PULSE RESPONSE OF NON-LINEAR NON-STATIONARY VIBRATIONAL SYSTEMS WITH $N$ DEGREES OF FREEDOM

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The paper deals with an approximate analysis of non-linear, non-stationary vibrational systems with multiple degrees of freedom subjected to a pulse excitation. The non-stationary system parameters, which may include masses, restoring forces or damping, are considered to be slowly varying functions of time. A general procedure for obtaining the first order approximate solution is presented through an application of the Bogoliubov-Mitropolsky technique. The special case of systems with two degrees of freedom is studied in detail. An illustrative example is included and the results are compared with fourth-order Runge-Kutta numerical solutions.

1. INTRODUCTION

The analysis of non-linear vibrational systems is of fundamental importance in many areas of mechanics and engineering design. An important class of these systems are those with time dependent parameters, generally known as non-stationary systems. Often such systems are subjected to various types of pulse loading and the analysis of the transient behavior may play an important role in design.

The response of non-linear vibrational systems subjected to a pulse excitation has been studied by several authors. Bapat and Srinivasan [1-3] have found the expressions for time periods and maximum displacements for a class of undamped non-linear stationary systems subjected to a step excitation. Step response of damped non-linear systems with constant parameters has been considered by Sinha and Srinivasan [4], Anderson [5] and Reed [6], among others. The response due to an arbitrary pulse excitation has been analyzed by Ariaratnam [7] and Bauer [8]. They have used the Poincaré-Lighthill perturbation method to determine the approximate response of second order non-linear systems with a single degree of freedom. Srirangarajan and Srinivasan [9] and Srirangarajan [10] have presented approximate analyses for the pulse response of certain third order systems through the application of a generalized averaging technique [4, 5] and the well-known Krylov-Bogoliubov-Mitropolsky method [11], respectively. The pulse response of non-linear systems with multiple degrees of freedom has been discussed by Bauer [12] and Rangacharyulu et al. [13]. However, these studies are restricted to non-linear systems with constant parameters only.

The analysis of non-linear, non-stationary systems has also received considerable attention. Nayfeh and Mook [14], Evan-Iwanowski [15], Bogoliubov and Mitropolsky
[11] and Mitropolsky [16] have studied several problems associated with such systems. Systems with single as well as multiple degrees of freedom have been considered. Nevertheless, the emphasis has been placed on systems subjected to harmonic excitations only. Recently Sinha and Chou [17] and Olberding and Sinha [18] have presented approximate analyses for non-linear, non-stationary vibrational systems with a single degree of freedom subjected to a step and an arbitrary pulse, respectively. Olberding and Sinha [18] used the Bogoliubov-Mitropolsky technique and found that the first order solutions were simple and compared well with a fourth-order Runge-Kutta solutions.

The purpose of this investigation is to present a general procedure for determining the pulse response of non-linear, non-stationary vibrational systems with multiple degrees of freedom. The system parameters as well as the external pulse are assumed to be slowly varying functions of time. First order approximate solutions are obtained through an application of the Bogoliubov-Mitropolsky method. Explicit results are obtained for systems with two degrees of freedom. An example is included to illustrate the procedure.

2. EQUATIONS OF MOTION AND THE GENERAL ANALYSIS

In many cases, the equations of motion for a non-linear system with $N$ degrees of freedom and slowly varying system parameters subjected to arbitrary pulse excitations can be reduced to the form

$$\frac{d}{dt} \left[ m_{ij}(\tau) \frac{dq_j}{dt} \right] + b_{ij}(\tau)q_j + \varepsilon f_i \left( q_j, \frac{dq_j}{dt} \right) = g_i(\tau), \quad i, j = 1, 2, \ldots, N,$$

where the summation convention has been used.

The generalized co-ordinates, and the non-stationary inertia and stiffness terms are represented by $q_j$, $m_{ij}(\tau)$ and $b_{ij}(\tau)$, respectively. $f_i$ are smooth non-linear functions and $\varepsilon$ is a small parameter characterizing the non-linearity. The non-stationary parameters $m_{ij}(\tau)$, $b_{ij}(\tau)$ and the excitation forces $g_i(\tau)$ are sufficiently smooth functions of the "slow time" $\tau$ defined as $\tau = \varepsilon t$. $m_{ij}(\tau)$ and $b_{ij}(\tau)$ are assumed to be positive definite for all $\tau$ within the time interval under consideration. Without any loss of generality, the initial conditions are taken as

$$q_j = \frac{dq_j}{dt} = 0 \quad \text{at} \quad t = 0.$$

Upon introducing the transformation

$$q_j = y_j + p_j,$$

equation (1) takes the form

$$\frac{d}{dt} \left[ m_{ij}(\tau) \frac{dy_j}{dt} \right] + \frac{d}{dt} \left[ m_{ij}(\tau) \frac{dp_j}{dt} \right] + b_{ij}(\tau)[y_j + p_j] + \varepsilon f_i \left( y_j + p_j, \frac{dy_j}{dt}, \frac{dp_j}{dt} \right) = g_i(\tau).$$

