

Periodic Flap Control of a Helicopter Blade in Forward Flight

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Abstract: The primary objective of this paper is to present new periodic control strategies for the control of flapping motion of an individual helicopter rotor blade in forward flight, which is represented by a differential equation with periodic coefficients. First, an algebraic procedure based on Chebyshev polynomial expansion is employed to control the periodic flapping motion. In this approach, the state vector and the elements of the periodic system matrix have been expanded in terms of shifted Chebyshev polynomials of the first kind over the principal period. Later, optimal control theory in conjunction with Floquet theory has been used to design full-state and observer-state feedback controllers. In the second method, the feedback controllers have been designed in the time-invariant domain through the application of a linear, invertible, periodic transformation known as Liapunov-Floquet (L-F) transformation to the periodic system model. It is shown that by using both methods the periodic control gains can be obtained as explicit functions of time and, therefore, a control scheme more suitable for real-time implementation can be achieved. The design procedures have been found to be much simpler in character when compared to those techniques that have appeared in the literature.

Key Words: Helicopter rotor blade, flap control, Liapunov-Floquet transformation, Chebyshev polynomials

1. INTRODUCTION

It is well-known that the equations describing the dynamics of helicopter rotor blades in forward flight contain coefficients that are periodic functions of blade azimuth angle. These equations are obtained as a result of force and moment equilibrium, and the control equation is obtained as a result of the moment trim about an equilibrium position of the helicopter. Typically, the feedback controllers for controlling helicopters have been obtained by approximating the periodic equations of motion by a set of linear ordinary differential equations with constant coefficients (Straub and Warmbrodt, 1985; Straub, 1987). This implicitly means that the advance ratio of the helicopter is assumed to be very small; however, the typical values of the advance ratio for present-day helicopters are in the order of 0.3-0.5, which is not considered to be small. Therefore, such design methods cannot adequately control helicopters that fly with high advance ratio. Another viable approach has been to use the well-known Floquet theory in the design of active control of periodic systems (Calico and Wiesel, 1984; Webb, Calico, and Wiesel, 1991; McKillip, 1985; Knospe and Haviland, 1992). Calico and Wiesel (1984) and Webb, Calico, and Wiesel (1991) discuss a modal control technique suitable for periodic systems that modify the system Poincaré exponents either one

by one (a scalar approach) or by a pair (a vector approach). However, these approaches run into difficulties when one tries to modify all the unstable roots to stable one in one attempt. It is claimed that the process of changing a single or pair of unstable roots to stable one has to be obtained in a sequence one after the other to obtain the final controlled system. Obviously, this method appears to be time consuming and further reported to have numerical instabilities when the modal control is used more than once. McKillip (1985) presented an implicit-model-following controller design for periodic systems using Floquet theory. However, in this technique the solution to the periodic Riccati equation has to be obtained numerically, which is a very exhaustive process by itself. Yet, another procedure based on pulse response method (Knospe and Haviland, 1992) has also been reported for vibration control of helicopters. In this approach, a discrete model of a periodic system has been considered and an open-loop control for each cycle is calculated using the plant and uncontrolled vibration state estimates. This method has a severe drawback in that it initiates the rigid-body motion of the helicopter, which opposes the pilot's control. Also, it is pointed out that the algorithm exhibits a regular instability and hence it is not a useful tool. Last, Calise, Wasikowski, and Schrage (1992) analyzed an output feedback control approach that is time-invariant in nature. In their study, the Liapunov-Floquet theory was used to transform the original periodic system matrix into an equivalent time-invariant system matrix; however, due to the presence of the input matrix in the control problem, the system continues to be periodic even after the transformation. At this stage, they neglect the time-varying periodic terms in the input matrix, keeping the constant terms only, and use theory of constant gain output feedback to decide upon the controller designs. Later, to account for the neglected time-varying terms, they incorporate iterative processes to improve on their controller gains. Therefore, their control design procedure has to be achieved in several stages that are tedious and cumbersome.

Over the years, though there has been a persistent research effort in getting the best active control algorithm for helicopter rotor blades, it can be said that this exercise still eludes the researchers because of the periodically time-varying nature of the system. In this paper, we address procedures for active control of helicopters by using two different methods based on Floquet theory. In the first method, the states and the elements of the system matrix of the augmented equation of the problem were expanded in terms of the shifted Chebyshev polynomials of the first kind and the State Transition Matrix (STM) was obtained as a result of solving a set of linear algebraic equations. Later, the periodic control gains were computed in terms of the adjoint variables to minimize a given performance index. In the second approach, the Liapunov-Floquet (L-F) transformation pertaining to the given control problem was evaluated as a result of the Chebyshev expansion procedure described in the first approach. After applying the transformation to the periodic control problem, a dynamically similar system whose homogeneous part is time invariant was obtained; however, it should be noted that the problem still continued to be a periodic one due to the presence of the input matrix of the control equation. At this stage, an auxiliary system was constructed with the same system matrix as that of the transformed one, which was time invariant. Also, the auxiliary system included an arbitrary time-invariant input matrix. Later, the control design was performed in such a way that the error response between the auxiliary system and the transformed one was minimized using a linear least squares procedure. By using these two approaches, it is shown that the periodic control gains can be obtained as explicit functions of time, thereby making these schemes suitable for real-time implementations. The algorithms have been implemented to control the helicopter individual blade flapping motion about a

