STABILITY ANALYSIS OF SYSTEMS WITH PERIODIC COEFFICIENTS: AN APPROXIMATE APPROACH

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The paper deals with an approximate method of stability analysis for second order linear systems with periodic coefficients. The periodic functions are approximated during the first period of motion by a constant, a linear or a quadratic function of time such that the resulting approximate equations have known closed form solutions. The approximate equivalent equations are generated through an expansion of periodic coefficients into ultraspherical polynomials. The stability criteria is determined from the solution of approximate equivalent system and the generalized Floquet theory. The technique is quite general and does not require any restriction on the magnitudes of system parameters. In particular, the method has been applied to construct approximate stability chart for the Mathieu equation. A close agreement between the approximate and the exact result is found even for large values of system parameters.

1. INTRODUCTION

The study of systems governed by differential equations with time-dependent coefficients is of importance in widely diverse branches of mechanics, physics and other engineering sciences. The problem of generating approximate solutions for such systems has been studied by several authors. Usually the approach has been to utilize a small or large parameter and develop parameter perturbation, or to utilize a small or large co-ordinate and develop co-ordinate perturbations. An account of these techniques has been summarized by Nayfeh [1] and Yakubovich and Starzhinskii [2]. In the case of slowly varying coefficients the solutions can be obtained by the well-known WKBJ method. Recently Srinivasan and Sankar [3] have presented approximate solutions of such systems in terms of Bessel and Weber functions. Following their work and a note by Kreyszig [4], Sinha and Chou [5] have developed a method for constructing approximate equivalent systems for a general class of time-varying
systems such that the approximate solutions can be expressed in terms of known closed form solutions. The method has been successfully applied to study the transient response of general second order time-varying systems including the Mathieu equation and problems with turning points.

However, the problem of stability analysis for a general class of second order systems with periodic coefficients based on approximate methods is still open to question. It is well known that the WKBJ method cannot be used for stability analysis because of the tacit assumption that all solutions are stable (see, for example, reference [6]). On the other hand, perturbation methods can only be applied to determine the stability characteristics if the governing equation contains a suitable small parameter. Such analyses have been undertaken by Stoker [7] and Cesari [8].

The purpose of this investigation is to develop an approximate approach for determining the stability of second order linear systems with periodic coefficients through an application of the method proposed by Sinha and Chou [5].

In this approach the periodic coefficient appearing in the equation is expanded in ultraspherical polynomials to generate approximate equivalent equations corresponding to the original problem. The expansion is restricted to a constant, a linear or a quadratic function such that the solutions of these equivalent equations can be expressed in terms of trigonometric or special functions. Once the approximate solutions are obtained, the stability criteria can be established through an application of Floquet theory. At this point, it should be observed that a method similar to the constant approximation has been used by other authors [9–11] to develop numerical solutions to the stability problem for the Schrödinger and Mathieu equations. In the approach suggested by Canosa and Oliveira [9] and Canosa [10], the potential function was approximated by a step function with an arbitrary but finite number of steps. In each step, the resulting differential equation was integrated exactly in terms of circular or hyperbolic functions. The solutions were then matched at the interface of each layer in order to obtain the eigenfunctions in the whole domain. The present method provides a systematic approach to construct three distinct approximate equivalent systems with increasing accuracy to obtain the stability criteria in closed forms. Emphasis has been placed on the development of the analytical solutions rather than the numerical solutions.

In particular, the method has been applied to study the stability of the Mathieu equation. The approximate stability chart obtained from the present method is compared with the exact chart available in the literature.

2. THE GOVERNING EQUATION AND THE METHOD OF APPROXIMATION

The canonical form of a second order differential equation with periodic coefficients can be written as

$$\ddot{x} + G(t)x = 0, \quad G(t) = G(t+T),$$

where $T$ is the period of $G(t)$.

The development of approximate solutions for equation (1) is based on the idea of replacing the original equation by an approximate equation

$$\ddot{x} + \bar{G}(t)x = 0, \quad \bar{G}(t) = \bar{G}(t+T),$$

and expecting that the solutions to equation (2) will be the approximate solutions to equation (1). The approximate equivalent function $\bar{G}(t)$ in equation (2) corresponding to the original function $G(t)$ can be obtained by minimizing the weighted mean-square error between the
two functions in the desired time interval \([0, T]\). One way to obtain such an approximate form of \(\bar{G}(t)\) is to expand \(G(t)\) in a series of ultraspherical polynomials and restrict the expansion up to the constant, the linear or the quadratic term in \(t\) depending on the type of approximation desired. Usually the total time interval (one period \([0, T]\)) has to be divided into two or three small intervals \([0, T_1], [T_1, T_2], \ldots, [T_{n-1}, T]\) to obtain a meaningful solution.

The general approximation procedure can be outlined in the following steps [5].

2.1. SELECTION OF SUB-INTERVALS

Divide the total period \([0, T]\) into \(n\) sub-intervals \([T_{k-1}, T_k]\), \(k = 1, 2, \ldots, n\) such that

\[
T = \sum_{k=1}^{n} \bar{T}_k, \tag{3}
\]

where \(\bar{T}_k = T_k - T_{k-1}\) with \(T_0 = 0\) and \(T_n = T\). Usually \(n\) can be restricted to 2 or 3.

2.2. CONSTRUCTION OF EQUIVALENT EQUATIONS

In each sub-interval \(\bar{T}_k\), \(G(t)\) is expanded in ultraspherical polynomials and the expansion is restricted up to a constant, a linear or a quadratic in \(t\) depending on the type of approximation desired such that

\[
G(t) \approx \bar{G}_k(t) = \begin{cases} 
A_{k0} \text{ (constant approx.)}, \\
A_{k1} + B_{k1} t \text{ (linear approx.)}, \\
A_{k2} + B_{k2} t + C_{k2} t^2 \text{ (quadratic approx.)}, 
\end{cases} \tag{4, 5, 6}
\]

where \(A_{ki}, B_{ki},\) and \(C_{ki}\) are coefficients of expansions in the interval \([T_{k-1}, T_k]\). The expansion of a function in ultraspherical polynomials is treated in detail in reference [5].