The initial conditions accordingly change to

$$y_j = -p_j \quad \text{and} \quad \frac{dy_j}{dt} = -\frac{dp_j}{dt} \quad \text{at} \quad t = 0.$$

As suggested in reference [18], the $p_j$ are chosen such that

$$\frac{d}{dt} \left[ m_{ij}(\tau) \frac{dp_j}{dt} \right] + b_{ij}(\tau)p_j = g_i(\tau).$$

Then equation (4) reduces to

$$\frac{d}{dt} \left[ m_{ij}(\tau) \frac{dy_j}{dt} \right] + b_{ij}(\tau)y_j + \varepsilon f_i \left( y_j + p_j, \frac{dy_j}{dt}, \frac{dp_j}{dt} \right) = 0.$$
It is observed from equation (7) that for a first order approximate solution for $y_j$, it suffices to obtain an approximate particular solution for $p_j$ which is independent of $\varepsilon$. If equation (6) is rewritten as

$$\varepsilon^2 m_y(\tau) \frac{d^2 p_j}{d\tau^2} + \varepsilon^2 \frac{d}{d\tau} [m_y(\tau)] \frac{dp_j}{d\tau} + b_j(\tau)p_j = g_j(\tau),$$

then it is evident that such a solution for $p_j$ is given by

$$p_j(\tau) = b_{ji}^{-1}(\tau)g_j(\tau).$$

Once the $p_j(\tau)$ are known, equation (7) takes the standard form

$$(d/d\tau)[m_{ij}(\tau) \frac{dy_j}{d\tau}] + b_{jj}(\tau)y_j + e\varepsilon_f[\tau, y_j, \frac{dy_j}{d\tau}] = 0.$$  

An approximate solution of the above equation is obtained through an application of the Bogoliubov-Mitropolsky method [16]. This is accomplished via a generalization of the approach suggested by Butenin [19] for systems with constant parameters.

First, consider the unperturbed system by setting $\varepsilon = 0$ and considering $\tau$ as a constant parameter in equation (10), viz.,

$$m_y(\tau) \frac{d^2 y_j}{d\tau^2} + b_j(\tau)y_j = 0.$$  

Since $m_y(\tau)$ and $b_j(\tau)$ are treated as stationary, a normal mode solution can be assumed as

$$y_j = \sum_{k=1}^{N} \alpha_{jk} a_k \cos \psi_k,$$  

and

$$\frac{dy_j}{d\tau} = -\sum_{k=1}^{N} \alpha_{jk} \omega_k a_k \sin \psi_k, \quad j = 1, 2, \ldots, N,$$

where

$$\psi_k = \omega_k(\tau)t + \theta_k,$$

and $a_k$ and $\theta_k$ are constants. The “normal frequencies” $\omega_k(\tau)$ are obtained from the frequency equation

$$\det [b_{ij}(\tau) - m_{ij}(\tau)\omega^2(\tau)] = 0,$$

while the quantities $\alpha_{jk}(\tau)$ are determined from

$$[b_{ij}(\tau) - m_{ij}(\tau)\omega^2(\tau)]\alpha_{jk}(\tau) = 0, \quad j, k = 1, 2, \ldots, N.$$  

Since $\alpha_{jk}$ represent the amplitude ratios, $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1N}$ can be set to unity without any loss of generality. $\omega_k(\tau)$ are assumed to be distinct.

In order to include the effect of the non-linearities, a solution of equation (10) can be assumed in the same form as given by equation (12), but $\alpha_{jk}, a_k$ and $\theta_k$ are allowed to be functions of time. This is possible if one is looking for an approximate solution valid up to the order of $\varepsilon$ only [16, 18, 19]. Then equation (12) yields

$$\frac{dy_j}{dt} = \sum_{k=1}^{N} \alpha_{jk} \cos \psi_k \frac{da_k}{dt} + \sum_{k=1}^{N} a_k \cos \psi_k \frac{d\alpha_{jk}}{dt}$$

$$- \sum_{k=1}^{N} \alpha_{jk} a_k \omega_k \sin \psi_k - \sum_{k=1}^{N} \alpha_{jk} a_k \sin \psi_k \frac{d\theta_k}{dt}, \quad j = 1, 2, \ldots, N.$$  

