On The Response of Linear Time-Periodic Systems Subjected to Deterministic and Stochastic Excitations

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Abstract: In many situations, engineering systems modeled by a set of linear, second-order differential equations, with periodic damping and stiffness matrices, are subjected to external excitations. It has been shown that the fundamental solution matrix for such systems can be efficiently computed using a Chebyshev polynomial series solution technique. Further, it is shown that the Liapunov-Floquet transformation matrix associated with the system can be computed, and the original time-periodic system can be put into a time-invariant form. In this paper, these techniques are applied in finding the transient response of periodic systems subjected to deterministic and stochastic forces. Two formulations are presented. In the first formulation, the response of the original system is computed directly. In the second formulation, first the original system is transformed to a time-invariant form, and then the response is found by determining the response of the time-invariant system. Both formulations use the convolution integral to form an expression for the response. This expression can be evaluated numerically, symbolically, or through a Chebyshev polynomial expansion technique. Results for some time-invariant and periodic systems are included, as illustrative examples.

Key Words: Time-periodic, stochastic, Chebyshev polynomials, Liapunov-Floquet transformation

1. INTRODUCTION

The study of systems governed by ordinary differential equations with periodic coefficients is of great importance in many areas of science and engineering. Often, because of the environments in which these systems operate, they encounter deterministic and stochastic excitations. In the past, two major types of solution techniques have been employed to study this problem. The first is the purely numerical approach, which is applicable to general systems (Wan and Lakshmikantham, 1973; Fuh, Hong, Lin, and Prussing, 1983), and the second is to assume that the periodic terms are small, and then to apply approximate analytical techniques such as averaging or perturbation (see Dimentberg, 1988; Ibrahim, 1985). These problems can also be handled via Markov vector approach and moment equations method (Ibrahim, 1985).

The difficulty in finding the response of general periodic systems to deterministic or stochastic excitation lies primarily in efficiently and accurately computing the fundamental

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matrix. Recently, a Chebyshev series solution technique has been developed by Sinha and associates (Sinha and Wu, 1991; Joseph, Pandiyan, and Sinha, 1993; Wu and Sinha, 1994) that accomplishes this goal. The technique does not depend upon the periodic term having a small parameter and yields solutions valid over the system's first period, in the form of Chebyshev polynomials. Solutions for future time periods can then be readily obtained from Floquet theory. In this study, it is proposed that the fundamental matrix, as calculated via the Chebyshev method, be used in the formulation of the response of periodic systems to external excitations. Once the fundamental matrix has been obtained, there are two distinct ways in which the response can be computed. In the first approach, called the *direct method*, the Chebyshev representation of the fundamental matrix, and its inverse, are inserted into the response expression. The excitation terms are also expanded into Chebyshev polynomials, allowing the response to be evaluated by taking advantage of the Chebyshev product and integration matrices. This approach is efficient and can be useful in many case.

The second approach proposed in this paper is termed the *Liapunov-Floquet* (L-F) *transformation method*. It is well-known that once the fundamental matrix has been obtained, it can be used to extract the L-F transformation matrix. The L-F matrix is used to transform the original periodic system to a time-invariant form. An efficient technique for extracting this transformation matrix has been reported by Pandiyan, Bibb, and Sinha, (1993); Sinha and Joseph, (1994); and Sinha and Pandiyan, (1994). Their technique uses the Chebyshev method mentioned above to compute the fundamental matrix over the first period and from this the L-F transformation matrix is computed and expressed in Fourier series. The periodic system is transformed into a time-invariant system, at which point well-known methods may be applied. However, the transformation does complicate the forcing vector; in the case of stochastic excitation, if the forces were originally stationary, they become cyclostationary. In the present study, the convolution integral is utilized to form an expression for the transient response. This expression can be evaluated through numerical integration, which is suitable for most forms of excitation, or through symbolic integration, which is limited to some special, but common, cases.