Thus the original differential equation (1) is approximated in each sub-interval \(\bar{T}_k\) as

\[
\ddot{x}_k + \bar{G}(t) \dot{x}_k = 0, \quad t \in [T_{k-1}, T_k], \quad k = 1, 2, \ldots, n, \tag{7}
\]

where \(\bar{G}(t)\) is given by equations (4), (5) or (6). To preserve the continuity of the solution, the initial conditions on one interval (say the \(k\)th) are determined from the initial values on the preceding interval ((\(k-1\)th)), or

\[
x_k(T_{k-1}) = x_{k-1}(T_{k-1}), \quad \dot{x}_k(T_{k-1}) = \dot{x}_{k-1}(T_{k-1}). \tag{8}
\]

\(k = 1\) corresponds to the initial conditions imposed on equation (1). Hence the complete approximate solution of equation (2) for \(t \in [0, T]\) can be written as

\[
x = \sum_{k=1}^{n} \left[ H(t - T_{k-1}) - H(t - T_k) \right] x_k, \tag{9}
\]

where \(H(\ )\) is the unit step function.

2.3. CLOSED FORM SOLUTIONS FOR VARIOUS FORMS OF \(\bar{G}_k(t)\)

2.3.1. Case I: Constant approximation

If \(\bar{G}_k(t)\) in equation (7) is given by equation (4), the solution can be written as

\[
x_k = M_{k0} \cos(A_{k0})^{1/2} t + N_{k0} \sin(A_{k0})^{1/2} t, \quad A_{k0} > 0, \tag{10}
\]

\[
x_k = M_{k0} \cosh(A_{k0})^{1/2} t + N_{k0} \sinh(A_{k0})^{1/2} t, \quad A_{k0} < 0, \tag{11}
\]

where \(M_{k0}\) and \(N_{k0}\) are the integration constants.
2.3.2. **Case II: Linear approximation**

If $G_k(t)$ is approximated by a linear function as given by equation (5), then by equation (7),

$$
\ddot{x}_k + (A_{k1} + B_{k1} t) x_k = 0, \quad t \in [T_{k-1}, T_k].
$$

(12)

With the transformation of variables

$$
z_k = (A_{k1} + B_{k1} t) B_{k1}^{-2/3}, \quad \xi_k = \frac{2}{3} z_k^{1/2},
$$

the solution of equation (12) is

$$
x_k = (z_k)^{1/2} [M_{k1} J_{1/3}(\xi_k) + N_{k1} J_{-1/3}(\xi_k)].
$$

(13)

where $M_{k1}$ and $N_{k1}$ are the integration constants.

2.3.3. **Case III: Quadratic approximation**

When $G_k(t)$ is given by equation (6), equation (7) yields

$$
\ddot{x}_k + (A_{k2} + B_{k2} t + C_{k2} t^2) x_k = 0, \quad t \in [T_{k-1}, T_k].
$$

(14)

Equation (14) can be transformed to

$$
\ddot{x}_k - (\delta_k) x_k = 0,
$$

(15)

where

$$
t = C_1 \dot{z}_k + C_2, \quad C_1 = [\pm \frac{1}{2} C_{k2}]^{1/2}, \quad C_2 = -B_{k2}/2C_{k2}
$$

$$
\delta_k = A_{k2} C_1^2 + B_{k2} C_2 C_1^2 + C_{k2} C_1^2 C_2.
$$

If equation (15) reduces to

$$
\ddot{x}_k - (\frac{1}{4} \delta_k^2 + \delta_k) x_k = 0,
$$

(16)

the solution is [5, 12]

$$
x_k = M_{k2} U(\delta_k, z_k) + N_{k2} V(\delta_k, z_k),
$$

(17)

where $M_{k2}$ and $N_{k2}$ are integration constants and $U(\delta_k, z_k)$ and $V(\delta_k, z_k)$ are the parabolic cylinder functions.

In the event when equation (15) is of the form

$$
x_k + (\frac{1}{4} \delta_k^2 - \delta_k) x_k = 0,
$$

(18)

then

$$
x_k = M_{k2} W(\delta_k, z_k) + N_{k2} W(\delta_k, -z_k),
$$

(19)

where $W(\delta_k, z_k)$ and $W(\delta_k, -z_k)$ are Weber functions. Parabolic cylinder functions and Weber functions are tabulated.

Hence for all the three cases, $x_k(t)$ has been determined for $t \in [T_{k-1}, T_k]$. The complete solution $x(t)$ for one period $[0, T]$ can be obtained from equation (9).

As an example, consider the Mathieu equation

$$
\ddot{x} + A [1 + m \cos \omega t] x = 0.
$$

(20)

When $A [1 + m \cos \omega t]$ is expanded in ultraspherical polynomials in the interval $[T_{k-1}, T_k]$ and if the expansion is restricted to one term only (i.e., the constant term), then $A_{k0}$
two functions in the desired time interval \([0, T]\). One way to obtain such an approximate form of \(G(t)\) is to expand \(G(t)\) in a series of ultraspherical polynomials and restrict the expansion up to the constant, the linear or the quadratic term in \(t\) depending on the type of approximation desired. Usually the total time interval (one period \([0, T]\)) has to be divided into two or three small intervals \([0, T_1], [T_1, T_2], \ldots, [T_{n-1}, T]\) to obtain a meaningful solution.

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In each sub-interval \(\bar{T}_k\), \(G(t)\) is expanded in ultraspherical polynomials and the expansion is restricted up to a constant, a linear or a quadratic in \(t\) depending on the type of approximation desired such that

\[
G(t) \simeq \bar{G}_k(t) = \begin{cases} 
A_{k0} \text{(constant approx.)}, \\
A_{k1} + B_{k1} t \text{(linear approx.)}, \\
A_{k2} + B_{k2} t + C_{k2} t^2 \text{(quadratic approx.)},
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where \(A_{ki}, B_{ki}\) and \(C_{ki}\) are coefficients of expansions in the interval \([T_{k-1}, T_k]\). The expansion of a function in ultraspherical polynomials is treated in detail in reference [5].