The argument $\tau$ has been dropped for brevity.
By requiring the derivatives of \( y_j \) to be of the same form as given by equation (13), the following condition is obtained:

\[
\sum_{k=1}^{N} \alpha_{jk} \cos \psi_k \frac{d\alpha_k}{dt} - \sum_{k=1}^{N} \alpha_{jk} a_k \sin \psi_k \frac{d\theta_k}{dt} + \sum_{k=1}^{N} a_k \cos \psi_k \frac{d\alpha_{jk}}{dt} = 0, \quad j = 1, 2, \ldots, N.
\]

(18)

Substituting for \( \frac{dy_j}{dt} \) in equation (7) and utilizing the results of equations (16) and (18) yields

\[
- \sum_{j,k=1}^{N} m_{ij}\alpha_{jk} \omega_k \sin \psi_k \frac{d\alpha_k}{dt} - \sum_{j,k=1}^{N} m_{ij}\alpha_{jk} \omega_k \cos \psi_k \frac{d\theta_k}{dt} = \sum_{j,k=1}^{N} \frac{d}{dt}[m_{ij}\alpha_{jk} \omega_k] a_k \sin \psi_k - \varepsilon_j \left( \tau, y_j, \frac{dy_j}{dt} \right), \quad i = 1, 2, \ldots, N.
\]

(19)

Equations (18) and (19) represent a set of \( 2N \) simultaneous equations in \( \frac{d\alpha_k}{dt} \) and \( \frac{d\theta_k}{dt} \) and may be rearranged in the matrix form

\[
\begin{bmatrix}
\mathbf{P}_1 & \mathbf{P}_2 \\
\mathbf{P}_3 & \mathbf{P}_4
\end{bmatrix}
\begin{bmatrix}
\frac{d\alpha}{dt} \\
\frac{d\theta}{dt}
\end{bmatrix}
= \begin{bmatrix}
\mathbf{u}_1 \\
\mathbf{u}_2
\end{bmatrix},
\]

(20)

where

\[
\begin{align*}
\mathbf{P}_1 &= \begin{bmatrix}
\alpha_{11} \cos \psi_1 & \cdots & \alpha_{1N} \cos \psi_N \\
\vdots & & \vdots \\
\alpha_{N1} \cos \psi_1 & \cdots & \alpha_{NN} \cos \psi_N
\end{bmatrix}, \\
\mathbf{P}_2 &= \begin{bmatrix}
-\alpha_{11} a_1 \sin \psi_1 & \cdots & -\alpha_{1N} a_N \sin \psi_N \\
\vdots & & \vdots \\
-\alpha_{N1} a_1 \sin \psi_1 & \cdots & -\alpha_{NN} a_N \sin \psi_N
\end{bmatrix}, \\
\mathbf{P}_3 &= \begin{bmatrix}
-\sum_{j=1}^{N} m_{1j} \alpha_{j1} \omega_1 \sin \psi_1 & \cdots & -\sum_{j=1}^{N} m_{1j} \alpha_{jN} \omega_N \sin \psi_N \\
\vdots & & \vdots \\
-\sum_{j=1}^{N} m_{Nj} \alpha_{j1} \omega_1 \sin \psi_1 & \cdots & -\sum_{j=1}^{N} m_{Nj} \alpha_{jN} \omega_N \sin \psi_N
\end{bmatrix}, \\
\mathbf{P}_4 &= \begin{bmatrix}
-\sum_{j=1}^{N} m_{1j} \alpha_{j1} \omega_1 \cos \psi_1 & \cdots & -\sum_{j=1}^{N} m_{1j} \alpha_{jN} \omega_N \cos \psi_N \\
\vdots & & \vdots \\
-\sum_{j=1}^{N} m_{Nj} \alpha_{j1} \omega_1 \cos \psi_1 & \cdots & -\sum_{j=1}^{N} m_{Nj} \alpha_{jN} \omega_N \cos \psi_N
\end{bmatrix},
\end{align*}
\]

\[
\mathbf{u}_1 = \begin{bmatrix}
\frac{N}{k=1} \frac{d\alpha_{1k}}{dt} a_k \cos \psi_k \\
\vdots \\
\frac{N}{k=1} \frac{d\alpha_{Nk}}{dt} a_k \cos \psi_k
\end{bmatrix},
\]