2. EQUATIONS OF MOTION

Consider the n degree-of-freedom, time-periodic, second-order system:

$$M\ddot{x} + C(t)\dot{x} + K(t)x = f(t) \tag{1}$$

where $C(t)$ and $K(t)$ are periodic with period T, and $f(t)$ is a deterministic or stochastic forcing vector. To use the Chebyshev polynomial technique, time t in equation (1) is normalized:

$$\begin{gathered} \tilde{t} = \tfrac{t}{T} \qquad \dot{x}(t) = \tfrac{\dot{x}(\tilde{t})}{T} \qquad \ddot{x}(t) = \tfrac{\ddot{x}(\tilde{t})}{T^2} \qquad \Rightarrow \\ \tfrac{M}{T^2}\ddot{x} + \tfrac{C(\tilde{t})}{T}\,\dot{x} + K(\tilde{t})x = \tilde{M}\ddot{x} + \tilde{C}(\tilde{t})\dot{x} + K(\tilde{t})x = f(\tilde{t}) \end{gathered} \tag{2}$$

or in the state-space form,

$$\dot{y} = A(\tilde{t})y + \begin{Bmatrix} 0 \\ \tilde{M}^{-1}f(\tilde{t}) \end{Bmatrix};$$

$$A(\tilde{t}) = A(\tilde{t}+1) = \begin{bmatrix} 0 & I \\ -\tilde{M}^{-1}K(\tilde{t}) & -\tilde{M}^{-1}\tilde{C}(\tilde{t}) \end{bmatrix}, \tag{3}$$

$$y(\tilde{t}) = \begin{Bmatrix} x(t) \\ T\dot{x}(t) \end{Bmatrix}.$$

3. SOLUTION OF LINEAR TIME-PERIODIC SYSTEMS VIA CHEBYSHEV POLYNOMIALS

Following Sinha and Wu (1991), the solution vector y, and the fundamental matrix $\Psi(\tilde{t})$ and its inverse, are calculated using the Chebyshev series solution technique. In this technique, the solution vector $y(\tilde{t})$ and the periodic matrix $A(\tilde{t})$ are expanded over the interval [0, 1] in terms of the shifted Chebyshev polynomials, as shown below.

$$\begin{aligned} y_i(\tilde{t}) &\approx \sum_{r=0}^{m-1} b_r^i s_r^*(\tilde{t}) \equiv s^{*^T}(\tilde{t})b^i, \quad i = 1,2,........,2n \\ A_{ij}(\tilde{t}) &\approx \sum_{r=0}^{m-1} d_r^{ij} s_r^*(\tilde{t}) \equiv s^{*^T}(\tilde{t})d^{ij}, \quad i,j = 1,2,...,2n , \end{aligned} \tag{4}$$

where

$$\begin{aligned} b^i &= \left\{b_0^i\, b_1^i \,...\, b_{m-1}^i\right\}^T \quad d^{ij} = \left\{d_0^{ij}\, d_1^{ij} \,...\, d_{m-1}^{ij}\right\}^T \\ s^{*^T}(\tilde{t}) &= \left\{s_0^*(\tilde{t})\, s_1^*(\tilde{t}) \,...\, s_{m-1}^*(\tilde{t})\right\}^T . \end{aligned} \tag{5}$$

Here, b_r^i are unknown expansion coefficients of $y_i(\tilde{t})$, d_r^{ij} are known expansion coefficients of A_{ij}, and $s_r^*(\tilde{t})$ are the shifted Chebyshev polynomials of the first kind. For convenience in algebraic manipulation, a $2n\times$ $2nm$ Chebyshev polynomial matrix is defined as

$$\hat{S}^*(\tilde{t}) = I_{2n\times 2n} \otimes s^{*^T}(\tilde{t}) \tag{6}$$

where $\otimes$ represents the Kronecker product (Bellman, 1970, see Appendix B), and $I_{2n\times 2n}$ is the $2n \times 2n$ identity matrix. Using the definitions given by equations (5) and (6), $y(\tilde{t})$ and $A(\tilde{t})$ can be rewritten as

