{aligned}
$$

With the following nondimensional quantities,

- $\tau = t\sqrt{\dfrac{k}{mL^2}}$ the normalized time;
- $\gamma = \dfrac{mL^2\Omega^2}{k}$ the normalized rotating angular velocity;
- $\delta = \sqrt{\dfrac{c^2}{mL^2k}}$ the normalized damping factor;

- $\varepsilon = \frac{L_1}{L}$ the normalized amplitude of the base excitation;
- $\Psi = \omega\sqrt{\frac{mL^2}{k}}$ the normalized frequency of the base excitation,

equation (1) becomes

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} q_1'' \\ q_2'' \end{bmatrix} + \delta\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1' \\ q_2' \end{bmatrix} + \begin{bmatrix} 2+\gamma & -1-\gamma \\ -1-\gamma & 1+\gamma \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$
$$+\,\varepsilon(\gamma + \Psi^2)\sin\Psi\tau\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{h}(\tau) \tag{2}$$

where the primes denote the derivatives with respect to the normalized time τ, and the forcing function $\mathbf{h}(\tau)$ is

$$\mathbf{h}(\tau) = -\Psi\varepsilon\sqrt{\gamma}\begin{bmatrix} 4 \\ 2 \end{bmatrix}\cos\Psi\tau. \tag{3}$$

The natural frequencies of the system given by equation (2) without any damping or parametric excitation are computed from the frequency equation

$$det\left(\begin{bmatrix} 2+\gamma & -1-\gamma \\ -1-\gamma & 1+\gamma \end{bmatrix} - \omega_{1,2}^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = 0, \tag{4}$$

which yields

$$\omega_{1,2} = \sqrt{\frac{6+5\gamma \pm \sqrt{(6+5\gamma)^2 - 4(1+\gamma)}}{2}}. \tag{5}$$

It is observed that since $\gamma \geq 0$, equation (5) always yields distinct positive values for ω_1 and ω_2.

In the following, various specific cases are first discussed:

(a) Nonrotating Systems

For the case of the nonrotating system, $\gamma = 0$, the system has only an oscillatory movement, and equation (2) becomes

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} q_1'' \\ q_2'' \end{bmatrix} + \delta\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1' \\ q_2' \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$
$$+\varepsilon\Psi^2\sin\Psi\tau\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

The natural frequencies for $\gamma = 0$ can be easily computed from equation (5) as $\omega_1 = 0.4142$

and $\omega_2 = 2.4142$ and the normalized modal matrix is given by

$$\mathbf{S} = \begin{bmatrix} 0.382683 & 0.92388 \\ 0.541196 & -1.30656 \end{bmatrix}. \tag{7}$$

In terms of the principal coordinates η_1 and η_2, defined by

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{S} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \tag{8}$$

equation (6) becomes

$$\begin{bmatrix} \eta_1'' \\ \eta_2'' \end{bmatrix} + \delta \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1' \\ \eta_2' \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$
$$+ \varepsilon \Psi^2 \sin \Psi\tau \begin{bmatrix} 0.585786 & 0 \\ 0 & 3.41421 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{9}$$

We note that system (9) represents a set of decoupled linear differential equations. This was expected since the damping and the periodic part of the stiffness matrix can be expressed as linear combinations of the mass matrix and the constant part of the stiffness matrix. Because equations (9) are decoupled, the stability analysis can be performed independently for η_1 and η_2.

(b) No Base Excitation ($\Psi = 0$)

If the normalized frequency $\Psi = 0$, then the base has no vibrating motion, and the whole system is rotating with a constant angular velocity γ. In this case equation (2) takes the form

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1'' \\ q_2'' \end{bmatrix} + \delta \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \end{bmatrix} + \begin{bmatrix} 2+\gamma & -1-\gamma \\ -1-\gamma & 1+\gamma \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{10}$$

which has only constant coefficients.