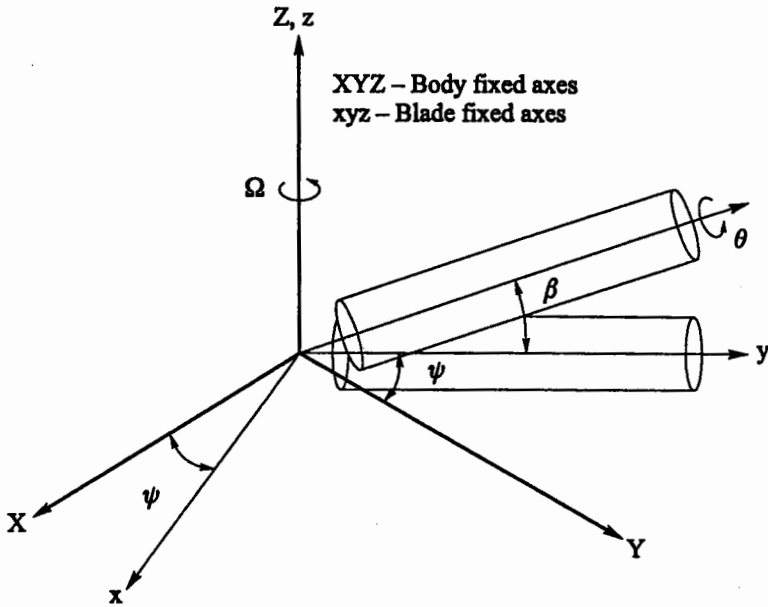


Figure 1. Geometry of a helicopter rotor blade.

moment trim condition. It can be seen that the algorithms are simple, behave well, and do not show numerical instabilities. Specifically, the procedures do not have multistage design processes and therefore no iterations involved. A more elaborate control of a full flap-lag motion of the individual helicopter rotor blade will be the topic of discussion of our future paper.

2. ROTOR BLADE MODEL

The equations of motion for a centrally hinged, spring-restrained, rigid, single helicopter rotor blade (c.f. Figure 1) in forward flight can be written as (Johnson, 1980; Papavassiliou, Friedmann, and Venkatesan, 1994)

$$\begin{Bmatrix} \beta' \\ \beta'' \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ A_{21}(\psi) & A_{22}(\psi) \end{bmatrix} \begin{Bmatrix} \beta \\ \beta' \end{Bmatrix} + \begin{bmatrix} 0 \\ b_{21}(\psi) \end{bmatrix} \theta + \begin{bmatrix} 0 \\ f(\psi) \end{bmatrix}, \quad (1)$$

where β is the flap angle and θ is the pitch angle of the rotor blade. In the above equation, prime refers to differentiation with respect to ψ . The elements of the various matrices of equation (1) are

$$A_{21}(\psi) = - \left\{ 1 + \omega_F^2 + \gamma \left(\frac{\mu}{6} \cos \psi + \frac{\mu^2}{8} \sin 2\psi \right) \right\}$$

$$A_{22}(\psi) = -\gamma \left(\frac{1}{8} + \frac{\mu}{6} \sin \psi \right)$$

$$b_{21}(\psi) = \gamma \left(\frac{1}{8} + \frac{\mu}{3} \sin \psi + \frac{\mu^2}{4} \sin^2 \psi \right)$$

$$f(\psi) = -\lambda \left(\frac{1}{6} + \frac{\mu}{4} \sin \psi \right).$$

The various parameters in the above elements are defined in the following way ω_F is the nonrotating flap frequency, γ is the lock number defined as the ratio of the aerodynamic to inertia forces of the blade, $\psi = \Omega\tau$ is the blade azimuth angle, μ is the rotor advance ratio defined as the ratio of the forward speed to the rotor tip speed, Ω is the rotor speed in rad/s, and λ is the total inflow. Equation (1) is derived by enforcing a moment equilibrium condition at the root of the blade with simplifying assumptions such as the inflow is uniform and the effects of noncirculatory lift and reverse flow are negligible. The corresponding control equation in forward flight can be obtained by using a moment trim procedure as shown by Wu and Sinha (1994). Let β_e and θ_e be the equilibrium states of the flap and pitch for a given moment trim condition, respectively. The form of β_e and θ_e are described by the truncated Fourier series as follows:

$$\beta_e(\psi) = \beta_0 + \beta_c \cos \psi + \beta_s \sin \psi$$

$$\theta_e(\psi) = \theta_0 + \theta_c \cos \psi + \theta_s \sin \psi + \theta_\beta \beta_e,$$

where β_0 is the coning angle independent of ψ ; β_c and β_s generate once-per-revolution variations of the flap angle; θ_0 is the collective pitch angle; θ_c and θ_s are cyclic pitch angles; and θ_β is pitch-flap coupling parameter. When these states are perturbed by $\tilde{\beta}(\psi)$ and $\tilde{\theta}(\psi)$, respectively, the perturbed states can be written as

$$\beta(\psi) = \beta_e + \tilde{\beta}(\psi)$$

$$\theta(\psi) = \theta_e + \theta_\beta \tilde{\beta}(\psi) + \tilde{\theta}(\psi).$$

Therefore, the perturbed equations of motion can be derived as

$$\begin{Bmatrix} \tilde{\beta}' \\ \tilde{\beta}'' \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ A_{21}(\psi) + \theta_\beta b_{21}(\psi) & A_{22}(\psi) \end{bmatrix} \begin{Bmatrix} \tilde{\beta} \\ \tilde{\beta}' \end{Bmatrix} + \begin{bmatrix} 0 \\ b_{21}(\psi) \end{bmatrix} \tilde{\theta}. \quad (2)$$