$$y(\tilde{t}) = \hat{S}^*(\tilde{t})\bar{b} \quad A(\tilde{t}) = \hat{S}^*(\tilde{t})D \quad A(\tilde{t})y(\tilde{t}) = \hat{S}^*(\tilde{t})Q_D\bar{b} \tag{7}$$

where $\overline{b} = \{\{b^1\}^T\{b^2\}^T ... \{b^{2n}\}^T\}^T$ is a $2nm \times 1$ vector, D = $[d^{i1}\ d^{i2}\ ...\ d^{i2n}]$, $i = 1, 2, ..., 2n$ is a $2nm \times 2n$ matrix, and Q_D is the $2nm \times 2nm$ product operation matrix associated with D (see Appendix A).

The integral form of the homogeneous part of equation (3) is

$$y(\tilde{t}) - y(0) = \int_0^{\tilde{t}} A(\xi) y(\xi) d\xi \tag{8}$$

where ξ is a dummy variable. Substituting equation (7) into equation (8), and then following the approach of Sinha and Wu (1991), the initial condition is written as $y(0) = \hat{S}^*(\tilde{t})\overline{y}_0$, and one can obtain a set of linear algebraic equations of the form:

$$\left[I_{2nm \times 2nm} - \hat{Z}\right]\overline{b} = \overline{y}_0 \; ; \quad \hat{Z} = \overline{G} Q_D \, . \tag{9}$$

$\overline{G}$ is given by

$$\left[\overline{G}\right] = I_{2n \times 2n} \otimes [G], \tag{10}$$

where $[G]$ is the $m \times m$ Chebyshev integration matrix (see Appendix A). Once the b^i are obtained from equation (9), the solution for $y(\tilde{t})$ is given by equation (4), and is simply a convergent power series in $\tilde{t}$.

To form the fundamental matrix $\Psi(\tilde{t})$ associated with the system, one must obtain a set of solutions $y_i(\tilde{t})$, i = 1, 2, ..., 2n, corresponding to the $2n$ initial conditions: $y(0) = \{1, 0, ..., 0\}, \{0, 1, 0, ..., 0\}, ..., \{0, ..., 0, 1\}$. Let the coefficient vector associated with the i^{th} initial condition be denoted $\overline{b}_i$. Then the fundamental matrix is given by

$$\Psi(\tilde{t}) = \hat{S}^*(\tilde{t})\overline{B} = \overline{B'}[\hat{S}^*(\tilde{t})]^T \tag{11}$$

where $\overline{B} = [\overline{b}_1\ \overline{b}_2\ ...\ \overline{b}_{2n}]$. $\overline{B'}$ is obtained by rotating the m-long column vectors, which make up $\overline{B}$ into row vectors. Note that $\Psi(0) = I_{2n \times 2n}$. The above expression for the fundamental matrix is only valid over $0 \leq \tilde{t} \leq 1$, because this is the interval over which the shifted Chebyshev polynomials are defined. The inverse of the fundamental matrix, $\Psi^{-1}(\tilde{t})$, is known to be the transpose of the fundamental matrix of the adjoint system (Yakubovich and Starzhinskii, 1975; Sinha and Joseph, 1994):

$$\xi = -A^T(\tilde{t})\xi \tag{12}$$

and can be obtained in the same manner as described for $\Psi(\tilde{t})$. The inverse of the fundamental matrix is expressed as:

$$\Psi^{-1}(\tilde{t}) = \hat{S}^*(\tilde{t})\overline{R} = \overline{R'}[\hat{S}^*(\tilde{t})]^T \, . \tag{13}$$

$\overline{R'}$ is obtained by rotating the m-long column vectors, which make up $\overline{R}$ into row vectors.

4. RESPONSE BY DIRECT METHOD

In the *direct method*, the response expressions are derived via the convolution integral technique. Functions appearing in the response expressions are written in terms of Chebyshev polynomials, and the expressions are then evaluated by using the Chebyshev product and integration matrices.