(c) The General Case ($\gamma \neq 0$, $\Psi \neq 0$)

Several numerical values of the normalized rotating frequency were considered. Here, the results for a typical value of $\gamma = 0.666$ are presented. The eigenvalues of the system computed from equation (5) are $\omega_1 = 0.42675$ and $\omega_2 = 3.02454$ and the modal matrix is

$$\mathbf{S} = \begin{bmatrix} 0.406823 & 0.913507 \\ 0.506683 & -1.32033 \end{bmatrix}. \tag{11}$$

In term of the principal coordinates, equation (2) becomes

$$\boldsymbol{\eta}'' + \varepsilon \mathbf{F_o} \boldsymbol{\eta}' + (\mathbf{B} + \varepsilon \mathbf{E} \sin \Psi \tau)\boldsymbol{\eta} = \mathbf{S}^T \mathbf{h}(\tau), \tag{12}$$

where the following notations have been used

$$\boldsymbol{\eta} = [\eta_1, \eta_2]^T, \qquad \mathbf{B} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix},$$

$$\mathbf{F_o} = \begin{bmatrix} f_{o_{11}} & f_{o_{12}} \\ f_{o_{21}} & f_{o_{22}} \end{bmatrix} = \frac{\delta}{\varepsilon} \begin{bmatrix} 0.175477 & 0.148565 \\ 0.148565 & 5.82452 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = (0.666 + \Psi^2) \begin{bmatrix} 0.587739 & 0.0742826 \\ 0.0742826 & 3.41226 \end{bmatrix}. \tag{13}$$

Since the right hand side term of equation (12) is bounded and periodic with the same frequency Ψ (as the coefficients), the stability analysis can be carried out from the homogeneous equation.

3. Stability Analysis

3.1. *The Numerical Approach*

In the state space form, the homogeneous part of equation (12) can be rewritten as

$$\boldsymbol{\theta}' = \mathbf{A}(\varepsilon, \gamma, \Psi, \tau)\boldsymbol{\theta}, \tag{14}$$

where

$$\boldsymbol{\theta} = [\boldsymbol{\eta}, \boldsymbol{\eta}']^T,$$

$$\mathbf{A}(\varepsilon, \gamma, \Psi, \tau) = \begin{bmatrix} \mathbf{0} & \mathbf{I_2} \\ -(\mathbf{B} + \varepsilon \mathbf{E} \sin \Psi \tau) & \varepsilon \mathbf{F_o} \end{bmatrix},$$

and $\mathbf{I_2}$ is the identity matrix of second order. A linear system with periodic coefficients is asymptotically stable if and only if all the eigenvalues of the fundamental matrix $\boldsymbol{\Phi}(\varepsilon, \gamma, \Psi, \tau)$ evaluated at the end of the principal period lie within the unit circle. The fundamental matrix, $\boldsymbol{\Phi}$ can be computed numerically, by integrating the matrix differential equation

$$\dot{\boldsymbol{\Phi}} = \mathbf{A}(\varepsilon, \gamma, \Psi, \tau)\, \boldsymbol{\Phi}, \tag{15}$$

with $\boldsymbol{\Phi}(0) = \mathbf{I_4}$. The integration was performed over the period $T = 2\pi/\Psi$ and the eigenvalues of $\boldsymbol{\Phi}(\varepsilon, \gamma, \Psi, T)$ are evaluated using the subroutine DIVPAG (which implements the Gear method) from IMSL Library [7]. If one of the eigenvalues has a magnitude greater then the unity, the system given by equation (14) is unstable. The system is neutrally stable if there are no unstable eigenvalues and some eigenvalues are placed on the unit circle.

Using this procedure, stability charts were drawn in the $(\Psi - \varepsilon)$ plane for several possible cases or damping-rotation parameters combinations. For low excitation frequencies, the integration step had to be kept small in order to obtain the stability region. A discussion of results is included in Section 3.4.

