PARAMETRIC EXCITATION

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Introduction

Excitation of a dynamic system is called parametric when the effect of the forcing appears as a coefficient of a variable in the governing equations of motion. This phenomenon leads to the family of time-varying (or nonautonomous) equations, whose coefficients are explicit functions of time. In most of the applications the nature of the explicit time dependence is periodic or can be modeled to be periodic as a good approximation to the true excitation. As a very simple example, consider a child playing on a swing. In order to drive the swing the child leans back and forth rhythmically, providing excitation that destabilizes the vertical equilibrium position of the swing. Although the excitation is not perfectly periodic, it is still a sufficiently accurate model of the motion.

Time-periodic systems play an important role in sciences and engineering, and have been the subject of investigation since 1868, when Mathieu first studied such a system, while analyzing the vibrations of an elliptical membrane. Since that time, the study of time-periodic equations has found applications in many areas such as dynamic stability of structures, circuit theory, systems and control, satellite dynamics, rotating shafts, helicopter rotor blades, quantum mechanics, and biomechanics (human or animal locomotion or modeling the function of the heart). Some physical examples are shown in **Figure 1**. In many situations modeling of these systems leads to nonlinear differential equations with periodic coefficients.

In general, it is not possible to obtain exact solutions of time-periodic nonlinear systems, except in a few very special cases. However, in most applications, it may suffice to obtain approximate solutions or information about the stability. The most natural idea is to somehow eliminate the explicit time dependence, and make the system autonomous and amenable to applications of a large number of theories developed for time-invariant systems. This is the basic motivation of point-mapping techniques, which are mainly used in stability and bifurcation theory. In the averaging method an approximate time-invariant system is constructed by averaging the equation over the period. Perturbation techniques seek approximate solutions under various conditions such as resonance, or limit cycle solutions after bifurcations, by assuming the periodic and nonlinear terms of the equation to be small perturbations of a linear time-invariant system. The Liapunov–Floquet transformation (L–F transformation) approach makes the linear part of a periodic equation time-invariant and thus retains the stability and bifurcation characteristics of the original system in the entire parameter space. The resulting equation can then be studied via standard techniques. The method of time-dependent normal form reduction aims to simplify the nonlinear equations by reducing the number of nonlinear terms to a minimum, even make them linear or time-invariant in some cases. The goal of time-periodic center manifold reduction is to reduce the dimension of the systems for the purpose of stability analysis in critical cases. These are the most current tools of analysis. In the following sections each method will be described and discussed briefly.

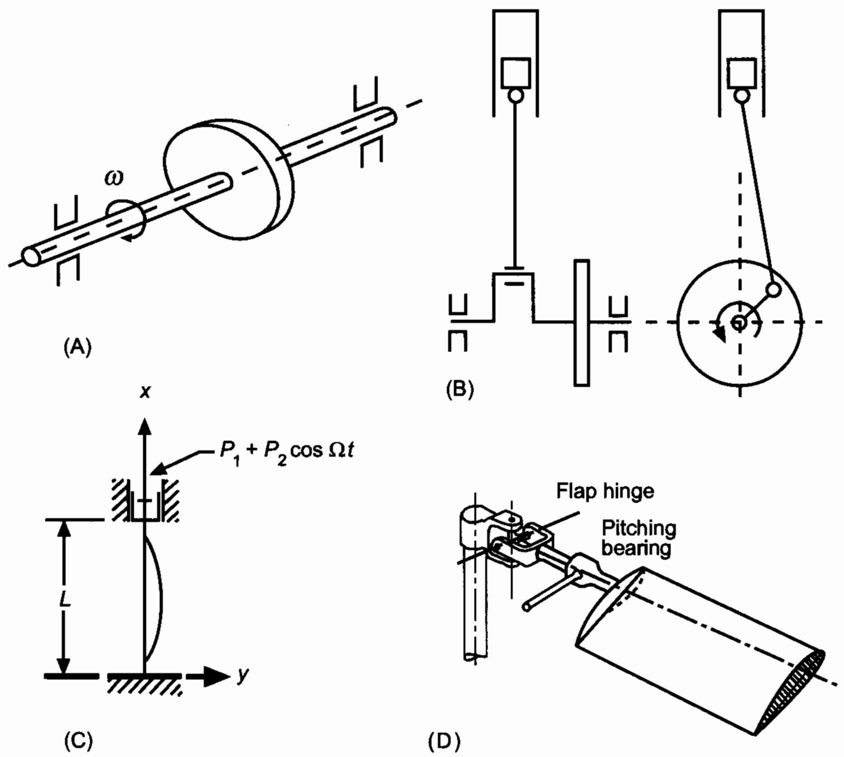

Figure 1 (A) Rotor-bearing system; (B) connecting rods; (C) periodically loaded column; (D) helicopter rotor blade.