Thus the original differential equation (1) is approximated in each sub-interval \(\bar{T}_k\) as

\[
\ddot{x}_k + G(t)x_k = 0, \quad t \in [T_{k-1}, T_k], \quad k = 1, 2, \ldots, n, \tag{7}
\]

where \(\bar{G}(t)\) is given by equations (4), (5) or (6). To preserve the continuity of the solution, the initial conditions on one interval (say the \(k\)th) are determined from the initial values on the preceding interval ((\(k - 1\))th), or

\[
x_k(T_{k-1}) = x_{k-1}(T_{k-1}), \quad \dot{x}_k(T_{k-1}) = \dot{x}_{k-1}(T_{k-1}). \tag{8}
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\(k = 1\) corresponds to the initial conditions imposed on equation (1). Hence the complete approximate solution of equation (2) for \(t \in [0, T]\) can be written as

\[
x = \sum_{k=1}^{n} [H(t - T_{k-1}) - H(t - T_k)] x_k, \tag{9}
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\]

\[
x_k = M_{k0} \cosh(A_{k0})^{1/2}t + N_{k0} \sinh(A_{k0})^{1/2}t, \quad A_{k0} < 0, \tag{11}
\]

where \(M_{k0}\) and \(N_{k0}\) are the integration constants.
2.3.2. Case II: Linear approximation

If $\bar{G}_k(t)$ is approximated by a linear function as given by equation (5), then by equation (7),

$$\ddot{x}_k + (A_{k1} + B_{k1} t)x_k = 0, \quad t \in [T_{k-1}, T_k]. \quad (12)$$

With the transformation of variables

$$z_k = (A_{k1} + B_{k1} t) B_{k1}^{-2/3}, \quad \xi_k = \frac{2}{3} z_k^{3/2},$$

the solution of equation (12) is

$$x_k = (z_k)^{1/2} [M_{k1} J_{1/3}(\xi_k) + N_{k1} J_{-1/3}(\xi_k)], \quad (13)$$

where $M_{k1}$ and $N_{k1}$ are the integration constants.

2.3.3. Case III: Quadratic approximation

When $\bar{G}_k(t)$ is given by equation (6), equation (7) yields

$$\ddot{x}_k + (A_{k2} + B_{k2} t + C_{k2} t^2)x_k = 0, \quad t \in [T_{k-1}, T_k]. \quad (14)$$

Equation (14) can be transformed to

$$\ddot{x}_k - \left( \frac{1}{4} \xi_k^2 + \delta_k \right) x_k = 0, \quad (15)$$

where

$$t = C_1 z_k + C_2, \quad C_1 = [\pm \sqrt{C_{k2}}]^{1/2}, \quad C_2 = -B_{k2}/2C_{k2}$$

$$\delta_k = A_{k2} C_1^2 + B_{k2} C_2 C_1^2 + C_{k2} C_1^2 C_2^2.$$

If equation (15) reduces to

$$\ddot{x}_k - \left( \frac{1}{4} \xi_k^2 + \delta_k \right) x_k = 0, \quad (16)$$

the solution is [5, 12]

$$x_k = M_{k2} U(\delta_k, z_k) + N_{k2} V(\delta_k, z_k), \quad (17)$$

where $M_{k2}$ and $N_{k2}$ are integration constants and $U(\delta_k, z_k)$ and $V(\delta_k, z_k)$ are the parabolic cylinder functions.

In the event when equation (15) is of the form

$$x_k + (\frac{1}{4} \xi_k^2 - \delta_k) x_k = 0, \quad (18)$$

then

$$x_k = M_{k2} W(\delta_k, z_k) + N_{k2} W(\delta_k, -z_k), \quad (19)$$

where $W(\delta_k, z_k)$ and $W(\delta_k, -z_k)$ are Weber functions. Parabolic cylinder functions and Weber functions are tabulated.

Hence for all the three cases, $x_k(t)$ has been determined for $t \in [T_{k-1}, T_k]$. The complete solution $x(t)$ for one period $[0, T]$ can be obtained from equation (9).

As an example, consider the Mathieu equation

$$\ddot{x} + A[1 + m \cos \omega t] x = 0. \quad (20)$$

When $A[1 + m \cos \omega t]$ is expanded in ultraspherical polynomials in the interval $[T_{k-1}, T_k]$ and if the expansion is restricted to one term only (i.e., the constant term), then $A_{k0}$
corresponding to equation (4) is given as [5]

$$A_{k0} = A [1 + mA_x(\omega a_k) \cos(\omega b_k)],$$

(21)

where

$$a_k = (T_k - T_{k-1})/2, \quad b_k = (T_k + T_{k-1})/2,$$

(22)

$$A_x(\omega a_k) = \Gamma (\lambda + 1) J_\lambda (\omega a_k)/(\omega a_k)^{\lambda}.$$

(23)

Hence, up to the one term approximation, equation (20) is replaced by

$$\dot{x}_k + A [1 + mA_x(\omega a_k) \cos(\omega b_k)] x_k = 0, \quad t \in [T_{k-1}, T_k].$$

(24)

If the expansion is limited up to a linear term in $t$, then $A_{k1}$ and $B_{k1}$ in equation (12) take the forms [5]

$$A_{k1} = A + mA_x(\omega a_k) \cos(\omega b_k) + \omega b_k A_{\lambda+1}(\omega a_k) \sin(\omega b_k),$$

$$B_{k1} = -mA_x A_{\lambda+1}(\omega a_k) \sin(\omega b_k).$$

(25)

Similarly an expansion up to quadratic terms in $t$ leads to the following values of $A_{k2}$, $B_{k2}$ and $C_{k2}$ for equation (14) [5]:

$$A_{k2} = A [1 + m \cos(\omega b_k) [A_x(\omega a_k) - \frac{1}{2} b_k^2 - \frac{1}{2} a_k^2/(\lambda + 1)] A_{\lambda+2}(\omega a_k)]$$

$$+ m(\omega b_k) A_{\lambda+1}(\omega a_k) \sin(\omega b_k)],$$

$$B_{k2} = mA [\omega^2 b_k A_{\lambda+2}(\omega a_k) \cos(\omega b_k) - \omega A_{\lambda+1}(\omega a_k) \sin(\omega b_k)],$$

$$C_{k2} = -\left(\frac{mA^2}{2}\right) A_{\lambda+2}(\omega a_k) \cos(\omega b_k).$$

(26)

For the constant or the quadratic approximation, the first period $[0, T]$ can be divided into three sub-intervals while for the linear approximation two sub-intervals will be sufficient to obtain a meaningful result.