3.2. The Method of Hill's Determinant

In this method, one can use the fact that the system has periodic solutions with the period T and $2T$ at the boundaries of the instability region. The instability regions are bounded by two solutions with the same period and stability regions are bounded by two solutions with different periods. Following [8], trial solutions with period T and $2T$ are assumed as

$$\eta_i(\tau) = \sum_{k=1,3,5,..} \left(A_{ik} \sin \frac{k\Psi\tau}{2} + B_{ik} \cos \frac{k\Psi\tau}{2} \right), \quad i = 1,2, \tag{16}$$

and

$$\eta_i(\tau) = \frac{B_{o_i}}{2} + \sum_{k=2,4,6,...} \left(A_{ik} \sin \frac{k\Psi\tau}{2} + B_{ik} \cos \frac{k\Psi\tau}{2} \right), \quad i = 1,2, \tag{17}$$

respectively. These solutions are substituted in equations (9) and (12), and the coefficients of $\sin(k\Psi\tau/2)$ and $\cos(k\Psi\tau/2)$ are equated. A homogenous system of linear algebraic equations in unknowns coefficients A_{ik} and B_{ik} is found. This system has a nontrivial solution if its determinant is equal to zero. This procedure results in an expression in ε and Ψ which provides the stability boundary. The computation can be done in symbolic form using MATHEMATICA [9]. As an example the expressions of the determinant for an undamped and nonrotating system is given in Appendix 1, where only the first three terms of the trial solutions (16) and (17) are considered. When the odd trial solution (16) is used, the following resonance conditions are found: $\Psi_i^* = 2\omega_1, 2\omega_2, 2\omega_1/3, 2\omega_2/3, 2\omega_1/5, 2\omega_2/5$. When the even trial solution (17) is used, the resonance conditions obtained are $\Psi_i^* = \omega_1, \omega_2, \omega_1/2, \omega_2/2, \omega_1/3, \omega_2/3$.

3.3. Conditions for Combination Resonances

In order to obtain analytical conditions for combination resonances of the type $\omega_1 \pm \omega_2$, the approach developed by Hsu [10] is followed.

First, the homogeneous part of equation (12) is rewritten in the form

$$\frac{d\,\eta_i}{d\,\tau} = \eta_i'$$

$$\frac{d\,\eta_i'}{d\,\tau} + \omega_i^2 \eta_i = g_i(\eta_j, \eta_j', \tau), i = 1,2. \tag{18}$$

The first order approximation solution of the equation (18) is given by

$$\eta_i = A_i(\tau)\cos\omega_i\tau + B_i(\tau)\sin\omega_i\tau + \varepsilon\eta_i^{(1)}(\tau),$$

$$\eta_i' = \omega_i(-A_i(\tau)\sin\omega_i\tau + B_i(\tau)\cos\omega_i\tau) + \varepsilon\frac{d\,\eta_i^{(1)}}{d\,\tau}. \tag{19}$$

Substituting equations (19) in equation (18), one can obtain

$$\eta_i^{(1)} = -0.5\sum_{j=1}^{2}\left[\frac{e_{ij}A_j\sin(\Psi+\omega_j)\tau - e_{ij}B_j\cos(\Psi+\omega_j)\tau}{\omega_i^2 - (\Psi+\omega_j)^2}\right.$$

$$+ \left. \frac{e_{ij} A_j \sin(\Psi - \omega_j)\tau - e_{ij} B_j \cos(\Psi - \omega_j)\tau}{\omega_i^2 - (\Psi - \omega_j)^2} \right] \tag{20}$$

$$+ \sum_{j=1}^{2} f_{o_{ij}} \omega_j \frac{B_j \cos \omega_j \tau - A_j \sin \omega_j \tau}{\omega_i^2 - \omega_j^2} i = 1, 2,$$

where e_{ij} and $f_{o_{ij}}$ are defined in equation (13). From equation (20) it is observed that with the first approximation solution, the $\eta_i^{(1)}$ are unbounded if $\Psi = \omega_1 \pm \omega_2$, $\Psi = 2\omega_1$ and $\Psi = 2\omega_2$.