\[
\mathbf{u}_2 = \begin{bmatrix}
\frac{N}{k=1} \frac{d\alpha_{1k}}{dt} a_k \sin \psi_k \\
\vdots \\
\frac{N}{k=1} \frac{d\alpha_{Nk}}{dt} a_k \sin \psi_k
\end{bmatrix}.
\]
The solution of equation (20) yields the $2N$ quantities, $\frac{da_k}{dt}$ and $\frac{d\theta_k}{dt}$, representing the rates of change of amplitudes and phase, respectively. In general, for systems with a large number of degrees of freedom, it may be quite cumbersome to obtain explicit expressions for these quantities. However, these may be represented in functional forms as

$$\frac{da_k}{dt} = F_k(a_1, a_2, \ldots, a_N, \psi_1, \psi_2, \ldots, \psi_N, e, \tau),$$

$$\frac{d\theta_k}{dt} = G_k(a_1, a_2, \ldots, a_N, \psi_1, \psi_2, \ldots, \psi_N, e, \tau), \quad k = 1, 2, \ldots, N. \tag{21a}$$

Following the Bogoliubov–Mitropolsky method [11, 16], one can average the right hand sides of equations (21a) and (21b) over a period of $2\pi$ to yield

$$\frac{da_k}{dt} = eF_k^*(a_1, a_2, \ldots, a_N, \tau),$$

$$\frac{d\theta_k}{dt} = eG_k^*(a_1, a_2, \ldots, a_N, \tau), \quad k = 1, 2, \ldots, N, \tag{22a}$$

where

$$F_k^* = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} F_k(\psi_1, \psi_2, \ldots, \psi_N, e, \tau) \, d\psi_1 \, d\psi_2 \cdots \, d\psi_N, \tag{23a}$$

$$G_k^* = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} G_k(\psi_1, \psi_2, \ldots, \psi_N, e, \tau), \quad k = 1, 2, \ldots, N. \tag{23b}$$

Equation (22a) represents a set of $N$ first order non-linear non-autonomous differential equations in $a_k$ and, in general, a closed form solution for such a system is not possible. However, under certain special circumstances, significant simplifications may be possible. Nevertheless, a simple numerical scheme can always be employed to integrate the amplitude equations. Once the $a_k$ are known, they can be substituted in the phase equation (22b) and the $\theta_k$ can be determined by a single quadrature. Since the $\alpha_{jk}$ are known from equation (16), the solution for $y_j$ can then be readily obtained from equations (12), (13) and (5). Finally, the total solution $q_j$ can be found from equation (3).

In the following sections, a system with two degrees of freedom is analyzed in detail. The amplitude and phase equations are derived in explicit forms and it is shown that closed form solutions can be obtained in certain special cases. An example is considered for illustration.

### 3. SYSTEMS WITH TWO DEGREES OF FREEDOM

For a system with two degrees of freedom, equation (1) reduces to the form

$$\frac{d}{dt} \left[ m_{11}(\tau) \frac{dq_1}{dt} \right] + \frac{d}{dt} \left[ m_{12}(\tau) \frac{dq_2}{dt} \right] + b_{11}(\tau)q_1 + b_{12}(\tau)q_2 + e f_1 \left( q_1, q_2, \frac{dq_1}{dt}, \frac{dq_2}{dt} \right) = g_1(\tau), \tag{24}$$

$$\frac{d}{dt} \left[ m_{21}(\tau) \frac{dq_1}{dt} \right] + \frac{d}{dt} \left[ m_{22}(\tau) \frac{dq_2}{dt} \right] + b_{21}(\tau)q_1 + b_{22}(\tau)q_2 + e f_2 \left( q_1, q_2, \frac{dq_1}{dt}, \frac{dq_2}{dt} \right) = g_2(\tau). \tag{25}$$
The initial conditions are
\[ q_1 = q_2 = dq_1/dt = dq_2/dt = 0 \quad \text{at} \quad t = 0. \]  \hfill (26)

The solutions for \( p_1 \) and \( p_2 \) are obtained from equation (9) as
\[ p_1(\tau) = \left[ g_1(\tau)b_{22}(\tau) - g_2(\tau)b_{12}(\tau) \right] / \left[ b_{11}(\tau)b_{22}(\tau) - b_{12}(\tau)b_{21}(\tau) \right], \]  \hfill (27)
\[ p_2(\tau) = \left[ g_2(\tau)b_{11}(\tau) - g_1(\tau)b_{21}(\tau) \right] / \left[ b_{11}(\tau)b_{22}(\tau) - b_{12}(\tau)b_{21}(\tau) \right]. \]  \hfill (28)

Note that \( b_{ij}(\tau) \) is positive definite by assumption and therefore \( p_1(\tau) \) and \( p_2(\tau) \) remain finite for all \( \tau \).