Since for an articulated rotor, θ_β is zero, the perturbed equation (2) can be simplified to equation (1) except that β is replaced by $\tilde{\beta}$. Normalizing with $\psi = \Omega\tau$ and subsequently with $\Omega\tau = 2\pi t$, equation (1) can be rewritten as

$$\begin{Bmatrix} \dot{\bar{\beta}} \\ \ddot{\bar{\beta}} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ \bar{A}_{21}(t) & \bar{A}_{22}(t) \end{bmatrix} \begin{Bmatrix} \bar{\beta} \\ \dot{\bar{\beta}} \end{Bmatrix} + \begin{bmatrix} 0 \\ \bar{b}_{21}(t) \end{bmatrix} \bar{\theta}, \quad (3)$$

where the elements $\bar{A}_{21}(t)$, $\bar{b}_{21}(t)$, and $\bar{A}_{22}(t)$ are equal to

$$\bar{A}_{21}(t) = -(2\pi)^2 \left\{ 1 + \omega_F^2 + \gamma \left(\frac{\mu}{6} \cos 2\pi t + \frac{\mu^2}{8} \sin 4\pi t \right) \right\}$$

$$\bar{A}_{22}(t) = -2\pi\gamma \left(\frac{1}{8} + \frac{\mu}{6} \sin 2\pi t \right)$$

$$\bar{b}_{21}(t) = (2\pi)^2\gamma \left(\frac{1}{8} + \frac{\mu}{3} \sin 2\pi t + \frac{\mu^2}{4} \sin^2 2\pi t \right).$$

The over dot in the above equation refers to differentiation with respect to time 't'. For simplicity, the above equation can be written in a standard matrix form along with the output equation as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \quad ; \quad \mathbf{A}(t) = \mathbf{A}(t+T) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t), \end{aligned} \quad (4)$$

where $\mathbf{x} = [\bar{\beta}, \dot{\bar{\beta}}]^T$, $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are simply system and input matrices of equation (3), $\mathbf{u}(t) = \bar{\theta}(t)$, \mathbf{y} is assumed to be $\bar{\beta}$, and therefore $\mathbf{C}(t) = [1, 0]$. In the following, the control design procedures suitable for a general $n \times n$ periodic system are described.

3. CONTROL VIA CHEBYSHEV EXPANSION

Consider a linear quadratic regulator (LQR) problem to minimize the performance index

$$\mathbf{J} = \frac{1}{2} \mathbf{x}^T(t_f) \mathbf{S}(t_f) \mathbf{x}(t_f) + \frac{1}{2} \int_0^{t_f} [\mathbf{x}^T(t) \mathbf{P}(t) \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{P}_1(t) \mathbf{u}(t)] dt \quad (5)$$

subject to satisfying the differential equation (4). $\mathbf{S}(t_f)$ and $\mathbf{P}(t)$ are symmetric positive semi-definite matrices, $\mathbf{P}_1(t)$ is a symmetric positive definite matrix, and t_f is the time at which the final state constraints are to be met. Following the standard development of optimal control theory given by Kwakernaak and Sivan (1972), the control vector to minimize the system Hamiltonian

$$\mathbf{H}^* = \frac{1}{2} [\mathbf{x}^T(t) \mathbf{P}(t) \mathbf{x}(t) + \mathbf{u}^T(t) \mathbf{P}_1(t) \mathbf{u}(t)] + \mathbf{p}^T(t) [\mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)] \quad (6)$$

can be obtained as

$$\mathbf{u}(t) = -\mathbf{P}_1^{-1}(t)\mathbf{B}^T(t)\mathbf{p}(t), \quad (7)$$

where the adjoint variable $\mathbf{p}(t)$ and the state variable $\mathbf{x}(t)$ satisfy the augmented equation

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{p}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A}(t) & \mathbf{L}(t) \\ -\mathbf{P}(t) & -\mathbf{A}^T(t) \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{Bmatrix}, \quad (8)$$

where $\mathbf{L}(t) = -\mathbf{B}(t)\mathbf{P}_1^{-1}(t)\mathbf{B}^T(t)$. Equation (8) represents a $2n \times 2n$ periodic differential system, and adapting the technique developed by Joseph, Pandiyan, and Sinha (1993), the system can be completely converted into an algebraic one by expanding the unknown state and adjoint vectors and the known elements of the system matrix of equation (8) in terms of Chebyshev polynomials. The solution for the algebraic equations provides the State Transition Matrix (STM), $\Phi(t, 0)$, of the system. The questions of accuracy and efficiency of such a technique in computing the STM have been addressed, depicting the stability and response characteristics of the periodic systems in various articles by Sinha and Wu (1991); Pandiyan, Bibb, and Sinha (1993); and Joseph, Pandiyan, and Sinha (1993). To cite some important results, the number of terms in the Chebyshev expansion of the blade model is just about 20, which is found to be adequate while computing the largest characteristic exponent with 5-decimal accuracy. Moreover, the computational efficiency of such techniques increases significantly when the systems become larger and larger as shown by Joseph, Pandiyan, and Sinha (1993). This way, $\Phi(t, 0)$, the STM of equation (8), can be obtained in closed form as explicit functions of Chebyshev polynomials in time. Hence, the adjoint variable $\mathbf{p}(t)$ can be obtained as a function of partitioned matrices $\Phi_{ij}(t, 0)$'s, $i = 1, 2$ and $j = 1, 2$ of the STM, $\Phi(t, 0)$ (c.f. Kwakernaak and Sivan, 1972), and a symmetric positive definite matrix $\mathbf{S}(t_f)$ as given below.

$$\mathbf{p}(t) = (\Phi_{22} - \mathbf{S}(t_f)\Phi_{12})^{-1}(\mathbf{S}(t_f)\Phi_{11} - \Phi_{21})\mathbf{x}(t) = \mathbf{K}(t)\mathbf{x}(t). \quad (9)$$

Combining equations (7) and (9), the periodic gains can be obtained in an analytical form as

$$\mathbf{u}(t) = -\mathbf{P}_1^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) \quad (10)$$

Substituting equation (10) in equation (4), one can obtain the closed-loop system equation as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) - \mathbf{B}(t)\mathbf{P}_1^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\mathbf{x}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t). \end{aligned} \quad (11)$$