The behavior of the system in the vicinity of the combination resonance $\Psi = \omega_1 \pm \omega_2$ is investigated by introducing a detuning parameter λ, defined as

$$\Psi = \omega_1 \pm \omega_2 + \varepsilon\lambda. \tag{21}$$

The damped system is stable if [10]

$$\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \leq f_{o_{11}} + f_{o_{22}}, \tag{22}$$

where

$$\alpha = (f_{o_{11}} + f_{o_{22}})^2 - 4(f_{o_{11}} f_{o_{22}} + \lambda^2) \pm \frac{e_{12} + e_{21}}{\omega_1 \omega_2},$$

$$\beta = 4\lambda(f_{o_{11}} - f_{o_{22}}). \tag{23}$$

In particular, an undamped system ($f_{o_{11}} = 0$, $f_{o_{22}} = 0$, $\beta = 0$), is neutrally stable if

$$\alpha < 0. \tag{24}$$

3.4. *Discussion of Results*

Figures 2 through 5 show several stability charts of the system given by equation (18), obtained for different combinations of damping and rotation parameters. Figures 2a, 3a, 4a and 5a were obtained using the method of numerical integration. For low frequencies ($\Psi < 2\omega_1$), the method of Hill's determinant was used and the results are shown in figures 2b, 3b, 4b and 5b.

Case I: The Nonrotating System ($\gamma = 0$)

For the case of $\delta = 0$ (no damping) the result are shown in figure 2a. and 2b. The cross-hatched zone is neutrally stable, while the white region is unstable. We note the existence of several unstable strips inside the hatched zone. For the undamped system, these strips touch the vertical axis Ψ at points Ψ_i^* given by $2\omega_2$, ω_2, $\omega_2/3$, $\omega_2/2$, $2\omega_1$, ω_1, $2\omega_1/3$, $\omega_1/2$, $2\omega_1/5$, $2\omega_1/6$. This was indicated in Section 3.2.

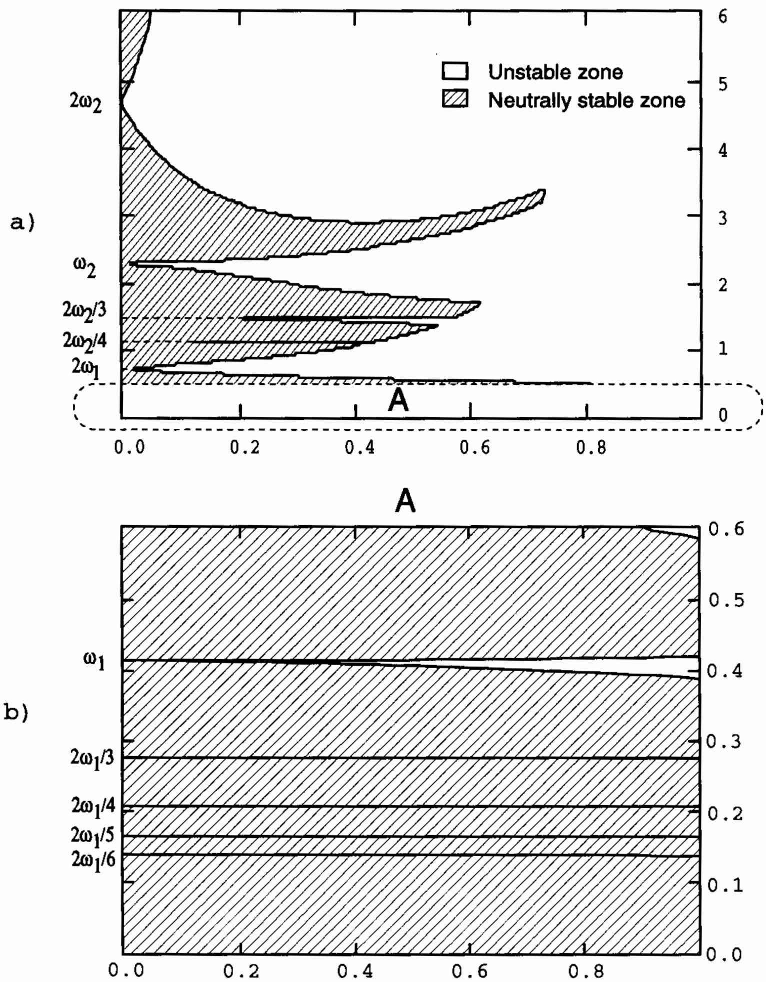

Figure 2. (a) Stability chart for undamped-nonrotating system. (b) Enlarged view of the A zone.