The resulting equations for \( y_1 \) and \( y_2 \) can be obtained from equation (10) with \( i,j = 1, 2 \), and the initial conditions are given by equation (5) as
\[ y_1(0) = -p_1(0), \quad y_2(0) = -p_2(0), \]  \hfill (29)
\[ dy_1(0)/dt = -dp_1(0)/dt, \quad dy_2(0)/dt = -dp_2(0)/dt. \]  \hfill (30)

As pointed out in section 2, \( \alpha_{11} \) and \( \alpha_{12} \) may be set to unity in equation (12) and the solutions can be assumed in the form
\[ y_1 = a_1 \cos \psi_1 + a_2 \cos \psi_2, \]  \hfill (31)
\[ y_2 = \alpha_{21}(\tau)a_1 \cos \psi_1 + \alpha_{22}(\tau)a_2 \cos \psi_2, \]  \hfill (32)

where
\[ \omega_1(\tau) = d\psi_1/dt = \omega_1(\tau) + d\theta_1/dt, \quad \omega_2(\tau) = d\psi_2/dt = \omega_2(\tau) + d\theta_2/dt. \]  \hfill (33)

Following the procedure outlined in section 2, one can obtain the amplitude and phase equations through an application of Cramer's rule, and, after some calculations, these can be put in the form of equation (22) as
\[ \frac{da_1}{d\tau} = \frac{\epsilon}{\Delta(\tau)} \left[ a_1 \frac{d\alpha_{21}(\tau)}{d\tau} - \frac{a_1 M_1(\tau)}{2\omega_1(\tau)M(\tau)} \left[ \frac{d[\omega_1(\tau)m_{11}(\tau)]}{d\tau} + \frac{\alpha_{21}(\tau)}{a_1} \frac{d[\omega_1(\tau)m_{12}(\tau)]}{d\tau} \right] f_{11}^* \right], \]  \hfill (34)
\[ \frac{da_2}{d\tau} = \frac{\epsilon}{\Delta(\tau)} \left[ -a_2 \frac{d\alpha_{22}(\tau)}{d\tau} - \frac{a_2 M_2(\tau)}{2\omega_2(\tau)M(\tau)} \left[ \frac{d[\omega_2(\tau)m_{11}(\tau)]}{d\tau} + \frac{\alpha_{22}(\tau)}{a_2} \frac{d[\omega_2(\tau)m_{12}(\tau)]}{d\tau} \right] f_{12}^* \right], \]  \hfill (35)
\[ \frac{d\theta_1}{d\tau} = \frac{\epsilon}{2a_1\Delta(\tau)\omega_1(\tau)M(\tau)} \left[ M_1(\tau)f_{11}^{**} - M_2(\tau)f_{21}^{**} \right] = -\omega_1(\tau) + \frac{d\psi_1}{d\tau}, \]  \hfill (36)
\[ \frac{d\theta_2}{d\tau} = \frac{\epsilon}{2a_2\Delta(\tau)\omega_2(\tau)M(\tau)} \left[ M_3(\tau)f_{22}^{**} - M_4(\tau)f_{12}^{**} \right] = -\omega_2(\tau) + \frac{d\psi_2}{d\tau}, \]  \hfill (37)

where
\[ \Delta(\tau) = [\alpha_{22}(\tau) - \alpha_{21}(\tau)], \quad M(\tau) = [m_{11}(\tau)m_{22}(\tau) - m_{12}(\tau)m_{21}(\tau)], \]  \hfill (38)
\[ M_1(\tau) = [m_{21}(\tau) + \alpha_{22}(\tau)m_{22}(\tau)], \quad M_2(\tau) = [m_{11}(\tau) + \alpha_{22}(\tau)m_{12}(\tau)], \]  \hfill (39)
\[ M_3(\tau) = [m_{11}(\tau) + \alpha_{21}(\tau)m_{12}(\tau)], \quad M_4(\tau) = [m_{21}(\tau) + \alpha_{21}(\tau)m_{22}(\tau)], \]  \hfill (40)
$$f_{ij}^* = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_i(a_1, a_2, \psi_1, \psi_2, \tau) \sin \psi_j \, d\psi_1 \, d\psi_2, \quad (43)$$

$$f_{ij}^{**} = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_i(a_1, a_2, \psi_1, \psi_2, \tau) \cos \psi_j \, d\psi_1 \, d\psi_2, \quad i, j = 1, 2. \quad (44)$$