4.1. Deterministic Response

With the fundamental solution matrix and its inverse known, the convolution integral is used to formulated the response $y(\tilde{t})$ due to the forcing vector f. The response is given by

$$y(\tilde{t}) = y(\eta + N) = \Psi(\eta)y(N) + \Psi(\eta)\int_0^\eta \Psi^{-1}(s)\left\{\begin{matrix} 0 \\ \tilde{M}^{-1}f(s+N)\end{matrix}\right\} ds, \tag{14}$$

where N is an integer, and $0 \le \eta \le 1$. To illustrate the direct method of evaluating response expressions, the integral in equation (14) is considered. The forcing vector $f(\tilde{t})$ is expanded over $N \le \tilde{t} \le N+1$,

$$f(\tilde{t}) = f(s+N) = [\hat{S}^*(s)]^T\{c\}\,, \tag{15}$$

where $\{c\}$ is the coefficient vector associated with $f(\tilde{t})$ and $[\Psi^{-1}(s)]$ is expressed as shown in equation (13). The integral is then evaluated as follows:

$$\begin{aligned}
&\int_0^\eta \Psi^{-1}(s)\left\{\begin{matrix} 0 \\ \tilde{M}^{-1}f(s+N)\end{matrix}\right\} ds \\
&= \int_0^\eta [\hat{S}^*(s)][\overline{R}]\left\{\begin{matrix} 0_n \\ \tilde{M}^{-1}[\hat{S}^*_{n\times n}(s)]\{c\}\end{matrix}\right\} ds \\
&= \int_0^\eta [\hat{S}^*(s)][\overline{R}]\begin{bmatrix} 0 & 0 \\ 0 & \tilde{M}^{-1}\end{bmatrix}\left\{\begin{matrix} [\hat{S}^*_{n\times n}(s)]0_{nm} \\ [\hat{S}^*_{n\times n}(s)]\{c\}\end{matrix}\right\} ds \\
&= \int_0^\eta [\hat{S}^*(s)]\Phi[\hat{S}^*(s)]\left\{\begin{matrix} 0_{nm} \\ \{c\}\end{matrix}\right\} ds \\
&= \int_0^\eta \Phi'[\hat{S}^*(s)]^T[\hat{S}^*(s)]\left\{\begin{matrix} 0_{nm} \\ \{c\}\end{matrix}\right\} ds \\
&= \int_0^\eta [\hat{S}^*(s)]Q_{\Phi'}\left\{\begin{matrix} 0_{nm} \\ \{c\}\end{matrix}\right\} ds \\
&= [\hat{S}^*(\eta)]\overline{G}^T Q_{\Phi'}\left\{\begin{matrix} 0_{nm} \\ \{c\}\end{matrix}\right\} = [\hat{S}^*(\eta)]\{h\},
\end{aligned} \tag{16}$$

where $\{h\}$ is the resulting coefficient vector for the integral, $\overline{G}$ is given by equation (10), and $Q_{\Phi'}$ is the assembly of Chebyshev product matrices associated with the m-long row vectors of Φ' (see Appendix A). The prime in $[\Phi']$ carries the same meaning as described for equation (11). $\hat{S}^*_{n\times n}(\eta)$ is given by $I_{n\times n}\otimes s^{*^T}(\eta)$, whereas $\hat{S}^*(\eta)$ is given by equation (6).

For brevity, no examples are presented in this section. However, Example 2 in Section 6.2.3 considers the effect of a non-zero mean value for the forcing vector. This is equivalent to a deterministic force being superimposed on a zero-mean stochastic excitation.

4.2. Stochastic Response

Consider the case in which the forcing vector in equation (1) is stochastic and can be written as:

$$f(t) = m_f(t) + f_0(t), \tag{17}$$

where $m_f(t)$ (an explicit function of time) is the mean of f, and $f_0(t)$ is a zero-mean random process. Because the system is linear, its response to these two components of the force can be considered separately. The mean of the response, $m_y(t)$, can be determined from the mean of the excitation, $m_f(t)$, using the approach described for deterministic excitation. The portion of the mean-square response that is due to $m_f(t)$ can be easily computed by taking the outer product of $m_y(t)$. Throughout the rest of this section, only the mean-square response due to $f_0(t)$ is considered. This zero-mean excitation is assumed to have the form:

$$f_0(\tilde{t}) = \{p_1(\tilde{t})g_1(\tilde{t})......p_n(\tilde{t})g_n(\tilde{t})\}^T, \tag{18}$$

where p_i and g_i are modulation functions and zero-mean stationary noise, respectively.