It was also observed that the unstable strips become narrower and narrower when the resonance frequencies decreases. Resonance combinations of the type $\omega_1 \pm \omega_2$ cannot occur in this case because for the nonrotating system, equation (9) represents a set of decoupled differential equations.

Figures 3a and 3b show results for the case when δ is no longer zero, but 0.0163. The cross-hatched zone with lines slanted at 135^o represents the stable region while the white zone is unstable. In the case of a damped system, the unstable zones do not start at the vertical axis $\varepsilon = 0$. This is expected for a Mathieu type system. Using the analysis

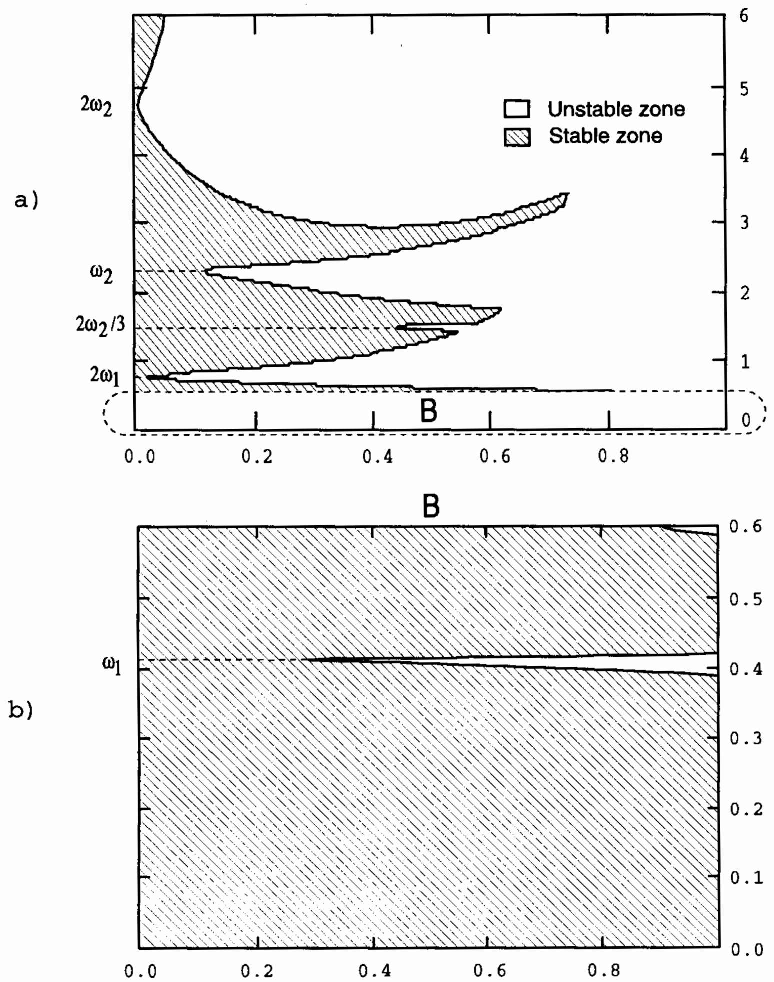

Figure 3. (a) Stability chart for damped-nonrotating system. (b) Enlarged view of the B zone.

presented in Section 3.2, it was found that resonances corresponding to $2\omega_1/3$ and $2\omega_i/k$, ($i = 1, 2$ and $k \geq 4$) do not appear.

Case II: The Rotating System ($\gamma = 0.666$)

For $\delta = 0$ and $\gamma = 0.666$, the results are shown in figures 4a and 4b. Comparing figures 4a and 2a, one can see that a new unstable zone corresponding to a resonance frequency of $\Psi = \omega_1 + \omega_2$ has appeared due to rotation. Resonance condition corresponding to