To obtain output feedback control design, the dual of equation (4) that represents the error dynamics between the standard observer equation and the system (4) can be considered. The gain matrix $\tilde{\mathbf{G}}(t)$ of the dual system should be selected such that the error vector

asymptotically goes to zero as time tends to infinity. The dual system of equation (4) is given by

$$\begin{aligned} \dot{\bar{z}}(t) &= \mathbf{A}^T(t)\bar{z}(t) + \mathbf{C}^T(t)\mathbf{u}(t) \\ \bar{y}(t) &= \mathbf{B}^T(t)\bar{z}(t). \end{aligned} \tag{12}$$

The STM of the above system can be computed following the procedure given by Joseph, Pandiyan, and Sinha (1993) using the Chebyshev expansion procedure, and therefore the periodic gain of the dual system can be computed as

$$\bar{\mathbf{p}}(t) = (\bar{\Phi}_{22} - \mathbf{S}(t_f)\bar{\Phi}_{12})^{-1}(\mathbf{S}(t_f)\bar{\Phi}_{11} - \bar{\Phi}_{21})\bar{z}(t) = \bar{\mathbf{G}}(t)\bar{z}(t), \tag{13}$$

where $\bar{\Phi}$ and $\bar{\mathbf{p}}$ are the STM and adjoint variable for the dual system, respectively. It should be noted that the observer gain $\mathbf{G}(t)$ is the transpose of the gain $\bar{\mathbf{G}}(t)$ and, therefore, the observer-based closed-loop system can be written as (Kwakernaak and Sivan, 1972)

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\bar{\mathbf{x}}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}(t) & \mathbf{M}(t) \\ \mathbf{G}(t)\mathbf{C}(t) & \mathbf{A}_r(t) \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{Bmatrix}, \tag{14}$$

where $\mathbf{A}_r(t) = \mathbf{A}(t) - \mathbf{G}(t)\mathbf{C}(t) - \mathbf{B}(t)\hat{\mathbf{K}}(t)$, $\mathbf{M}(t) = -\mathbf{B}(t)\hat{\mathbf{K}}(t)$, $\hat{\mathbf{K}}(t) = \mathbf{P}_1^{-1}\mathbf{B}^T(t)\mathbf{K}(t)$ and $\bar{\mathbf{x}}(t)$ are the observed states of the system (4).

4. CONTROL VIA L-F TRANSFORMATION

The STMs of periodic systems have been computed fairly well using the Chebyshev expansion procedure as explicit functions of time (Sinha and Wu, 1991; Joseph, Pandiyan, and Sinha, 1993). Therefore, applying Floquet theory, the STM can be factored into a product of a periodic matrix and an exponential matrix. The periodic matrix is known as the Liapunov-Floquet (L-F) transformation matrix and has been found to be useful in transforming a periodic system matrix into a time-invariant one, thereby allowing the system to be dealt with in the time-invariant domain. It has been shown that the L-F transformation matrix for any arbitrary periodic system can be computed using the Chebyshev expansion procedure and, therefore, it is possible to obtain a Fourier representation of the L-F transformation (Pandiyan, Bibb, and Sinha, 1993). Once the L-F transformation matrix $\mathbf{Q}(t)$ is computed and the transformation

$$\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t) \tag{15}$$

is applied, the control equation (4) can be converted to the following form

$$\dot{\mathbf{z}}(t) = \mathbf{R}\mathbf{z}(t) + \mathbf{Q}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t), \tag{16}$$

where \mathbf{R} is an $n \times n$ constant matrix. Following the approach of Sinha and Joseph (1994), linear control can be obtained in time-invariant domain by considering an auxiliary system of the form

$$\dot{\mathbf{z}}(t) = \mathbf{R}\bar{\mathbf{z}}(t) + \bar{\mathbf{B}}\mathbf{v}(t), \quad (17)$$

where $\bar{\mathbf{B}}$ is a constant matrix such that $(\mathbf{R}, \bar{\mathbf{B}})$ is a complete controllable pair. The control law for the time-invariant equation (17) can be written as

$$\mathbf{v}(t) = -\bar{\mathbf{K}}\bar{\mathbf{z}}(t). \quad (18)$$

The time-invariant gain matrix $\bar{\mathbf{K}}$ can be obtained by applying standard procedures such as pole-placement technique or optimal control theory. Defining the error between systems (16) and (17) as $e = \mathbf{z}(t) - \bar{\mathbf{z}}(t)$, the error dynamics can be written as

$$\dot{e}(t) = (\mathbf{R} - \bar{\mathbf{B}}\bar{\mathbf{K}})e(t) + \mathbf{Q}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) + \bar{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t). \quad (19)$$

It can be noted that since $(\mathbf{R} - \bar{\mathbf{B}}\bar{\mathbf{K}})$ is a stability matrix with eigenvalues having negative real parts, error equation (19) is an asymptotically stable system. However, the error e can be made zero as time tends to infinity only when the response due to forcing terms $\mathbf{Q}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) + \bar{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t)$ is negligible. In this way, the two systems described by equations (16) and (17) can be made equivalent only if $\mathbf{Q}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) + \bar{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t)$ is minimized to zero in the least squares sense. That is,

$$\mathbf{Q}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t) = -\bar{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t). \quad (20)$$

For this purpose, a weighted least squares estimation of \mathbf{u} can be computed as explained below. Let ϵ be the error vector from equation (20) defined by

$$\epsilon = -(\mathbf{B}(t)\mathbf{u}(t) + \mathbf{Q}(t)\bar{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t)). \quad (21)$$