Problem Formulation

The general form of nonlinear time-periodic systems is given by the equation:

$$\dot{x} = f(x, t) \quad [1]$$

where x and f are n real vectors and $f(x, t) = f(x, t + T)$. T is called the principal period of the system. If we restrict ourselves to studying moderately large vibrations only, then we can expand the above equation in Taylor series about an equilibrium position or a known periodic solution of eqn [1] and obtain:

$$\dot{x} = \mathbf{A}(t)x + f_2(x, t) + \cdots + f_k(x, t) + O\left(\left|x^{k+1}\right|, t\right) \quad [2]$$

or:

$$\dot{x} = \mathbf{A}(t)x + F(x, t) \quad [3]$$

where $\mathbf{A}$ is an $n \times nT$-periodic matrix, and the vector functions f_i, $i = 2...k$, contain ith order monomials of x and are T-periodic in time. F is suitably defined in terms of f_i's. The index k is the highest order considered. The goal is either to predict stability and analyze bifurcations or to find the system response (most often steady-state solutions). Stability of the equilibrium position or the known periodic solution can be studied by Floquet theory if in the linear equation:

$$\dot{x} = \mathbf{A}(t)x \quad [4]$$

matrix $\mathbf{A}$ is not critical, that is, it does not have Floquet multipliers on the unit circle. If the system is critical, a nonlinear analysis is necessary. Floquet theory is also useful for finding the response of linear time-periodic equations.

Floquet Theory

Consider the linear periodic system given by eqn [4]. Let $\Phi(t)$ denote the fundamental solution matrix or state transition matrix (STM) that contains n linearly independent solutions of eqn [4] with the initial conditions $\Phi(\mathbf{0}) = \mathbf{I}$ where $\mathbf{I}$ is the $n \times n$ unit matrix. Then the following statements hold:

1. $\Phi(t + T) = \Phi(T)\Phi(t) \quad 0 \le t \le T$ and, consequently
2. $\Phi(t + jT) = \Phi(T)^j\Phi(t)$ $0 \le t \le T, j = 2, 3 \ldots$
3. $x(t) = \Phi(t)x(0)$ $t \ge 0$

These results imply that, if the solution is known for the first period, it can be constructed for all time t. The matrix $\Phi(T)$ is called the Floquet transition matrix (FTM). The next statement considers the stability of eqn [4].

4. Let $\zeta_i (i = 1, \ldots, n)$ denote the eigenvalues of $\Phi(T)$. System (4) is asymptotically stable if all ζ_i's lie inside the unit circle of the complex plane. The system is unstable if at least one of the eigenvalues of the FTM has magnitude greater than one. The eigenvalues ζ_i are called the Floquet multipliers. The above statements briefly summarize the results of Floquet theory.

The Necessity of Nonlinear Analysis: Bifurcations and other Nonlinear Phenomena

In nonlinear systems a number of phenomena can occur which are impossible to predict from the linear analysis. Linear theory can predict the instants when a system goes through a change of stability characteristics, called bifurcation, but it cannot predict what kind of motion follows and whether or not the system is stable in critical situations. The nonlinear phenomena of time-periodic systems include cyclic bifurcations of equilibria and periodic orbits. When one Floquet multiplier of the system leaves the unit circle of the complex plane at -1, a period doubling (or flip) bifurcation occurs, the equilibrium or limit cycle changes its stability characteristic and a new limit cycle with twice the period of the original is born. When a multiplier crosses the unit cycle at $+1$, there are several possibilities, depending on the structure of the nonlinearities. Transcritical bifurcation occurs in the presence of quadratic terms, while cubic terms without any quadratics imply symmetry-breaking bifurcation. In transcritical bifurcation a stable and an unstable limit cycle (or an equilibrium point and a limit cycle) that exist together, switch their stability. Symmetry-breaking bifurcation is similar to flip bifurcation, without the doubling of the period. The cyclic-fold bifurcation, which also occurs when a multiplier becomes $+1$, is a phenomenon when a stable and an unstable cycle collide and disappear. For instance, two cyclic-fold bifurcations cause the jump phenomenon of resonance. If a pair of complex multipliers leave the unit circle, a quasiperiodic solution arises in a secondary Hopf (or Neimann–Sacker) bifurcation. The stability and bifurcation diagram is shown in **Figure 2**. Chaos can occur in these systems as well, through common routes, a series of period-doubling bifurcations or through secondary Hopf bifurcations. Intermittent chaos is also often observed. Features such as limit cycles and chaos observed in nonlinear systems have great significance in both theory and practical

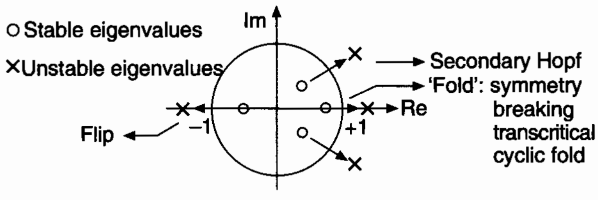

Figure 2 Stability and bifurcation diagram.

applications and can be treated only by means of nonlinear analysis techniques.