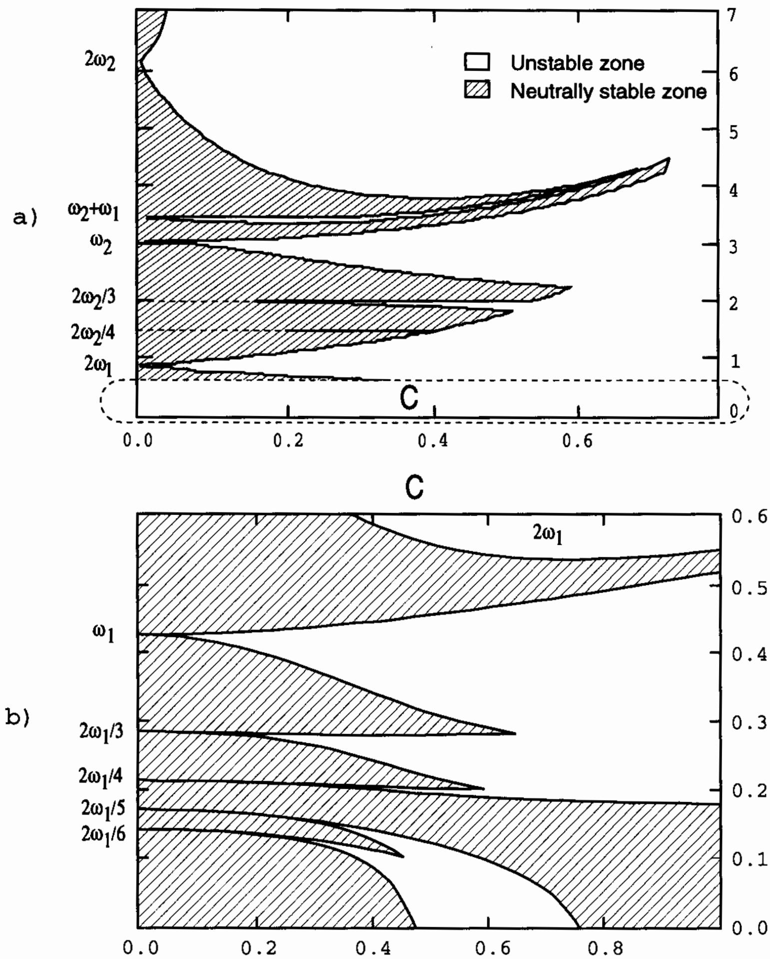

Figure 4. (a) Stability chart for undamped-rotating system. (b) Enlarged view of the C zone.

$\Psi = \omega_2 - \omega_1$ does not exist in this case. From the condition given by equation (24), it is found that the system is always neutrally stable ($\alpha = -0.0025 < 0$ for a detuning parameter $\lambda = 0.001$). Comparing figures 2b and 4b one can see that the system rotation can significantly change the structure of the stability chart for low values of Ψ.

For the case of $\delta = 0.0163$ and $\gamma = 0.666$, the results are shown in figure 5a and 5b. Because the system has viscous damping, once again the unstable regions do not start at $\varepsilon = 0$ line. The stability conditions for the case of combination resonance $\Psi = \omega_1 + \omega_2$ is given by equation (22) in Section 3.3. This is verified by numerical computation. From the condition (23), since $0.0783 < 0.0978$, the combination resonance $\omega_2 - \omega_1$ is stable.