The mean square response $R^y(\tilde{t},\tilde{t}) = E[y(\tilde{t})y^T(\tilde{t})]$ is given by:

$$\begin{aligned}
R^y(\tilde{t},\tilde{t}) &= R^y(\eta+N,\eta+N) = \Psi(\eta)\left[\sum_{i=1}^{4} term_i\right]\Psi^T(\eta)\\
term_1 &= R^y(N,N)\\
\left.\begin{matrix} term_2 \\ term_3^T \end{matrix}\right\} &= \int_0^{\eta}\begin{bmatrix} 0_{n\times 2n} \\ \tilde{M}^{-1}E[f_0(s+N)y^T(N)]\end{bmatrix}^T [\Psi^{-1}(s)]^T ds \\
term_4 &= \int_0^{\eta}[\Psi^{-1}(s_1)] \\
&\quad \int_0^{\eta}\begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \tilde{M}^{-1}R^{f_0}(s_1+N,s_2+N)[\tilde{M}^{-1}]^T\end{bmatrix}[\Psi^{-1}(s_2)]^T ds_2 ds_1
\end{aligned} \tag{19}$$

where,

$$\begin{aligned}
E[y(N)f_0^T(\eta+N)] &= \Psi(1)E[y(N-1)f_0^T(\eta+N)] \\
&+ \Psi(1)\int_0^1 \Psi^{-1}(s)\begin{bmatrix} 0_{n\times n} \\ \tilde{M}^{-1}R^{f_0}(s+N-1,\eta+N)\end{bmatrix} ds.
\end{aligned} \tag{20}$$

$Term_1$ is the initial condition of the stochastic response. $Term_2$, $term_3$, and $term_4$ depend on the autocorrelation of the forcing vector, R^{f_0}, and consequently must be evaluated according to the type of excitation being considered. Equation (19) can be evaluated numerically as long as all elements of R^{f_0} are well-behaved. Alternatively, all functions can be expanded into Chebyshev polynomials, which allows us to evaluate the expression by using the Chebyshev product and integration matrices, as demonstrated for the case of deterministic excitation. Following this approach, for any interval of time $N \leq \tilde{t} \leq N+1$, the right-hand side of equation (19) can be reduced to the following form:

$$[\hat{S}^*(\eta)][\overline{B}]\left([\hat{S}^*(\eta)]([H_1]+[H_2]+[H_3]+[H_4])\right)[\overline{B}]^T[\hat{S}^*(\eta)]^T, \tag{21}$$

where $\eta = \tilde{t} - N$, and

$$term_k = [\hat{S}^*(\eta)]^T[H_k] \ ; \quad k = 1,2,3,4. \tag{22}$$

The $[H_k]$ are coefficient matrices resulting from integration and product operations and are obtained through procedures similar to the one demonstrated by equation (16) and to those reported by Sinha and associates (Sinha and Wu, 1991; Joseph et al., 1993; Wu and Sinha, 1994). Note, however, that the result of equation (16) is in terms of a coefficient vector, whereas the results of $term_1$ through $term_4$ are in terms of coefficient matrices.