Note that if $f_i$ ($i = 1, 2$) are functions of $q_j$ ($j = 1, 2$) only, then $f_{ij}^*$ are identically zero and equations (33)-(36) can be integrated by a single quadrature. On the other hand, if the $f_i$ do not depend on $q_j$, then $f_{ij}^{**}$ vanish and the total phase $\psi_k$ ($k = 1, 2$) can be obtained from

$$\psi_k = \int_0^t \omega_k(\tau) \, d\tau, \quad k = 1, 2. \quad (45)$$

The initial conditions on $a_1(t), a_2(t), \psi_1(t)$ and $\psi_2(t)$ can be easily obtained from equations (27)-(31) as

$$a_1(0) = \frac{1}{\Delta(0)} \left[ \left[ p_2(0) - \alpha_{22}(0)p_1(0) \right]^2 + \omega_1^2(0) \left[ \frac{dp_2(0)}{dt} - \alpha_{22}(0) \frac{dp_1(0)}{dt} \right]^2 \right]^{1/2}, \quad (46)$$

$$a_2(0) = \frac{1}{\Delta(0)} \left[ \left[ p_2(0) - \alpha_{21}(0)p_1(0) \right]^2 + \omega_2^2(0) \left[ \frac{dp_2(0)}{dt} - \alpha_{21}(0) \frac{dp_1(0)}{dt} \right]^2 \right]^{1/2}, \quad (47)$$

$$\psi_1(0) = \cos^{-1} \left\{ \frac{\left[ p_2(0) - \alpha_{22}(0)p_1(0) \right]}{\Delta(0)} a_1(0) \right\}, \quad (48)$$

$$\psi_2(0) = \cos^{-1} \left\{ -\frac{\left[ p_2(0) - \alpha_{21}(0)p_1(0) \right]}{\Delta(0)} a_2(0) \right\}. \quad (49)$$

In the following, an important case of equations (24) and (25) is considered for which considerable simplification occurs.

3.1. DYNAMICALLY DECOUPLED PROPORTIONAL SYSTEMS

Let us assume that $m_{12}(\tau) = m_{21}(\tau) = 0$, $m_{22}(\tau)$ is proportional to $m_{11}(\tau)$, and $b_{22}(\tau)$, $b_{12}(\tau)$ and $b_{21}(\tau)$ are proportional to $b_{11}(\tau)$. Then it can be easily shown that $\omega_2(\tau)$ is proportional to $\omega_1(\tau)$ and $\alpha_{22}(\tau)$ and $\alpha_{21}(\tau)$ are constants: i.e., independent of $\tau$. Under these conditions, equations (33)-(36) simplify to

$$\frac{da_1}{dt} = \frac{\varepsilon}{2\omega_1(\tau)m(\tau)} \left\{ -a_1 \frac{d[\omega_1(\tau)m(\tau)]}{d\tau} + \frac{1}{\Delta} \left[ \alpha_{22} f_{11}^{**} - \frac{1}{\gamma} f_{21}^{**} \right] \right\}, \quad (50)$$

$$\frac{da_2}{dt} = \frac{\varepsilon}{2\omega_2(\tau)m(\tau)} \left\{ -a_2 \frac{d[\omega_2(\tau)m(\tau)]}{d\tau} + \frac{1}{\Delta} \left[ \frac{1}{\gamma} f_{22}^{**} - \alpha_{21} f_{12}^{**} \right] \right\}, \quad (51)$$

$$\frac{d\psi_1}{dt} = \omega_1(\tau) + \frac{\varepsilon}{2a_1\Delta \omega_1(\tau)m(\tau)} \left[ \alpha_{22} f_{11}^{**} - \frac{1}{\gamma} f_{21}^{**} \right], \quad (52)$$

$$\frac{d\psi_2}{dt} = \omega_2(\tau) + \frac{\varepsilon}{2a_2\Delta \omega_2(\tau)m(\tau)} \left[ \frac{1}{\gamma} f_{22}^{**} - \alpha_{21} f_{12}^{**} \right], \quad (53)$$

where

$$m_{22}(\tau) = \gamma m_{11}(\tau) = \gamma m(\tau). \quad (54)$$

Similarly, other special cases which lead to considerable simplification can also be considered.
4. AN EXAMPLE