Hence for the constant or the quadratic approximation, equation (3) takes the form

$$T = \frac{2\pi}{\omega} = \sum_{k=1}^{3} \bar{T}_k,$$

(27)

where

$$\bar{T}_1 = \pi/2\omega, \quad t \in [0, \pi/2\omega]; \quad \bar{T}_2 = \pi/\omega, \quad t \in [\pi/2\omega, 3\pi/2\omega];$$

$$\bar{T}_3 = \pi/\omega, \quad t \in [3\pi/2\omega, 2\pi/\omega],$$

whereas for the linear approximation

$$T = \frac{2\pi}{\omega} = \sum_{k=1}^{2} \bar{T}_k,$$

(28)

where

$$\bar{T}_1 = \pi/\omega, \quad t \in [0, \pi/\omega]; \quad \bar{T}_2 = \pi/\omega, \quad t \in [\pi/\omega, 2\pi/\omega].$$

In equation (27) $\bar{T}_3$ can also be taken as $\pi/\omega$ if the periodic coefficient in equation (20) is expanded in the interval $[3\pi/2\omega, 5\pi/2\omega]$.

Once the intervals have been determined, $a_k$ and $b_k$ for each interval can be found from equation (22).

Thus one can find the approximate closed form fundamental solutions at the end of period $T$ to determine the stability of the system via Floquet theory. The procedure is indicated in section 3.
3. STABILITY ANALYSIS

Although the stability analysis of equation (1) has been presented in detail in the literature [7], the theory is discussed here briefly in the framework of present analysis.

Let $x_1(t)$ and $x_2(t)$ be the two fundamental solutions of equation (1) such that at $t = 0$

$$x_1(0) = 1, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 1.$$  \hspace{1cm} (29)

The general solution of equation (1) can be written as

$$x(t) = A_1 x_1(t) + A_2 x_2(t),$$  \hspace{1cm} (30)

where $A_1$ and $A_2$ are arbitrary constants.

Due to the periodicity of the solutions, one also has

$$x_1(t + T) = ax_1(t) + bx_2(t), \quad x_2(t + T) = cx_1(t) + dx_2(t),$$  \hspace{1cm} (31)

where $a$, $b$, $c$ and $d$ are constants.

It can be shown [6, 7] that for a normal solution $x(t)$ one has $x(t + T) = \sigma x(t)$ with $\sigma$ a root of

$$\begin{vmatrix} \sigma - a & -c \\ -b & \sigma - d \end{vmatrix} = 0,$$  \hspace{1cm} (32)

which can be written in the form

$$\sigma^2 - \sigma(a + d) + 1 = 0,$$  \hspace{1cm} (33)

where

$$a = x_1(T), \quad b = \dot{x}_1(T), \quad c = x_2(T), \quad d = \dot{x}_2(T).$$  \hspace{1cm} (34)

The characteristic equation (33) yields

$$\sigma = \frac{1}{2}(a + d) \pm j[1 - (a + d)^2/4]^{1/2},$$  \hspace{1cm} (35)

where $j = \sqrt{-1}$. One can set

$$\sigma = e^{\mu T} = \cos \mu T + j \sin \mu T,$$  \hspace{1cm} (36)

and comparison of equations (35) and (36) gives

$$\cos \mu T = \frac{1}{2}(a + d) = \frac{1}{2}[x_1(T) + \dot{x}_2(T)].$$  \hspace{1cm} (37)

$\mu$ is real for $|\sigma| < 1$, and the solution is stable if

$$|x_1(T) + \dot{x}_2(T)| < 2;$$  \hspace{1cm} (38)

also $\mu = \pm \pi/2T$ and $|\sigma| = 1$ if

$$|x_1(T) + \dot{x}_2(T)| = 2.$$  \hspace{1cm} (39)

For $\mu$ an imaginary quantity, $|\sigma| > 1$, (and an unstable solution), one must have

$$|x_1(T) + \dot{x}_2(T)| > 2.$$  \hspace{1cm} (40)

The theory developed in this section is utilized to obtain the stability charts for the Mathieu equation and its corresponding approximate equations.

4. APPLICATION TO THE STABILITY ANALYSIS OF THE MATHIEU EQUATION

The Mathieu equation is the most widely studied example of a periodic-coefficient differential equation. Because of its technical importance, the equation has been studied in
detail and the stability chart is available in the literature. In this section the stability chart for the exact equation is constructed by using a numerical technique and compared with the stability charts obtained from the approximate equations based on the method outlined in section 2.

4.1. STABILITY CHART FOR THE EXACT EQUATION (A NUMERICAL APPROACH)

The method adopted here is the straightforward integration of the Mathieu equation

\[ \ddot{x} + A(1 + m \cos t)x = 0, \quad A, m \geq 0, \]  \hspace{1cm} (41)

subject to two sets of initial conditions given by equation (29). The computation of \( \mu \) requires that the two fundamental solutions \( x_1 \) and \( \dot{x}_2 \) be evaluated at the end of the first period \( T = 2\pi \). Thus equation (37) yields

\[ \cos 2\pi \mu = \frac{1}{2}[x_1(2\pi) + \dot{x}_2(2\pi)]. \]  \hspace{1cm} (42)

If \( -1 < \cos(2\pi \mu) < 1 \), then \( \mu \) is real and can be obtained from

\[ \mu = \frac{1}{2} \arccos(\cos v), \quad -1 < \cos v < 1, \quad 0 < \mu < \frac{1}{2}, \]  \hspace{1cm} (43)

where \( v = 2\pi \mu \).