Let $\hat{\mathbf{u}}(t)$ denote the best estimate of $\mathbf{u}(t)$ and be chosen in such a way that the weighted error defined by

$$\mathbf{J}' = \epsilon^T \bar{\mathbf{W}} \epsilon \quad (22)$$

is minimized. The weighting matrix $\bar{\mathbf{W}}$ in equation (22) is assumed to be positive definite. The condition to minimize the weighting error is given by (from equation (22))

$$\frac{\partial \mathbf{J}'}{\partial \mathbf{u}} = 0, \quad (23)$$

which provides the weighted least squares estimate $\hat{\mathbf{u}}$ of \mathbf{u} . Therefore,

$$\hat{\mathbf{u}}(t) = -[\mathbf{B}^T(t)\bar{\mathbf{W}}\mathbf{B}(t)]^{-1} \mathbf{B}^T(t)\bar{\mathbf{W}}\mathbf{Q}(t)\bar{\mathbf{B}}\bar{\mathbf{K}}\mathbf{z}(t). \quad (24)$$

When $\bar{W} = I$, the identity matrix, then

$$\hat{u}(t) = -B^*(t)Q(t)\bar{B}\bar{K}z(t). \quad (25)$$

Here $B^*(t)$ is the generalized inverse of $B(t)$ such that $B(t) = B(t)B^*(t)B(t)$. Taking $\hat{u}(t)$ for $u(t)$ as $\hat{u}(t)$ is the best estimate of $u(t)$ and noting that $z(t) = Q^{-1}(t)x(t)$, equation (25) takes the form

$$u(t) = -B^*(t)Q(t)\bar{B}\bar{K}Q^{-1}(t)x(t) = -K(t)x(t), \quad (26)$$

which provides the control law for the linear time periodic system (4). Substituting equation (26) in equation (4), one can obtain the closed-loop system equation as

$$\begin{aligned} \dot{x}(t) &= (A(t) - B(t)K(t))x(t) \\ y(t) &= C(t)x(t). \end{aligned} \quad (27)$$

Observer designs can also be performed for periodically time-varying systems using the L-F transformation approach on similar lines as discussed above. Following the standard observer design approach, the dual system of equation (4) representing the error dynamics between the system and the corresponding observer equation is considered. The gains of the dual system \hat{G} have been selected such that the error vector between the system and the observer states asymptotically goes to zero as time tends to infinity. To obtain the \hat{G} using the L-F transformation approach, first the STM of the dual system (12) has to be obtained using the Chebyshev expansion technique and the L-F transformation matrix $\bar{Q}(t)$ is extracted. Applying the transformation

$$\bar{z}(t) = \bar{Q}(t)z(t), \quad (28)$$

equation (12) can be converted to the following form

$$\dot{\bar{z}}(t) = \bar{R}z(t) + \bar{Q}^{-1}(t)C^T(t)u(t), \quad (29)$$

where \bar{R} is an $n \times n$ constant matrix. Following the full-state feedback control design methodology, the control law using time-invariant approach corresponding to equation (29) can be written as

$$u(t) = -C^*(t)\bar{Q}(t)\bar{C}\bar{K}\bar{Q}^{-1}(t)\bar{z}(t) = -\hat{G}(t)\bar{z}(t), \quad (30)$$

where $C^*(t)$ is the generalized inverse of C^T and \bar{C} is an arbitrary constant input matrix. Therefore, the gain matrix of the observer can be obtained as $\bar{G} = \hat{G}^T$. Again, \bar{K} has been obtained as a result of applying time-invariant control design techniques. Hence the observer-based closed-loop system can be written as

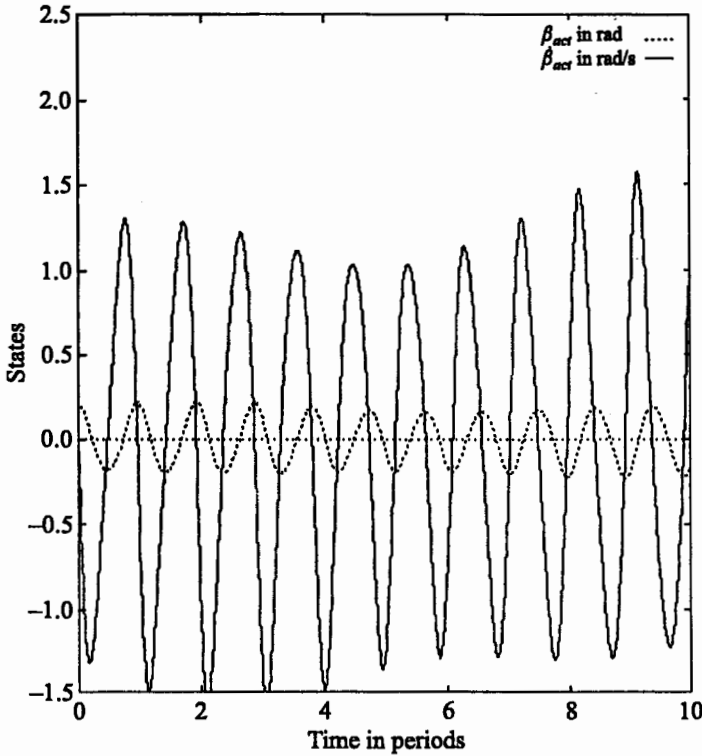


Figure 2. Uncontrolled states of flapping motion $\mu = 0.3$.

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\bar{\mathbf{x}}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}(t) & \mathbf{N}(t) \\ \bar{\mathbf{G}}(t)\mathbf{C}(t) & \mathbf{A}_0(t) \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \bar{\mathbf{x}} \end{Bmatrix} \quad (31)$$

where $\mathbf{A}_0(t) = \mathbf{A}(t) - \bar{\mathbf{G}}(t)\mathbf{C}(t) - \mathbf{B}(t)\hat{\mathbf{K}}(t)$ and $\mathbf{N}(t) = -\mathbf{B}(t)\mathbf{K}(t)$.