For illustration, consider the system shown in Figure 1. The restoring forces due to \(b(\tau)\) and \(k\) are assumed to be linear and cubic functions of the relative displacement, respectively. The equations of motion can then be written as

\[
\frac{d}{d\tau} \left[ m(\tau) \frac{dq_1}{d\tau} \right] + 2b(\tau)q_1 - b(\tau)q_2 + \varepsilon f_1(q_1, q_2) = g_1(\tau),
\]

\[
\frac{d}{d\tau} \left[ m(\tau) \frac{dq_2}{d\tau} \right] - b(\tau)q_1 + 2b(\tau)q_2 + \varepsilon f_2(q_1, q_2) = g_2(\tau),
\]

where

\[
\varepsilon f_1(q_1, q_2) = k[q_1^3 + (q_1 - q_2)^3] + c \frac{dq_1}{d\tau}, \quad \varepsilon f_2(q_1, q_2) = k(q_2 - q_1)^3.
\]

First, \(p_1(\tau)\) and \(p_2(\tau)\) are found from equations (27) and (28) as

\[
p_1(\tau) = \frac{2g_1(\tau) + g_2(\tau)}{3b(\tau)}, \quad p_2(\tau) = \frac{2g_2(\tau) + g_1(\tau)}{3b(\tau)}.
\]

The frequency equation

\[
det \begin{bmatrix}
2b(\tau) - m(\tau)\omega^2(\tau) & -b(\tau) \\
-b(\tau) & 2b(\tau) - m(\tau)\omega^2(\tau)
\end{bmatrix} = 0
\]

yields

\[
\omega_1^2(\tau) = \frac{b(\tau)}{m(\tau)}, \quad \omega_2^2(\tau) = \frac{3b(\tau)}{m(\tau)}.
\]

Further, equation (16) can be solved to obtain

\[
\alpha_{21} = 1, \quad \alpha_{22} = -1, \quad \Delta = -2.
\]

Therefore, the solutions for \(y_1\) and \(y_2\) can be written as

\[
y_1 = a_1 \cos \psi_1 + a_2 \cos \psi_2, \quad y_2 = a_1 \cos \psi_1 - a_2 \cos \psi_2.
\]

With these expressions for \(y_1\) and \(y_2\) in equations (57) and (58), the integrals in equations (43) and (44) can be evaluated to yield

\[
\varepsilon f_{11}^* = -ca_1\omega_1(\tau), \quad f_{12}^* = -ca_2\omega_2(\tau), \quad f_{21}^* = f_{22}^* = 0,
\]

\[
\varepsilon f_{11}^{**} = k[\frac{3}{2}a_1^3 + \frac{3}{2}a_1a_2^2 + 3p_1^2a_1], \quad f_{21}^{**} = 0,
\]

\[
\varepsilon f_{12}^{**} = k[\frac{3}{2}a_2^3 + \frac{3}{2}a_1^2a_2 + 3p_1^2a_2 + 6a_2^2 + 6a_2(p_1 - p_2)^2],
\]

\[
\varepsilon f_{22}^{**} = -k[6a_2^2 + 6a_2(p_1 - p_2)^2],
\]

where \(p_1(\tau)\) and \(p_2(\tau)\) are known from equation (59). Then from equations (50) and (51), it can be shown that

\[
a_j(t) = a_j(0) \left[ \frac{\omega_j(0)m(0)}{\omega_j(\tau)m(\tau)} \right]^{1/2} \exp \left[ -\frac{c}{4} \int_0^t \frac{dt}{m(\tau)} \right], \quad j = 1, 2.
\]
With $\gamma = 1$, the phase equations (52) and (53) take the forms

$$\frac{d\psi_1}{dt} = \omega_1(\tau) + \frac{k}{4 \omega_1(\tau) m(\tau)} \left[ \frac{1}{2} a_1^2 + \frac{3}{2} a_2^2 + 3 p_1^2 \right],$$

(66)

$$\frac{d\psi_2}{dt} = \omega_2(\tau) + \frac{k}{4 \omega_2(\tau) m(\tau)} \left[ \frac{1}{2} a_1^2 + \frac{3}{2} a_2^2 + 3 p_1^2 + 12 (p_1 - p_2)^2 \right].$$

(67)

Since $p_1$, $p_2$, $a_1$ and $a_2$ are known functions of $t$, equations (66) and (67) can be integrated by a single quadrature. Sometimes it may be possible to obtain closed form solutions for the phase equations as well. The initial conditions for $\psi_j$ are obtained from equations (46)-(49).