Tools of Nonlinear Analysis

Perturbation Methods

Perturbation theory has been widely used to obtain approximate solutions of many problems, in various branches of science and engineering. The basic idea is that if the problem at hand differs from a simple solvable problem only by small additional terms, then we could assume that the solutions of the two problems will be close as well. Therefore we can start from the solution of the simple problem, called the 'unperturbed problem' and try to find a series of corrections, in order to construct the solution of the original problem. The corrections are most often expressed as a power series of a small quantity called the 'perturbation parameter'. In the last step one needs to show that a few terms in the series will provide a useful approximate solution.

In the case of time-periodic nonlinear differential equations, perturbation techniques are only generally applicable if there exists a small parameter multiplying the periodic linear and nonlinear terms. Since the periodic linear terms as well as the nonlinear terms are small compared to the time-invariant linear part, the solution can be composed as slowly varying vibrations superimposed on the well-known periodic steady-state solution of the time-invariant linear system. The solution is written as a power series in terms of the small parameter. Substituting the series into the original equation and equating the coefficients of powers of the small parameter on both sides of the equations, a set of linear differential equations is obtained. These equations are solved to yield an approximate solution up to some desired power of the small parameter.

If eqn [3] can be written as:

$$\dot{x} = \mathbf{B}x + \varepsilon[A_1(t)x + F(x, t)] \tag{5}$$

where $\mathbf{B}$ is a constant matrix, $A_1(t)$ is the periodic part of matrix $\mathbf{A}(t)$, $F(x, t)$ is an n vector containing the nonlinear terms and ε is a small parameter. Then we seek a uniform expansion of the solution of the form:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \qquad [6]$$

After substituting this expression into eqn [5] and collecting the coefficients for powers of ε, we obtain a set of linear differential equations in x_i. The equation for ε^0 is obviously given by the solution of which is $x_0 = \mathbf{\Phi}(t)a$, where a is a constant vector and $\mathbf{\Phi}(t)$ is the fundamental solution matrix of $\dot{x} = \mathbf{B}x$. Now, we let $a = a(t)$ be the modulation, an unknown function of time. An approximation of $a(t)$ can be obtained from the equation corresponding to ε^1, by eliminating the resonant (so-called the secular) terms. This elimination is necessary in all problems with excitation in order to obtain a uniform expansion. Once again, to refine the approximation further, we assume that the coefficient of $a(t)$ is an unknown function of time, to be determined from the ε^2 equation, and so on, until we obtain a 'reasonably good' approximation.

A modification of this method called the method of multiple scales makes special use of the fact that the corrections superimposed on the solution of the linear time-invariant system are relatively 'slow' vibrations with much smaller frequencies. Because of the limitation on the size of the parameter ε, however, the applicability of these methods is limited. They also seem to yield poor approximations for the locations of bifurcation points as well as for steady-state solutions in the neighborhood of bifurcations for time-periodic equations.

Averaging Method

The averaging method is based on the same basic idea as the perturbation methods. Once again, the periodic and nonlinear terms are assumed to be small compared to the constant linear part and are treated as slowly varying perturbation of the time-invariant linear equation. The solution of the nonlinear problem can be treated as a slowly varying modulation added to the solution of the time-invariant part, which can be constructed as before. The aim of averaging is to obtain an approximate solution based on these assumptions. In eqn [5] if we let $x = \mathbf{\Phi}(t)z$, then it can be transformed into the standard form:

$$\dot{z} = \varepsilon\Phi^{-1}(t)g(\Phi(t)z, t) \qquad [8]$$

where $g(x, t) = \mathbf{A}_1(t)x + F(x, t)$ and the variable $z(t)$ represents the modulation. Since an exact solution of eqn [8] is not possible, the theorem of averaging allows us to construct the following averaged system:

$$\dot{\bar{z}} = \varepsilon\bar{g}(\bar{z}) \qquad [9]$$

where:

$$\bar{g}(\bar{z}) = \frac{1}{T}\int \mathbf{\Phi}^{-1}(t)g(\mathbf{\Phi}(t)z, t)\,\mathrm{d}t \qquad [10]$$

The error between $z(t)$ and $\bar{z}(t)$, can be expressed as:

$$z(t) - \bar{z}(t) = \mathrm{O}(\varepsilon), \quad \varepsilon \to 0, \quad t \in \left[0, \frac{1}{\varepsilon}\right] \qquad [11]$$

This implies that for small ε the solution of the averaged equation is a good approximation of the solution of eqn [5], but only in a finite time interval depending on ε. Eqn [9] is a quasilinear autonomous differential equation in the slowly varying averaged coefficient $\bar{z}$. Although it is quasilinear, in general, an exact analytical solution is still not possible, except in some cases the averaged equation may become solvable in a closed form. Further simplification is possible via normal form theory and in some cases an approximate analytical solution is possible. Like the perturbation methods, the applicability of the method is limited due to the small parameter assumption.