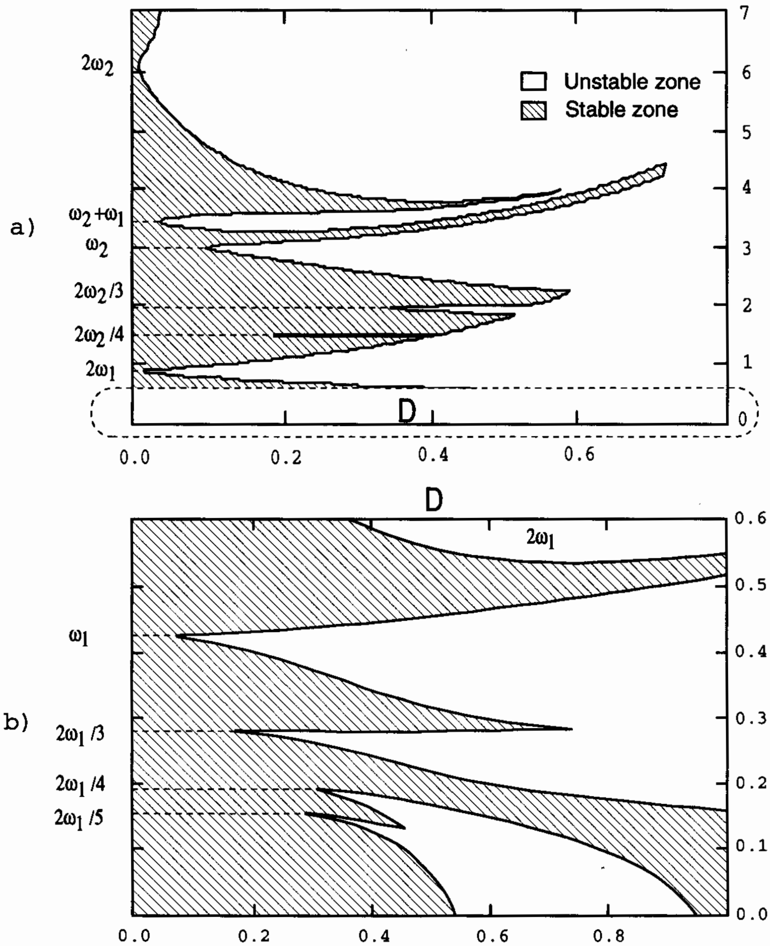

Figure 5. (a) Stability chart for damped-rotating system. (b) Enlarged view of the D zone

4. Conclusions

The stability charts for a parametrically excited rotating mechanical system with two-degrees-of-freedom has been presented. It is observed that all main and fractional parametric resonances occur in a nonrotating system. The additive or differential combination resonance cannot appear in this case since the equations can be decoupled in the principal coordinates. For the rotating system, additional instability zones corresponding to additive combination resonance, $\omega_1 + \omega_2$ appear and the stability region for low values of parametric excitation frequency is significantly modified. In particular, it is observed from figure 3b that for a damped nonrotating system only one dynamic resonance ($\Psi = \omega_I$) appears, whereas figure 5b shows the existence of several subharmonic resonances.

Acknowledgments

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Appendix 1

For the case of an undamped and nonrotating system, the expressions used to determine the stability boundaries in Hill's method (taking only the first three terms of the trial solutions) are as follows

- for the odd trial solution [*c.f.* equation (16)]

$$
\begin{gathered}
1 - 105\Psi^2 + 4010.125\Psi^4 - 15.\varepsilon^2\Psi^4 - 68920.3125\Psi^6 \\
+1192.5\varepsilon^2\Psi^6 + 557989.9335\Psi^8 - 32655.1875\varepsilon^2\Psi^8 \\
+57.25\varepsilon^4\Psi^8 - 2.0764 \times 10^6\Psi^{10} + 382198.375\varepsilon^2\Psi^{10} \\
-2773.5\varepsilon^4\Psi^{10} + 4.0657 \times 10^6\Psi^{12} - 1.929 \times 10^6\varepsilon^2\Psi^{12} \\
+39934.6875\varepsilon^4\Psi^{12} - 47.25\varepsilon^6\Psi^{12} - 4.4734 \times 10^6\Psi^{14} \\
+4.0836 \times 10^6\varepsilon^2\Psi^{14} - 212523.6562\varepsilon^4\Psi^{14} + 1085.625\varepsilon^6\Psi^{14} \\
+2.7819 \times 10^6\Psi^{16} - 4.2403 \times 10^6\varepsilon^2\Psi^{16} + 714656.1230\varepsilon^4\Psi^{16} \\
-12141.7968\varepsilon^6\Psi^{16} + 12.875\varepsilon^8\Psi^{16} - 931415.5323\Psi^{18} \\
+2.3546 \times 10^6\varepsilon^2\Psi^{18} - 1.0625 \times 10^6\varepsilon^4\Psi^{18} + 34332.4218\varepsilon^6\Psi^{18} \\
-121.125\varepsilon^8\Psi^{18} + 148922.6925\Psi^{20} - 735835.7036\varepsilon^2\Psi^{20} \\
+705202.5534\varepsilon^4\Psi^{20} - 66065.29\varepsilon^6\Psi^{20} + 1047.7031\varepsilon^8\Psi^{20} \\
-1.125\varepsilon^{10}\Psi^{20} - 8440.5094\Psi^{22} + 125110.6309\varepsilon^2\Psi^{22} \\
-207608.7249\varepsilon^4\Psi^{22} + 53131.3032\varepsilon^6\Psi^{22} - 1308.6328\varepsilon^8\Psi^{22} \\
+2.6875\varepsilon^{10}\Psi^{22} + 152.7601\Psi^{24} - 9027.1085\varepsilon^2\Psi^{24} \\
+22211.2011\varepsilon^4\Psi^{24} - 13155.8670\varepsilon^6\Psi^{24} + 826.8862\varepsilon^8\Psi^{24} \\
-13.1953\varepsilon^{10}\Psi^{24} + 0.0156\varepsilon^{12}\Psi^{24} = 0;
\end{gathered}
$$