For numerical considerations, it is assumed that $m(\tau)$ and $b(\tau)$ vary linearly with $\tau$ and the system is excited by step functions: i.e.,

$$m(\tau) = m_0 + m_1 \tau, \quad b(\tau) = b_0 + b_1(\tau),$$

(68)

$$g_1(\tau) = P u(\tau), \quad g_2(\tau) = \delta P u(\tau),$$

(69)

where $u(\tau)$ is defined as

$$u(\tau) = \begin{cases} 0, & \tau \leq 0^- \\ 1, & \tau \geq 0^+ \end{cases}.$$  

(70)

Figure 2. Response for the system \((d/d\tau)[(1 + 0.2 \tau) dq_1/d\tau] + (6 + 0.2 \tau) q_1 - (3 + 0.1 \tau) q_2 + \epsilon dq_1/d\tau + \epsilon (q_1^2 - q_1 q_2)] = 4u(\tau), \quad (d/d\tau)[(1 + 0.2 \tau) dq_2/d\tau] - (3 + 0.1 \tau) q_1 + (6 + 0.2 \tau) q_2 + \epsilon (q_2 - q_1)]^2 = u(\tau).\) Present analysis; -- -- --, Runge-Kutta solution; $\epsilon = 0.1$. (a) Displacement $q_1$; (b) displacement $q_2$. 
\( a_j(t) = a_j(0) \left[ \frac{m_0}{m_0 + m_1 t} \right]^{1/4(1+c/m_0)} \left[ \frac{b_0}{b_0 + b_1 t} \right]^{1/4}, \quad j = 1, 2. \) (71)

From equation (59), one obtains
\[
\begin{align*}
    p_1(\tau) &= P(2 + \delta) u(\tau)/3(b_0 + b_1 \tau), \\
    p_2(\tau) &= P(2\delta + 1) u(\tau)/3(b_0 + b_1 \tau).
\end{align*}
\] (72)

For such a system, the expressions for \( \psi_1 \) and \( \psi_2 \) can also be obtained in closed forms. However, they are rather lengthy and are therefore omitted. The initial conditions for equations (65)-(67) are computed from equations (46)-(49) to yield
\[
\begin{align*}
    a_1(0) &= P(\delta + 1)/2b_0, \\
    a_2(0) &= P(\delta - 1)/6b_0, \\
    \psi_1(0) &= \pi, \\
    \psi_2(0) &= 0. \quad (73)
\end{align*}
\]

Figures 2 and 3 show the responses of the system for some typical values of parameters. Numerical results obtained from a Runge-Kutta scheme are also presented for a comparison. Figures 2(a) and (b) show the response of a system for which \( m_1 \) and \( b_1 \) are positive and thus \( m(\tau) \) as well as \( b(\tau) \) increase with time. The response is significantly modified if \( m_1 \) becomes negative although \( b_1 \) remains positive. The results for negative \( m_1 \) are shown in Figures 3(a) and (b). In both cases the mass changes up to 40% during the time interval considered.

Figure 3. Response for the system
\[
\begin{align*}
    (d/dt)[(1-0.2\tau) dq_1/dt] + (6+0.2\tau) q_1 - (3+0.1\tau) q_2 + \epsilon dq_1/dt + e[q_1^3 + (q_1-q_2)^3] &= 4u(\tau), \\
    (d/dt)[(1-0.2\tau) dq_2/dt] - (3+0.1\tau) q_1 + (6+0.2\tau) q_2 + \epsilon(q_2-q_1)^3 &= u(\tau).
\end{align*}
\]
Present analysis; ---, Runge-Kutta solution; \( \epsilon = 0.1 \). (a) Displacement \( q_1 \), (b) displacement \( q_2 \).
5. DISCUSSION AND CONCLUSIONS

An approximate analysis of non-linear, non-stationary vibrational system with multiple degrees of freedom subjected to an arbitrary pulse excitation is presented. It has been shown that the well known Bogoliubov-Mitropolsky technique can be successfully employed for such problems if the non-linearity is small and the system parameters vary slowly with time. In general, the method yields a set of $N$ non-linear non-autonomous first order differential equations for the amplitudes which could be quite complicated. Thus, for large systems, it may not be possible to integrate them in closed forms. Nevertheless, once the amplitudes are known, each of the remaining $N$ equations for the phase can be integrated by a single quadrature. As shown in sections 3 and 4, considerable simplification occurs for certain systems with two degrees of freedom and it is possible to express the amplitudes and phase in closed forms.

Although the analysis has been restricted to the first order approximation only, higher order approximate solutions can always be generated if desired. However, as shown by Olberding and Sinha [18] for the case of a single degree of freedom, this may involve some tedious algebraic manipulations and there may not be any significant improvement in the accuracy. The first order analysis is simple and may prove to be useful in many practical situations arising in engineering design. It should be pointed out that the present analysis also provides information which can be used to investigate the stability of the system. The stability behavior can be predicted through a discussion of the amplitude equations.

REFERENCES