Liapunov–Floquet transformation

According to the Liapunov–Floquet theorem, the fundamental matrix (or STM) $\mathbf{\Phi}(t)$ of eqn [4] can be written as a product of two matrices as:

$$\mathbf{\Phi}(t) = \mathbf{L}(t)\mathrm{e}^{\mathbf{C}t} \qquad [12]$$

where $\mathbf{L}(t)$ is T-periodic and $\mathbf{C}$ is a constant matrix, both, in general, are complex. There also exists a real factorization of the same form as eqn [12], where $\mathbf{L}(t)$ is real $2T$-periodic and $\mathbf{C}$ is a real constant matrix. $\mathbf{L}(t)$ is called the (T or $2T$-periodic) Liapunov–Floquet (LF) transformation matrix. Applying the transformation:

$$x = \mathbf{L}(t)y \qquad [13]$$

results in:

$$\dot{y} = \mathbf{C}y \qquad [14]$$

which is a dynamically equivalent autonomous form of eqn [4], either complex or real. After applying eqn [13], the nonlinear system (eqn [2]) becomes:

$$\dot{y} = \mathbf{C}y + \mathbf{L}^{-1}\left(f_2(y, t) + \cdots + f_k(y, t) + \mathrm{O}\left(\left|y^{k+1}\right|, t\right)\right) \qquad [15]$$

Eqn [15] is dynamically equivalent to eqn [2] and it has a time-invariant linear part. This form allows one to apply the method of normal forms and/or center manifold reduction, which can provide considerable simplifications.

However, in general, it is not possible to obtain the L–F transformation in a closed form because for this purpose one needs to compute the fundamental solution matrix $\mathbf{\Phi}(t)$ of the linear periodic differential eqn [4] (cf., eqn [12]). On the other hand, recent developments suggest that the elements of the solution matrix $\mathbf{\Phi}(t)$ can be obtained as a sum of Chebyshev polynomials to a desired accuracy. The L–F transformation $\mathbf{L}(t)$ and matrix $\mathbf{C}$ can then be obtained by a factorization of $\mathbf{\Phi}(t)$. Since $\mathbf{L}(t)$ is a periodic function of time, it can be expressed as a Fourier series.

Time-dependent Normal Form Theory

The idea of normal form reduction is to simplify nonlinear equations by eliminating as many nonlinear terms as possible, while keeping the linear part (and therefore the stability and bifurcation characteristics) unchanged. This is achieved by a series of near-identity transformations. Because $\mathbf{A}(t)$ in eqn [2] is time-dependent, a direct application of normal form theory is not possible. Therefore, first the $2T$-periodic L–F transformation is used to transform eqn [2] to a vector field where the linear part is time-invariant. Then the resulting eqn [15] can be handled by the method of 'time-dependent normal forms' (TDNF). Eqn [15] can be written in the Jordan canonical form as:

$$\dot{y} = Jy + w_2(y,t) + \cdots + w_k(y,t) + \cdots + O\left(|y|^{k+1}, t\right) \qquad [16]$$

where J is the Jordan form of the real matrix $\mathbf{C}$ and w_r's $(r = 2 \ldots k)$ are $2T$-periodic functions which contain homogeneous monomials of y of order r. Now one can construct a sequence of transformations which successively removes the nonlinear terms w_r. The transformation is of the form:

$$y = v + h_r(v,t) \qquad [17]$$

where h_r is a formal power series in v of degree r with $2T$-periodic coefficients. Applying the above change of coordinates, one can establish the conditions for eliminating each nonlinear term, and solve for h_r. These conditions are expressed in the form of a set of partial differential equations for h_r, called the homological equation. Since the coefficients of w_r and the unknown nonlinear transformation h_r can be expressed in finite Fourier series, one only needs to solve a set of algebraic equations for the coefficients of h_r, instead of solving the differential equations. Following this procedure it is not always possible to eliminate all of the nonlinear terms. The nonlinear terms corresponding to the resonance condition remain in the normal form that is the simplest possible nonlinear form of eqn [2]. In some cases it can even be linear, or, although nonlinear, completely time-invariant. In many instances, the resulting normal form equations can be analyzed analytically and the stability and bifurcation characteristics can be examined. Usually, some algebraic manipulations and/or transformation to polar coordinates render closed-form solutions for simple nonlinear equations. Thus, the solutions of the original quasilinear time-periodic equations can be obtained in the original coordinates by substituting back all the transformations, including the L–F transformation. Usually these computations are tedious due to the presence of a large number of periodic terms in eqn [16]. In order to simplify the procedure, without any significant compromise in the quality of solutions, only time-invariant terms in w_r may be retained and the periodically varying terms may be neglected. Then eqn [16] takes the form:

$$\dot{\bar{y}} = J\bar{y} + \bar{w}_2(\bar{y}) + \cdots + \bar{w}_k(\bar{y}) + O\left(|\bar{y}|^{k+1}\right) \qquad [18]$$

which is autonomous and $\bar{y}$ and $\bar{w}_r(\bar{y})$ are the approximate states and nonlinear functions, respectively. Then the solution can be obtained by application of the time-independent normal form (TINF) theory.

Time-periodic center manifold reduction

In situations where some of the eigenvalues of matrix $\mathbf{C}$ in eqn [15] are critical, the stability can be discussed in the local center manifold via the time-periodic center manifold theorem. The idea of center manifold theorem is to reduce the dimension of the system to the minimum possible, i.e., to the dimension of the invariant manifold associated with the critical eigenvalues. All 'critical' dynamics take place in this manifold. The center manifold theorem states that if the reduced system on the center manifold is stable (unstable) then the original n-dimensional system is also stable (unstable). Due to the periodic nature of eqn [2], a direct application of the center manifold theorem is not possible, but it can certainly be applied to eqn [15]. Let us assume that eqn [15] has n_1 eigenvalues that are critical and n_2 eigenvalues that have negative real parts. Then it can be rewritten in the form:

$$\begin{pmatrix} \dot{y}_c \\ \dot{y}_s \end{pmatrix} = \begin{bmatrix} J_c & 0 \\ 0 & J_s \end{bmatrix} \begin{pmatrix} y_c \\ y_s \end{pmatrix} + \begin{pmatrix} w_{c2} \\ w_{s2} \end{pmatrix} + \begin{pmatrix} w_{c3} \\ w_{s3} \end{pmatrix} + \cdots \qquad [19]$$

where the subscripts c and s represent the critical and stable vectors, respectively. At this point a nonlinear relation is sought to decouple the nonlinear terms of the equations corresponding to the critical and stable states, in the form of:

$$y_s = h(y_c, t) \qquad [20]$$

such that $h(y_c, t)$ is a polynomial of y_c with $2T$-periodic unknown coefficients. The relation for y_s can be obtained by substituting eqn [19] into the set of n_2 equations associated with the stable eigenvalues in eqn [18]. The substitution results in a set of partial differential equations for the coefficients of h. Since these coefficients have to be periodic, we can assume a formal Fourier series form with unknown coefficients. Then the differential equations reduce to a set of linear algebraic equations for the Fourier coefficients. After solving for the coefficients, substitution of eqn [19] in eqn [18] decouples stable and critical states and hence the problem reduces to the investigation of stability of an n_1-dimensional system in the center manifold. If necessary, the center manifold equations can be simplified further by the time-dependent normal form theory as described earlier. For certain types of bifurcation they become completely time-invariant and solvable in a closed form.

Point-Mapping Techniques

For periodic systems, one could also represent the motion of the system as events in discrete time instants. The goal here is to eliminate the explicit time dependence from the equation by replacing the dynamic system as a set of difference equations sampled at period T. This process links the state of the system at the beginning of a period to the state of the system at the end of the period by a point-mapping transformation. The advantage of the discrete time approach is that by introducing this formulation, one eliminates the explicit continuous dependence on time. This feature allows, at least in principle, an easier determination of the various solutions and their stability properties. Another advantage is that difference equations are easy to simulate on digital computers. One of the main obstacles in applying point-mapping techniques to real problems is in obtaining the corresponding difference equations. In order to obtain the complete difference equation representation one has to find a solution of a nonautonomous nonlinear system of differential equations over one period. This is not possible except in a few special cases. Thus, in general, one is forced to accept an approximation to the point mapping.