- for the even trial solution [*c.f.* equation (17)]

$$1 - 168\Psi^2 + 10780\Psi^4 - 18.\varepsilon^2\Psi^4 - 335304\Psi^6$$

$$
\begin{aligned}
&+2378\varepsilon^2\Psi^6 + 5.3951\times 10^6\Psi^8 - 116202\varepsilon^2\Psi^8 + 77\varepsilon^4\Psi^8\\
&-4.5176\times 10^7\Psi^{10} + 2.6730\times 10^6\varepsilon^2\Psi^{10} - 7050\varepsilon^4\Psi^{10}\\
&+2.0096\times 10^8\Psi^{12} - 3.0553\times 10^7\varepsilon^2\Psi^{12} + 228971\varepsilon^4\Psi^{12}\\
&-36\varepsilon^6\Psi^{12} - 4.7898\times 10^8\Psi^{14} + 1.7314\times 10^8\varepsilon^2\Psi^{14}\\
&-3.2058\times 10^6\varepsilon^4\Psi^{14} + 3116.\varepsilon^6\Psi^{14} + 6.2453\times 10^8\Psi^{16}\\
&-4.9191\times 10^8\varepsilon^2\Psi^{16} + 2.046\times 10^7\varepsilon^4\Psi^{16} - 91824\varepsilon^6\Psi^{16}\\
&+4\varepsilon^8\Psi^{16} - 4.427\times 10^8\Psi^{18} + 7.366\times 10^8\varepsilon^2\Psi^{18}\\
&-7.8318\times 10^7\varepsilon^4\Psi^{18} + 1.0481\times 10^6\varepsilon^6\Psi^{18} - 336\varepsilon^8\Psi^{18}\\
&+1.6266\times 10^8\Psi^{20} - 5.8995\times 10^8\varepsilon^2\Psi^{20} + 1.6078\times 10^8\varepsilon^4\Psi^{20}\\
&-4.0169\times 10^6\varepsilon^6\Psi^{20} + 9352\varepsilon^8\Psi^{20} - 2.7433\times 10^7\Psi^{22}\\
&+2.4513\times 10^8\varepsilon^2\Psi^{22} - 1.7016\times 10^8\varepsilon^4\Psi^{22}\\
&+8.9875\times 10^6\varepsilon^6\Psi^{22} - 92448\varepsilon^8\Psi^{22} + 1.6796\times 10^6\Psi^{24}\\
&-4.7589\times 10^7\varepsilon^2\Psi^{24} + 9.0167\times 10^7\varepsilon^4\Psi^{24} - 1.1022\times 10^7\varepsilon^6\Psi^{24}\\
&+187236\varepsilon^8\Psi^{24} + 3.3592\times 10^6\varepsilon^2\Psi^{26} - 2.1555\times 10^7\varepsilon^4\Psi^{26}\\
&+6.4851\times 10^6\varepsilon^6\Psi^{26} - 136080.\varepsilon^8\Psi^{26} + 1.6796\times 10^6\varepsilon^4\Psi^{28}\\
&-1.3996\times 10^6\varepsilon^6\Psi^{28} + 32400.\varepsilon^8\Psi^{28} = 0.
\end{aligned}
$$

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