5. RESULTS AND DISCUSSION

The controller design methodologies discussed above based on full- and observer-state feedback have been implemented to control the flapping motion of an individual helicopter blade. The typical design parameters are taken as $\Omega = 200$ rpm, $\omega_F = 0.4$ rad/s, $\gamma = 5.0$ and μ in the range of 0.3 to 1.2. When controller design due to Chebyshev expansion procedure is considered, the symmetric positive semidefinite matrix $\mathbf{S}(t_f)$ is assumed to be an identity matrix, thereby all the states are assumed to be constrained at t_f and $\mathbf{P}(t)$ is assumed to be a null matrix. The control vector \mathbf{u} consists of a single variable θ , and, therefore, the symmetric positive definite matrix $\mathbf{P}_1(t)$ is a single value that is assumed to be a constant for simplicity. By varying these parameters, the control design performance characteristics can be suitably altered.

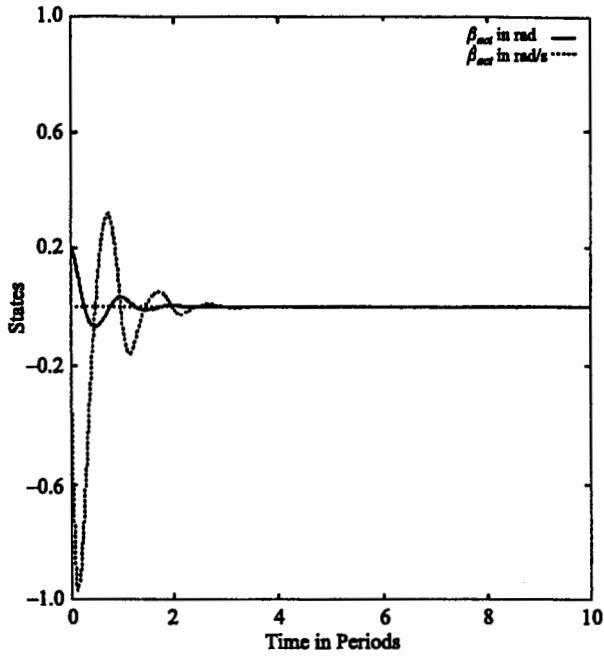


Figure 3. Controlled states-Full-state feedback. Chebyshev expansion approach; $\mu = 0.3$.

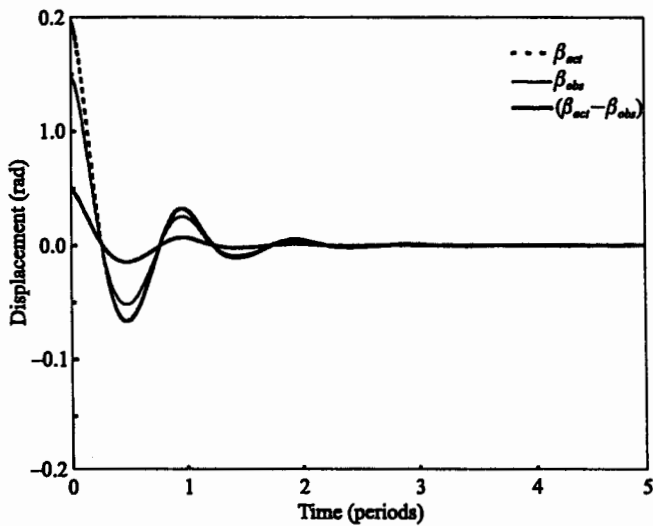


Figure 4a. Controlled flap angles-Observer-state feedback. Chebyshev expansion approach; $\mu = 0.3$.

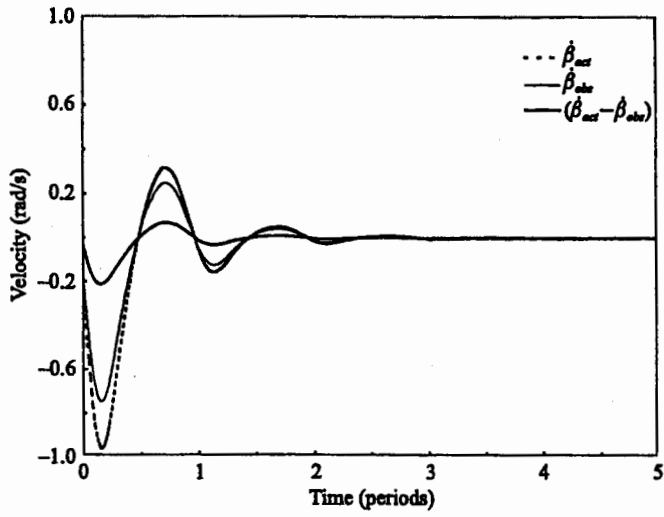


Figure 4b. Controlled flap rates-Observer-state feedback. Chebyshev expansion approach; $\mu = 0.3$.

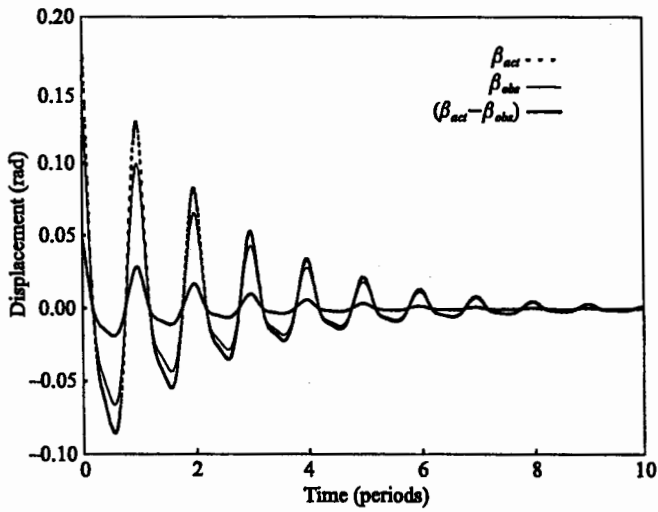


Figure 5a. Controlled flap angles-Observer-state feedback. Chebyshev expansion approach; $\mu = 1.2$.