Consider a nonlinear dynamic system described by eqn [1], i.e.:

$$\dot{x}(t) = f(t, x(t)) \qquad [21]$$

where x and f are real vectors and f is periodic in time with period T. The state of the system is observed at discrete instances of T units of time. The dynamic relationship between the state of the system at $t = nT$ and the state at $t = (n+1)T$ results in a set of difference equations which are independent of time. If such a relationship can be found, the discrete counterpart to the set of equations in eqn [21] can be written as:

$$x(n+1) = G(x(n)), \quad n = 1, 2, 3\ldots \qquad [22]$$

where $x(n+1)$ and $x(n)$ are the state of the system at $t = (n+1)T$ and $t = nT$, respectively. The representation of the system dynamics given by eqn [22] is called the Poincaré map. An equilibrium point or a T-periodic orbit of eqn [21] is represented as a fixed point of the map. A periodic solution of period KT consists of K distinct points $x*(j), j = 1 \ldots K$, such that:

$$\begin{aligned} &x^*(m+1) = G(x^*(m)), \quad m = 1, 2, \ldots K-1 \\ &x^*(1) = G^K(x^*(1)) \end{aligned} \qquad [23]$$

Approximations of the map are usually obtained numerically, for example, by the Runge–Kutta method for solving systems of ordinary differential equations. By keeping track of the various coefficients, an approximation up to any desired degree can be obtained. Point-mapping methods are mostly used to study the local as well as global stability and bifurcation characteristics of a system. Unlike the perturbation and averaging methods, this technique is not restricted to systems with small parameters. The main disadvantage of this method is that the construction of the Poincaré map requires knowledge of either the exact or an approximate solution of the original problem. Further, the discretized system dynamics cannot be transformed back to the original continuous state. Computational problems may also arise as the system dimension becomes larger.

Illustrative Examples

In this section some simple examples are considered to demonstrate the application of the methods discussed above. First, consider a Mathieu equation with cubic nonlinearity in the form:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\alpha + \beta \cos \omega t) & -\delta \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\varepsilon x_1^3 \end{Bmatrix} \qquad [24]$$

where δ, α, β, ω, and ε are the parameters of the system. Results for two sets of parameters are shown in **Figures 3** and **4**. As seen from **Figure 3**, when $\alpha(=10)$ is much larger than $\beta(=0.2)$, then the averaging method does provide meaningful results although it is not as accurate as the two normal form methods, viz., TINF and TDNF. However, when $\alpha(=0.5)$ is small compared to $\beta(=4)$, the averaging method incorrectly predicts unbounded solution while TINF and TDNF provide very good results (**Figure 4**). As an example of the application of time-dependant center manifold reduction, the bifurcation analysis of a base-excited simple pendulum shown in **Figure 5** is considered:

$$\ddot{\theta} + d\dot{\theta} + (a + b\sin\omega t)\sin\theta = 0 \qquad [25]$$

where $a = g/L$, $b = A\omega^2/L$, and $d = c/ML^2$. Writing $\{x_1\, x_2\} = \{\theta\, \dot{\theta}\}$ and expanding $\sin\theta$ in a Taylor series about the zero equilibrium point up to the cubic terms, yields:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\alpha + \beta\sin\omega t) & \alpha \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ (a + b\sin\omega t)\dfrac{x_1^3}{6} \end{Bmatrix} \qquad [26]$$

Letting a be the bifurcation parameter; for $b = 4$, $d = 0.31623$, and $\omega = 2$, the system undergoes a symmetry breaking bifurcation (one of the Floquet multipliers is $0.999999 \approx 1$, the other being 0.370292) at $a_c = 3.917787$. After applying the L–F and modal transformations, the time-dependent center manifold reduction is used to decouple the critical equation from the other. The center manifold equation is further simplified via normal form theory to yield:

$$\dot{v} = -0.4306v^3 \qquad (v \text{ is a new variable}) \qquad [27]$$

indicating that the equilibrium of the nonlinear system is asymptotically stable at the critical point. The versal deformation of eqn [27] yields the normal form of symmetry breaking bifurcation for this system as:

$$\dot{v} = \mu v - 0.4306v^3 \qquad [28]$$

where μ is the unfolding parameter. It is solvable in a closed from; the solution for the initial condition $v(0) = v_0$ being:

$$v(t) = \left(\left(\frac{1}{v_0^2} + \frac{-0.4306}{\mu}\right)e^{-2\mu t} - \frac{-0.4306}{\mu}\right)^{-1/2} \qquad [29]$$

Unlike the case of autonomous systems, here the normal form does not contain all the information