For \( \cos v > 1 \), the exponent \( \mu \) becomes complex and can be computed from

\[ \mu = j(1/2\pi) \arccosh(\cos v), \quad \cos v > 1, \quad \mu > j0. \]  \hspace{1cm} (44)

On the other hand, if \( \cos v < -1 \), \( \mu \) can be determined from

\[ \mu = -j(1/2\pi) \arccosh(\cos v), \quad \cos v < -1, \quad \mu < -j0. \]  \hspace{1cm} (45)

It is to be observed that the stable regions of the Mathieu equation are characterized by a real \( \mu \). The beginnings of the unstable regions are given by \( \cos 2\pi \mu = \pm 1 \). The corresponding values of \( \mu \) are \( \mu = 0 \) and \( \mu = \frac{1}{2} \).

Equation (41) was integrated by using a Runge–Kutta numerical scheme and the characteristic exponent \( \mu \) was computed by using equations (43)–(45) for various values of \( A \) and \( m \). The integrations were carried out with an accuracy of three figures in the numerical value of \( \mu \). Some special check runs were made with higher accuracy for a comparison with the numerical results available in the literature [6]. The stability boundaries obtained from these numerical calculations are shown with solid lines in Figure 1 of section 4.2. The graphical representations are accurate within a line-width.

4.2. STABILITY CHARTS FOR THE APPROXIMATE EQUATIONS

As described in section 2, equation (41) can be approximated by using a constant, a linear or a quadratic approximation. The fundamental solutions in each case can be obtained at the end of the first period (2\( \pi \)) and the characteristic exponent \( \mu \) can be once again determined from equations (43)–(45) as indicated in the following.

4.2.1. Case I: Constant approximation

Upon expanding \( \cos t \) in the sub-intervals \([0, \pi/2] \), \([\pi/2, 3\pi/2] \) and \([3\pi/2, 5\pi/2] \) and retaining the constant term only, equation (24) yields

\[ \ddot{x}_1 + A(1 + \alpha_1 m)x_1 = 0, \quad t \in [0, \pi/2], \]  \hspace{1cm} (46)

\[ \ddot{x}_2 + A(1 - \beta_1 m)x_2 = 0, \quad t \in [\pi/2, 3\pi/2], \]  \hspace{1cm} (47)

\[ \ddot{x}_3 + A(1 + \beta_1 m)x_3 = 0, \quad t \in [3\pi/2, 2\pi], \]  \hspace{1cm} (48)
Figure 1. Stability charts for Mathieu equation. (a) Constant approximation: ■, $\lambda = \frac{1}{2}$; ○, $\lambda = 10$; --, exact; (b) linear approximation: ■, $\lambda = \frac{1}{2}$; ○, $\lambda = 10$; --, exact; (c) quadratic approximation: △, $\lambda = 0$; ○, $\lambda = 10$; --, exact.
where
\[ x_1 = A_2 (\pi/4)/\sqrt{2}, \quad \beta_1 = A_2 (\pi/2). \]  

(49)

Approximate values for \( \mu \) can be calculated from the solutions of equations (46)-(48) subject to two sets of initial conditions,
\[ x_{11}(0) = 1, \quad \dot{x}_{11}(0) = 0, \quad x_{12}(0) = 0, \quad \dot{x}_{12}(0) = 1, \]  

(50)

where \( x_{11}(t) \) and \( x_{12}(t) \) are the two fundamental solutions of \( x_1(t) \). The continuity of solution requires
\[ x_{11}(\pi/2) = x_{21}(\pi/2), \quad x_{12}(\pi/2) = x_{22}(\pi/2), \]  

(51a)

\[ x_{21}(3\pi/2) = x_{31}(3\pi/2), \quad x_{22}(3\pi/2) = x_{32}(3\pi/2), \]  

(51b)

where \( x_{21}(t), x_{22}(t) \) and \( x_{31}(t), x_{32}(t) \) are the two fundamental solutions of \( x_2(t) \) and \( x_3(t) \), respectively.

The fundamental solutions of equations (46)-(48) can be readily written as
\[ x_{11}(t) = A_{11} \sin \alpha t + A_{21} \cos \alpha t, \quad x_{12}(t) = A_{12} \sin \alpha t + A_{22} \cos \alpha t, \]  

(52a)

\[ x_{21}(t) = B_{11} \sin \beta t + B_{21} \cos \beta t, \quad x_{22}(t) = B_{12} \sin \beta t + B_{22} \cos \beta t, \]  

(52b)

\[ x_{31}(t) = C_{11} \sin \gamma t + C_{21} \cos \gamma t, \quad x_{32}(t) = C_{12} \sin \gamma t + C_{22} \cos \gamma t, \]  

(52c)

where
\[ \alpha = [A(1 + x_1 m)]^{1/2}, \quad \beta = [A(1 - \beta_1 m)]^{1/2}, \quad \gamma = [A(1 + \beta_1 m)]^{1/2}, \]

(53)

the \( A \)'s, \( B \)'s and \( C \)'s are constants and it is assumed that \( \beta \) is real.