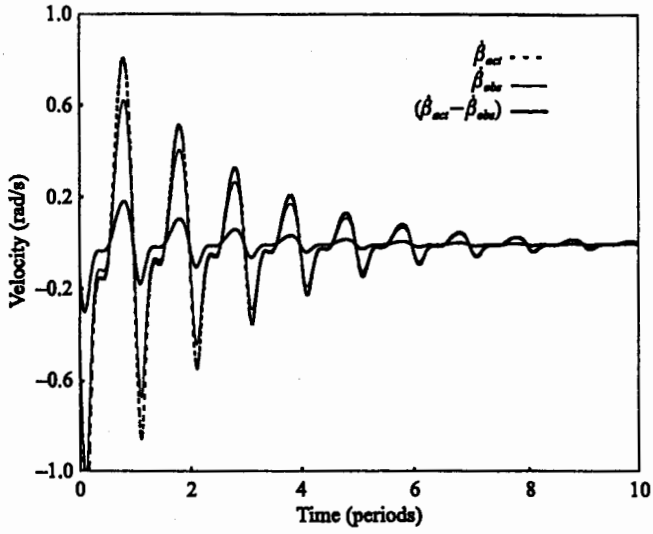


Figure 5b. Controlled flap rates-Observer-state feedback. Chebyshev expansion approach; $\mu = 1.2$.

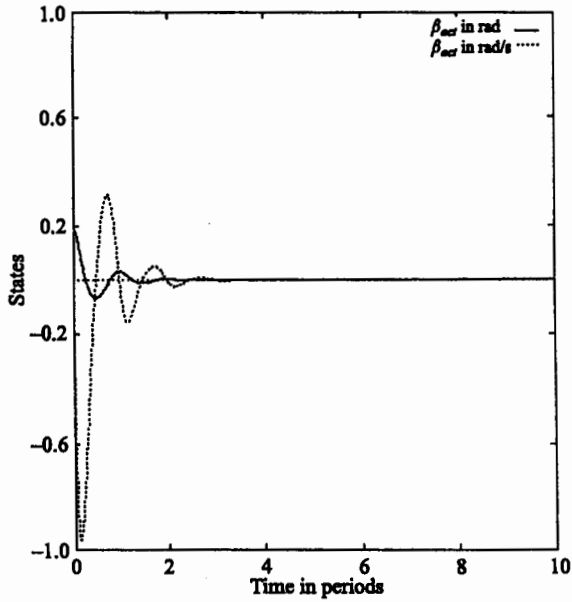


Figure 6. Controlled states-Full-state feedback. Liapunov-Floquet approach; $\mu = 0.3$.

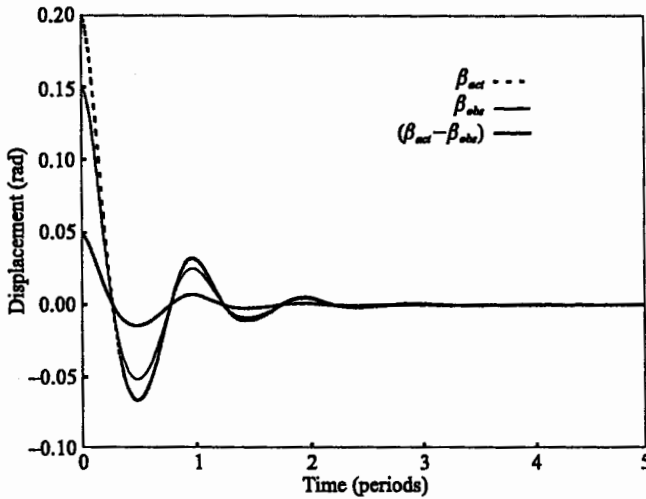


Figure 7a. Controlled flap angles-Observer-state feedback. Liapunov-Floquet approach; $\mu = 0.3$.

The uncontrolled states of the helicopter rotor blade when $\mu = 0.3$ are provided in Figure 2. The damping of the system can be noted to be very small, and the initial error does not die out even after 50 periods. Initially, the controllers have been designed using the Chebyshev expansion approach. In Figure 3, the controlled states of the flap motion have been plotted when a full-state feedback controller is incorporated. Assuming that the flap angle β is the output available for control, the observer-state feedback controller has also been designed. The response characteristics of actual and observed flap angles, namely, β_{act} and β_{obs} , and their difference are shown in Figure 4a, and the actual and observed flap angular velocities, namely, $\dot{\beta}_{act}$ and $\dot{\beta}_{obs}$, and their difference are shown in Figure 4b. These figures are for a helicopter rotor blade with an advance ratio $\mu = 0.3$. Similarly, the actual and observed flap angles and flap angular rates for a case with high advance ratio $\mu = 1.2$ are provided in Figures 5a and 5b. From these Figures, it can be seen that the $\beta_{act} - \beta_{obs}$ and $\dot{\beta}_{act} - \dot{\beta}_{obs}$ tend to zero as the time period advances, showing the convergence between the actual and observed states. It should be noted that the matrix $b(t)$ becomes singular (McKillip, 1985) when the advance ratio μ is around 1.3 and therefore the controllability of the problem is completely lost at those instances. The response of the controlled states with a full-state feedback controller based on the L-F transformation approach has been depicted in Figure 6 for $\mu = 0.3$. The actual and observed states of the controlled response of the system (4) using an observer via L-F transformation approach have been portrayed in Figures 7a and 7b for $\mu = 0.3$ and in Figures 8a and 8b for $\mu = 1.2$. Furthermore, it is also observed that the convergence characteristics are just similar to what is seen in the Chebyshev expansion approach.