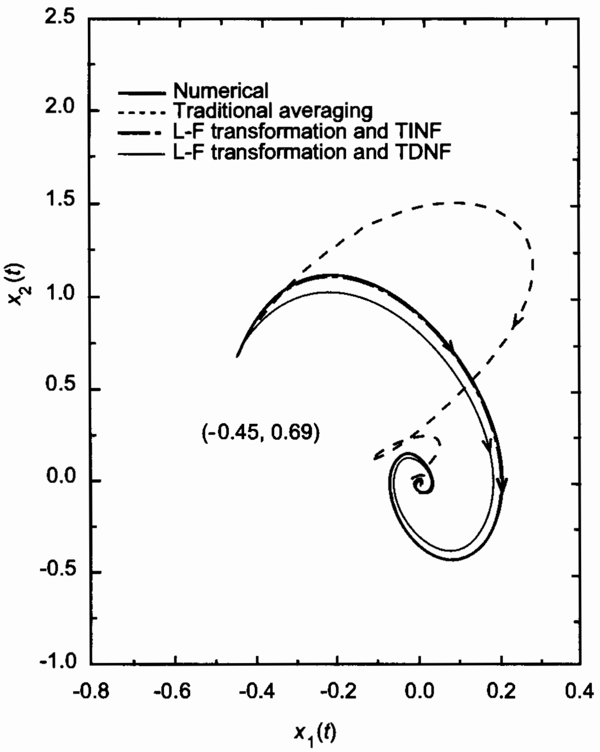

Figure 3 Solutions of Mathieu equation with cubic nonlinearity ($\alpha = 10.0$, $\beta = 0.2$, $\delta = 1.8974$, $\varepsilon = 3.0$).

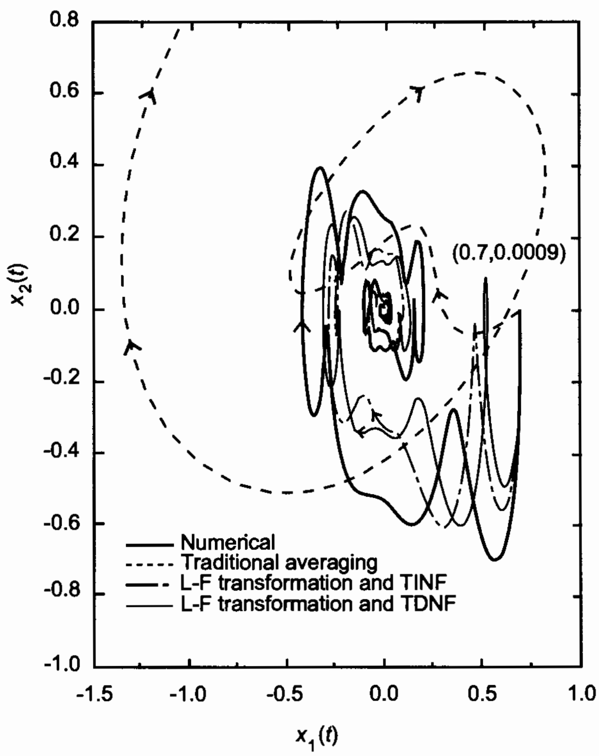

Figure 4 Solutions of Mathieu equation with cubic nonlinearity ($\alpha = 0.5$, $\beta = 4.0$, $\delta = 0.4243$, $\varepsilon = 3.0$).

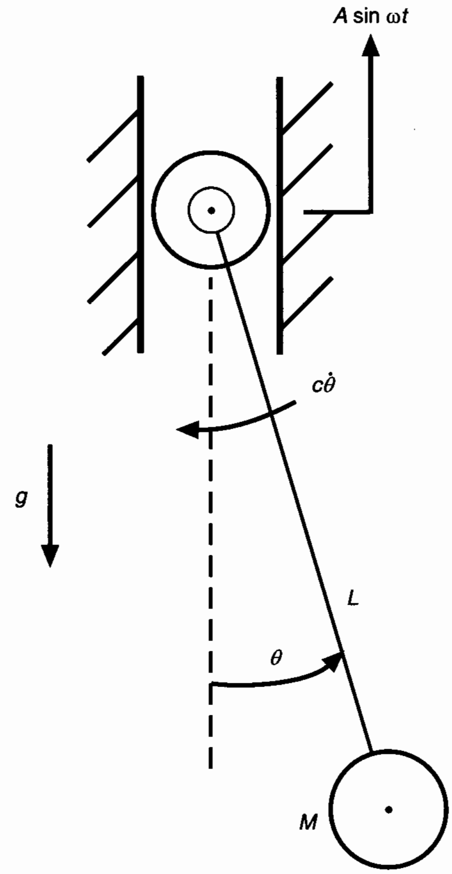

Figure 5 Base-excited simple pendulum.