The characteristic exponent \( \mu \) can be calculated from equations (42) and (52c) as
\[ \cos 2\pi \mu = \frac{1}{2} [(C_{11} \sin 2\gamma + C_{21} \cos 2\gamma) + \gamma (C_{12} \cos 2\gamma - C_{22} \sin 2\gamma)]. \]  

(54)

From equations (50), (51) and (52), the values of \( C_{11}, C_{21}, C_{12} \) and \( C_{22} \) can be calculated as
\[ C_{11} = (B_{11} \sin 3\beta + B_{21} \cos 3\beta) \sin 3\gamma /3 \]  
\[ + (\beta/\gamma)(B_{11} \cos 3\beta - B_{21} \sin 3\beta) \sin 3\gamma/2, \]
\[ C_{21} = (B_{21} \sin 3\beta + B_{11} \cos 3\beta) \cos 3\gamma/2 \]  
\[ - (\beta/\gamma)(B_{11} \cos 3\beta - B_{21} \sin 3\beta) \sin 3\gamma/2, \]
\[ C_{12} = (B_{12} \sin 3\beta + B_{22} \cos 3\beta) \sin 3\gamma/2 \]  
\[ + (\beta/\gamma)(B_{12} \cos 3\beta - B_{22} \sin 3\beta) \cos 3\gamma/2, \]
\[ C_{22} = (B_{22} \sin 3\beta + B_{12} \cos 3\beta) \cos 3\gamma/2 \]  
\[ - (\beta/\gamma)(B_{12} \cos 3\beta - B_{22} \sin 3\beta) \sin 3\gamma/2, \]  

(54)

where
\[ B_{11} = \cos \alpha \beta / \sin \beta - (\alpha/\beta) \sin \alpha \beta \cos \beta, \]
\[ B_{21} = \cos \alpha \beta / \cos \beta + (\alpha/\beta) \sin \alpha \beta \sin \beta, \]
\[ B_{12} = (1/\alpha) \sin \alpha \beta / \sin \beta + (1/\beta) \cos \alpha \beta \cos \beta, \]
\[ B_{22} = (1/\alpha) \sin \alpha \beta / \cos \beta + (1/\beta) \cos \alpha \beta \sin \beta. \]  

(55)
Since $A, m, \alpha_1$ and $\beta_1$ are positive quantities, $\alpha$ and $\gamma$ are real and the fundamental solutions for $x_1(t)$ and $x_3(t)$ are always given by equations (52a) and (52c) respectively. However, if $(1 - \beta_1 m) < 0$, the quantity $\beta$ becomes imaginary and the two fundamental solutions of $x_2(t)$ have the following forms:

$$x_{21}(t) = B'_{11} \sinh \beta t + B'_{21} \cosh \beta t, \quad x_{22}(t) = B'_{12} \sinh \beta t + B'_{22} \cosh \beta t.$$  \hspace{1cm} (56)

By proceeding in a similar fashion as before, the characteristic exponent $\mu$ can be determined from

$$\cos 2\pi \mu = \frac{1}{2} [(C_{11} \sin 2\gamma \pi + C_{21} \cos 2\gamma \pi) + \gamma (C_{12} \cos 2\gamma \pi - C_{22} \sin 2\gamma \pi)],$$  \hspace{1cm} (57)

where

\begin{align*}
C_{11} &= (B'_{11} \sinh 3\beta \pi/2 + B'_{21} \cosh 3\beta \pi/2) \sin 3\gamma \pi/2 \\
&\quad + (\beta/\gamma)(B'_{11} \cosh 3\beta \pi/2 + B'_{21} \sinh 3\beta \pi/2) \cos 3\gamma \pi/2, \\
C_{21} &= (B'_{11} \sinh 3\beta \pi/2 + B'_{21} \cosh 3\beta \pi/2) \cos 3\gamma \pi/2 \\
&\quad - (\beta/\gamma)(B'_{11} \cosh 3\beta \pi/2 + B'_{21} \sinh 3\beta \pi/2) \sin 3\gamma \pi/2, \\
C_{12} &= (B'_{12} \sinh 3\beta \pi/2 + B'_{22} \cosh 3\beta \pi/2) \sinh 3\gamma \pi/2 \\
&\quad + (\beta/\gamma)(B'_{12} \cosh 3\beta \pi/2 + B'_{22} \sinh 3\beta \pi/2) \cosh 3\gamma \pi/2, \\
C_{22} &= (B'_{12} \sinh 3\beta \pi/2 + B'_{22} \cosh 3\beta \pi/2) \cos 3\gamma \pi/2 \\
&\quad - (\beta/\gamma)(B'_{12} \cosh 3\beta \pi/2 + B'_{22} \sinh 3\beta \pi/2) \sin 3\gamma \pi/2, \\
B'_{11} &= - \cos \alpha \pi/2 \sinh \beta \pi/2 - (\alpha/\beta) \sin \alpha \pi/2 \cosh \beta \pi/2, \\
B'_{21} &= \cos \alpha \pi/2 \cosh \beta \pi/2 + (\alpha/\beta) \sin \alpha \pi/2 \cos \beta \pi/2, \\
B'_{12} &= -(1/\alpha) \sin \alpha \pi/2 \sinh \beta \pi/2 + (1/\beta) \cos \alpha \pi/2 \cosh \beta \pi/2, \\
B'_{22} &= (1/\alpha) \sin \alpha \pi/2 \cosh \beta \pi/2 - (1/\beta) \cos \alpha \pi/2 \cosh \beta \pi/2.  \hspace{1cm} (58)
\end{align*}

Equations (53) and (57) provide closed form expressions for the characteristic exponent $\mu$. It is seen that the results depend on the quantities $\alpha, \beta$ and $\gamma$ which are functions of $\lambda$, the parameter which determines the class of ultraspherical polynomials used in the expansion. It was observed from numerical calculations that $\lambda = \frac{1}{2}$ (Legendre polynomials) provides the closest agreement between the approximate and the exact values of $\mu$ for small values of $A$ and $m$. The approximate and the exact charts are shown in Figure 1(a) for $\lambda = \frac{1}{2}$. Partial results with $\lambda = 10$ are also shown on the diagram for a comparison.