Although the response of the system has been controlled by adjusting the damping levels of the flapping motion in this case, the applicability of the procedures to more elaborate models is just as simple and will be the focus of our forthcoming paper. The difficulty in controlling the unstable flapping response at high advance ratio $\mu = 1.3$ lies in the

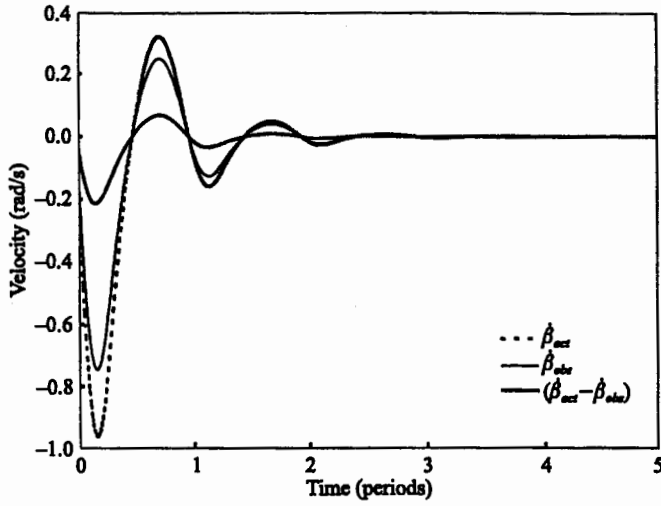


Figure 7b. Controlled flap rates-Observer-state feedback. Liapunov-Floquet approach; $\mu = 0.3$.

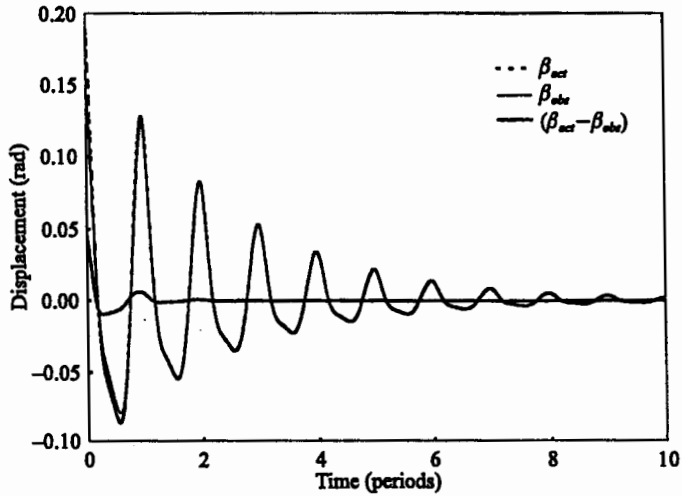


Figure 8a. Controlled flap angles-Observer-state feedback. Liapunov-Floquet approach; $\mu = 1.2$.

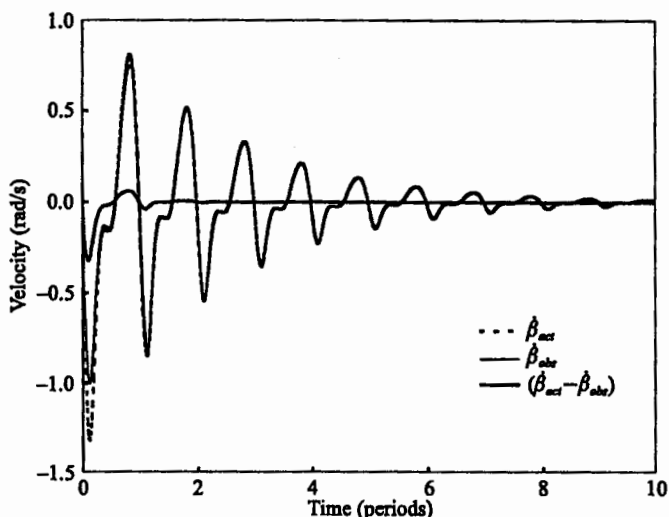


Figure 8b. Controlled flap rates-Observer-state feedback. Liapunov-Floquet approach; $\mu = 1.2$.

uncontrollability of the flap motion (due to the singular nature of the matrix $\mathbf{B}(t)$) rather than anything else. It is to be noted that the flapping motion crosses the boundary of the unit circle to become unstable only around $\mu = 1.3$. However, the above control design procedures are found to have no restrictions in controlling the unstable responses, which can be seen from other works reported elsewhere (Joseph, Pandiyani, and Sinha, 1993; Sinha and Joseph, 1994; Pandiyani and Sinha, 1994).

6. CONCLUSIONS

In this paper, development of a set of practical strategies in the control synthesis for Individual-Blade-Control (IBC) of a single helicopter blade is presented. The control of such systems has been hampered in the past due to the lack of suitable design techniques for periodically time-varying systems. The first technique is based on a Chebyshev expansion procedure together with optimal control techniques. The second method utilizes the Liapunov-Floquet transformation to convert a periodic system matrix into a time-invariant one and the control design is performed using standard control design techniques used for autonomous systems. It should be noted that the first method requires getting the STM of an augmented system, which is twice the size of the original system equation (4). However, the second method does work with the original size of the system. Both methods are capable of providing full-state and observer-based feedback designs and thereby render very practical tools in the control design of periodic systems. The application of these methods to control the flapping motion of an individual helicopter blade has been shown for low to high advance ratios. It is noticed that the flap equation of the helicopter rotor blade has a singularity in $\mathbf{B}(t)$ matrix at high advance ratio such as $\mu = 1.3$ and has been found to cross zero-reference at least for a couple of times (McKillip, 1985). This affects the controllability condition of any

periodic system, which requires that the $A(t)$ and $B(t)$ should be continuously differentiable for $n - 2$ times and $n - 1$ times (Silvermann and Medows, 1967), respectively. Except for such situations, the control design procedures have been found extremely effective over the entire operating range of advance ratio of helicopters. A more realistic model of helicopter rotor blades considering flap and lead-lag dynamics will be dealt with in our forthcoming article elsewhere.

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