about the bifurcation and the L–F transformation also plays an important role. From eqn [29] one can conclude the following about the dynamics in the neighborhood of the bifurcation. For $\mu < 0$ the $v = 0$ equilibrium is stable and there are no other limit sets in the neighborhood. For $\mu > 0$ the zero equilibrium becomes unstable and there is a stable nonzero equilibrium born. This equilibrium point becomes a stable T-periodic limit cycle in the original domain because the real L–F transformation is T-periodic in this case. So we say that the stability has been lost softly. This limit cycle is the boundary of the small attractive domain around the unstable equilibrium. By computing the amplitude of the limit cycle we have estimated the size of the stable region, and we can also estimate the rate of its growth as the bifurcation parameter goes further away from the critical value. The limit cycle is obtained by substituting the initial condition:

$$v_0 = -\left(\frac{\mu}{-0.4306}\right) \qquad [30]$$

into eqn [29] and applying the necessary transformations to yield the original state $x(t)$. However, one must establish a relationship between μ and η ($\equiv a - a_c$) which represents the change in the bifurcation parameter a. By performing a sensitivity analysis of the Floquet transition matrix (FTM), one can obtain a relationship between μ and η. However, the FTM must be computed in a symbolic form. One can also obtain such a relationship numerically by the curve-fitting technique. Using:

$$\mu = 0.865\eta - 1.635\eta^2 \qquad [31]$$

the postbifurcation limit cycles for $\eta = 0.01$ and $\eta = 0.1$ are shown in **Figures 6A** and **B**, respectively. Numerical results are included for a comparison.

Phenomena Associated with Time-periodic Nonlinear Systems

The time-periodic nonlinear systems contain typical features of parametric excitation and other phenomena associated with nonlinear systems. The para-

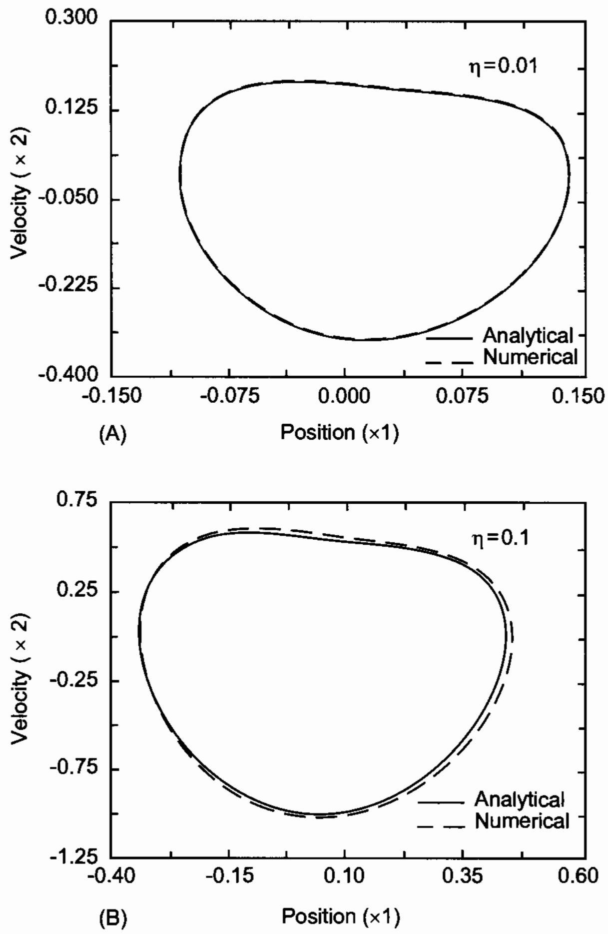

Figure 6 Postbifurcation limit cycles.

metric excitation can cause resonance which, in nonlinear systems, is associated with the jump phenomenon, the amplitude of vibration grows as the excitation frequency grows in the neighborhood of the resonant frequency, but at a certain frequency it drops suddenly, and the system vibrates with a much smaller amplitude. Near resonance another interesting feature, synchronization (or entrainment of frequencies) can also occur in these systems due to nonlinearities. The existence of limit cycles, quasiperiodic, and chaotic attractors is also observed in these systems.

These features can be analyzed by the use of the methods described above. Perturbation, averaging, point–mapping methods, and the L–F transformation with a subsequent use of normal form reduction have been employed to construct periodic, quasiperiodic, and even chaotic solutions. For bifurcation analysis, the L–F transformation combined with center manifold reduction and normal form theory appears to be very useful. The L–F transformation also appears to be a powerful tool in control of time-periodic systems. These are the current subjects of research.

Nomenclature

a	bifurcation parameter
$\mathbf{A}$	$n \times n$ T-periodic matrix
$\mathbf{B}$	constant matrix
$\mathbf{C}$	constant matrix
$\mathbf{I}$	$n \times n$ unit matrix
$\mathbf{L}(t)$	Liapunov–Floquet transformation matrix
T	principal period
$\mathbf{\Phi}(t)$	fundamental solution matrix
$\mathbf{\Phi}(T)$	Floquet transition matrix

See also: **Averaging**; **Critical damping**; **Dynamic stability**; **Nonlinear systems analysis**; **Nonlinear systems, overview**; **Perturbation techniques for nonlinear systems**.

Further Reading

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