4.2.2. Case II: Linear approximation

For the linear approximation, the Mathieu equation can be approximated by a set of two equations in the interval $[0, 2\pi]$. The linear approximation for $A[1 + m\cos t]$ in two sub-intervals $[0, \pi]$ and $[\pi, 2\pi]$ can easily be obtained from equation (25) and the approximate equivalent system is given by

\begin{align*}
\ddot{x}_1 + A[1 + m(\alpha_1 + \beta_1 t)] x_1 &= 0, \quad t \in [0, \pi], \hspace{1cm} (60a) \\
\ddot{x}_2 + A[1 + m(\alpha_2 + \beta_2 t)] x_2 &= 0, \quad t \in [\pi, 2\pi], \hspace{1cm} (60b)
\end{align*}

where

\begin{align*}
\alpha_{11} &= (\pi/2) A_{\lambda - 1}(\pi/2), & \beta_{11} &= - A_{\lambda + 1}(\pi/2), \\
\alpha_{21} &= -(3\pi/2) A_{\lambda - 1}(\pi/2), & \beta_{21} &= A_{\lambda + 1}(\pi/2).
\end{align*}
The characteristic exponent \( \mu \) can be calculated from equations (60) subject to initial conditions given by equation (50). The continuity of solution requires

\[
x_{11}(\pi) = x_{21}(\pi), \quad x_{12}(\pi) = x_{22}(\pi), \quad \dot{x}_{11}(\pi) = \dot{x}_{21}(\pi), \quad \dot{x}_{12}(\pi) = \dot{x}_{22}(\pi),
\]

where \( x_{11}(t), x_{12}(t) \) and \( x_{21}(t), x_{22}(t) \) are the two fundamental solutions of \( x_1(t) \) and \( x_2(t) \), respectively.

The fundamental solutions of equations (60) are

\[
x_{11}(\xi_1) = \left(\frac{2}{3}\xi_1\right)^{1/3} [M_{11}(J_{1/3}(\xi_1) + N_{11}J_{-1/3}(\xi_1))],
\]

\[
x_{12}(\xi_1) = \left(\frac{2}{3}\xi_1\right)^{1/3} [M_{12}(J_{1/3}(\xi_1) + N_{12}J_{-1/3}(\xi_1))],
\]

\[
x_{21}(\xi_2) = \left(\frac{2}{3}\xi_2\right)^{1/3} [M_{21}(J_{1/3}(\xi_2) + N_{21}J_{-1/3}(\xi_2))],
\]

\[
x_{22}(\xi_2) = \left(\frac{2}{3}\xi_2\right)^{1/3} [M_{22}(J_{1/3}(\xi_2) + N_{22}J_{-1/3}(\xi_2))],
\]

where

\[
\xi_1 = \left(\frac{3}{2}\right) [A + mA(\xi_{11} + \beta_{11} t)]^{3/2} (mA\beta_{11})^{-1},
\]

\[
\xi_2 = \left(\frac{3}{2}\right) [A + mA(\xi_{21} + \beta_{21} t)]^{3/2} (mA\beta_{21})^{-1},
\]

and the \( M \)’s and \( N \)’s are the integration constants.

One can rewrite the initial conditions and the continuity equations in terms of the new independent variables \( \xi_1 \) and \( \xi_2 \) and evaluate the fundamental solutions at \( t = 2\pi \) to obtain a closed form solution for the characteristic exponent \( \mu \). After some simple calculations one has

\[
\cos 2\pi \mu = \frac{1}{2} \left(\frac{2}{3}\delta_3\right)^{1/3} \left[ [M_{21}J_{1/3}(\delta_3) + N_{21}J_{-1/3}(\delta_3)] + \eta_3 [M_{22}J_{-2/3}(\delta_3) - N_{22}J_{2/3}(\delta_3)] \right],
\]

where

\[
M_{21} = \left(\frac{3}{2}\delta_2\right)^{-1/3} \left[ K_1 J_{2/3}(\delta_2) + K_2 J_{-1/3}(\delta_2) \right]/H(\delta_2),
\]

\[
N_{21} = \left(\frac{3}{2}\delta_2\right)^{-1/3} \left[ K_1 J_{-2/3}(\delta_2) + K_2 J_{1/3}(\delta_2) \right]/H(\delta_2),
\]

\[
M_{22} = \left(\frac{3}{2}\delta_2\right)^{-1/3} \left[ K_3 J_{2/3}(\delta_2) + K_4 J_{-1/3}(\delta_2) \right]/H(\delta_2),
\]

\[
N_{22} = \left(\frac{3}{2}\delta_2\right)^{-1/3} \left[ K_3 J_{-2/3}(\delta_2) - K_4 J_{1/3}(\delta_2) \right]/H(\delta_2),
\]

\[
K_1 = \left(\frac{3}{2}\delta_1\right)^{1/3} \left[ M_{11} J_{1/3}(\delta_1) + N_{11} J_{-1/3}(\delta_1) \right],
\]

\[
K_2 = \eta_1/\eta_2 \left(\frac{3}{2}\delta_1\right)^{1/3} \left[ M_{11} J_{-2/3}(\delta_1) - N_{11} J_{2/3}(\delta_1) \right],
\]

\[
K_3 = \left(\frac{3}{2}\delta_1\right)^{1/3} \left[ M_{12} J_{1/3}(\delta_1) + N_{12} J_{-1/3}(\delta_1) \right],
\]

\[
K_4 = \eta_1/\eta_2 \left(\frac{3}{2}\delta_1\right)^{1/3} \left[ M_{12} J_{-2/3}(\delta_1) - N_{12} J_{2/3}(\delta_1) \right],
\]

\[
M_{11} = \left(\frac{3}{2}\delta_0\right)^{-1/3} J_{2/3}(\delta_0)/H(\delta_0), \quad N_{11} = \left(\frac{3}{2}\delta_0\right)^{-1/3} J_{-2/3}(\delta_0)/H(\delta_0),
\]

\[
M_{12} = \eta_0 \left(\frac{3}{2}\delta_0\right)^{-1/3} J_{-1/3}(\delta_0)/H(\delta_0), \quad N_{12} = -\eta_0 \left(\frac{3}{2}\delta_0\right)^{-1/3} J_{1/3}(\delta_0)/H(\delta_0),
\]

\[
\delta_0 = \left(\frac{3}{2}\right) (A + mA\alpha_{11})^{3/2} (mA\beta_{11})^{-1}, \quad \eta_0 = (A + mA\alpha_{11})^{-1/2},
\]

\[
\delta_1 = \left(\frac{3}{2}\right) [A + mA(\alpha_{11} + \pi\beta_{11})]^{3/2} (mA\beta_{11})^{-1}, \quad \eta_1 = [A + mA(\alpha_{11} + \pi\beta_{11})]^{1/2},
\]

\[
\delta_2 = \left(\frac{3}{2}\right) [A + mA(\alpha_{21} + \pi\beta_{21})]^{1/2}, \quad \delta_3 = \left(\frac{3}{2}\right) [A + mA(\alpha_{21} + 2\pi\beta_{21})]^{3/2} (mA\beta_{21})^{-1}, \quad \eta_3 = [A + mA(\alpha_{21} + 2\pi\beta_{21})]^{1/2},
\]

and

\[
H(\ ) = [J_{-2/3}( ) J_{-1/3}( ) + J_{2/3}( ) J_{1/3}( )].
\]
Once again, the stability boundaries can be determined from equations (65) and (43)-(45). The results depend on the parameter which characterizes the class of polynomials used in approximation. It was observed that \( \lambda = \frac{1}{2} \) (Legendre polynomials) provides the closest agreement between the approximate and the exact values of \( \mu \). The approximate stability chart, thus obtained, is shown in Figure 1(b) along with the exact chart. The results with \( \lambda = 10 \) are also presented on the same diagram for a comparative study. It is seen that \( \lambda = \frac{1}{2} \) provides a good approximation to the exact results even for large values of \( A \) and \( m \). A comparison of Figures 1(a) and 1(b) clearly shows that the results due to linear approximation are much better than the results obtained by constant approximation.

4.2.3. Case III: Quadratic approximation

For the quadratic approximation, the Mathieu equation can be approximated by a set of three equations. With the same sub-intervals as for the constant approximation, the following set of approximate equations can be obtained with the help of equation (26):

\[
\begin{align*}
\dot{x}_1 + A [1 + m (\alpha'_1 + \beta'_1 t + \gamma'_1 t^2)] x_1 &= 0, \quad t \in [0, \pi/2], \\
\dot{x}_2 + A [1 + m (\alpha'_2 + \beta'_2 t + \gamma'_2 t^2)] x_2 &= 0, \quad t \in [\pi/2, 3\pi/2], \\
\dot{x}_3 + A [1 + m (\alpha'_3 + \beta'_3 t + \gamma'_3 t^2)] x_3 &= 0, \quad t \in [3\pi/2, 2\pi],
\end{align*}
\]

where

\[
\begin{align*}
\alpha'_1 &= (2)^{-1/2} [A_\lambda (\pi/4) + (\pi/4) A_{\lambda+1} (\pi/4) - \{\pi^2 (2\lambda + 1)/64 (\lambda + 1) \} A_{\lambda+2} (\pi/4)], \\
\beta'_1 &= (2)^{-1/2} [A_{\lambda+1} (\pi/4) + (\pi/4) A_{\lambda+2} (\pi/4)], \\
\gamma'_1 &= - (1/8)^{-1/2} A_{\lambda+2} (\pi/4), \\
\alpha'_2 &= - [A_\lambda (\pi/2) - \{\pi^2 (8\lambda + 7)/16 (\lambda + 1) \} A_{\lambda+2} (\pi/2)], \\
\beta'_2 &= - (\pi) A_{\lambda+2} (\pi/2), \\
\gamma'_2 &= (2)^{-1} A_{\lambda+2} (\pi/2), \\
\alpha'_3 &= [A_\lambda (\pi/2) - \{\pi^2 (32\lambda + 31)/16 (\lambda + 1) \} A_{\lambda+2} (\pi/2)], \\
\beta'_3 &= (2\pi) A_{\lambda+2} (\pi/2), \\
\gamma'_3 &= - (2)^{-1} A_{\lambda+2} (\pi/2).
\end{align*}
\]

As described in section 2, one can transform equations (66) into their corresponding standard forms and write the solutions in terms of parabolic cylinder functions or Weber functions. The continuity conditions are once again given by equations (51). Thus a closed form expression can be obtained to determine the characteristic exponent \( \mu \). However, the calculations become very cumbersome and extremely complicated. For this reason, equations (66) were directly integrated by using a fourth-order Runge-Kutta numerical scheme. The initial conditions for the two fundamental solutions are given by equation (50). It was found that for this type of approximation, \( \lambda = 0 \) (Chebyshev polynomials) provides the best agreement between the exact and the approximate boundaries. It is seen from Figure 1(c) that the approximate stability boundaries with \( \lambda = 0 \) are in excellent agreement with the exact boundaries. From Figures 1(b) and 1(c) it is observed that the quadratic approximation provides much better results than the linear approximation.

5. DISCUSSION AND CONCLUSION

As seen from Figure 1, the approximate stability boundaries are in close agreement with the exact boundaries. As one should expect, the quadratic approximation provides an excellent agreement with the exact analysis. As indicated from Figure 1(a), even a simple constant approximation with \( \lambda = \frac{1}{2} \) yields very good results for small values of \( m \).
In conclusion, the present investigation provides a technique which can be successfully applied to study the stability characteristics of general second order systems with periodic coefficients. An approximate system, equivalent to the original system, is constructed for the first period of motion through an application of the ultraspherical polynomial expansion technique. In most cases, these approximate systems provide closed form analytical expressions for the characteristic exponent governing the stability of the original problem. The method is quite general and does not require any restriction on the system parameters.

In particular, the technique has been successfully applied to construct the stability chart for Mathieu equation. The analysis has been carried out for fairly large values of system parameters appearing in the Mathieu equation to demonstrate the advantage of the present method over other techniques available in the literature.

Very recently Epstein and Barakat [13] have studied the perturbation solutions of the Carson–Cambi equation. It is well-known that this equation can be transformed to Hill's equation, and thus one can study the stability characteristics of this equation by using the technique presented here without putting any restriction on the system parameters.

It should be observed that the proposed technique can also be extended to develop algorithms for numerical solutions following the approach suggested by Canosa and Oliveira [9]. It is expected that the numerical algorithms based on the linear and the quadratic approximations should yield considerably better results than reported in reference [9].

Further work on this topic is suggested toward the questions of the best choice of polynomials and the error analysis.

REFERENCES