4.2.1. Modulated white noise. In this section the above procedure is applied to determine the mean square response $R^y(\tilde{t},\tilde{t})$ due to a modulated white noise excitation. The elements of the autocorrelation matrix $R^{f_0}(\tilde{t}_1,\tilde{t}_2)$ are assumed to be of the form:

$$R^{f_0}_{ij}(\tilde{t}_1,\tilde{t}_2) = R_{ij}\, p_i(\tilde{t}_1)\delta(\tilde{t}_1-\tilde{t}_2)p_j(\tilde{t}_2); \ i,j = 1,2,...,n. \tag{23}$$

With R^{f_0} in this form, it can be seen that $term_2$ and $term_3$ in equation (19) are equal to *zero*, and $term_4$ is the only remaining term that is affected by the autocorrelation matrix of the zero-mean input. Note that, in general, the delta function in equation (23) could take the form $\delta(t_1 - t_2 + c_{ij})$. Allowing for such shifts in the discontinuity of the force's autocorrelation significantly complicates the development of a general solution routine, and so this case is not considered in the present analysis. By the properties of the delta function, $term_4$ in equation (19) becomes:

$$term_4 = \int_0^\eta \Psi^{-1}(s_1)\begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \tilde{M}^{-1}P(s_1+N)[\tilde{M}^{-1}]^T \end{bmatrix}[\Psi^{-1}(s_1)]^T ds_1 \tag{24}$$

where

$$P_{ij}(s_1+N) = R_{ij}\, p_i(s_1+N)\, p_j(s_1+N).$$

The functions $p_k(s_1+N), k = 1,2,...,n$, are expanded into Chebyshev polynomials, and $term_4$ can then be evaluated by using the Chebyshev product and integration matrices. The form of the result is given by equation (22). Note that this expression is only valid over $N \leq \tilde{t} \leq N+1$. In performing these calculations, N is initially taken to be zero. Equation (19) can then be used to obtain the mean square response as one steps through time.

4.2.2. Separable correlation functions. In this section, excitations that give rise to separable correlation functions are considered. The zero-mean forcing vector is again taken to be in the form of equation (18) and each element of $R^{fo}(\tilde{t}_1, \tilde{t}_2)$, the autocorrelation matrix of the zero-mean process, has the form:

$$R_{ij}^{fo}(\tilde{t}_1, \tilde{t}_2) = p_i(\tilde{t}_1)\, R_{ij}^{g}(\tilde{t}_1 - N, \tilde{t}_2 - N)\, p_j(\tilde{t}_2), \tag{25}$$

where we have used the property that g is stationary. We assume that the random functions g_i have separable correlation functions. That is, each element R_{ij}^g of the autocorrelation matrix of g can be expressed as:

$$R_{ij}^{g}(\tilde{t}_1, \tilde{t}_2) = \begin{cases} \sum\limits_k v_{ijk}(\tilde{t}_1 - N) w_{ijk}(\tilde{t}_2 - N)\ ; & \tilde{t}_2 \geq \tilde{t}_1 \\ \sum\limits_k w_{ijk}(\tilde{t}_1 - N) v_{ijk}(\tilde{t}_2 - N)\ ; & \tilde{t}_1 \geq \tilde{t}_2 \end{cases}. \tag{26}$$

One example of such a function is the following, which is commonly used for description of band-limited excitation:

$$R_{ij}^{g}(\tilde{t}_1, \tilde{t}_2) = R_{ij} \exp(-\mu_{ij}\Omega_{ij}\,|\tilde{t}_1 - \tilde{t}_2|) \cos(\Omega_{ij}\,|\tilde{t}_1 - \tilde{t}_2|);\ i, j = 1, 2, ..., n. \tag{27}$$

As indicated by equation (26), the definition of R_{ij}^g is assumed to have a discontinuity in its definition at $\tilde{t}_1 = \tilde{t}_2$. Note that, in general, the exponential function in equation (27) could take the form $\exp(-\mu_{ij}\Omega_{ij}\,|\tilde{t}_1 - \tilde{t}_2 + c_{ij}|)$. However, for simplicity, this case is not considered in the present analysis.

We now consider the evaluation of $term_2$, $term_3$, and $term_4$. N is initially taken as *zero*, and the initial conditions are taken to be uncorrelated with the excitation. Equation (19) can then be used to obtain the mean square response as one steps through time.

Because the definition of $R^{fo}(s_1, s_2)$ has a discontinuity at $s_1 = s_2$, $term_4$ in equation (19) is evaluated as:

$$term_4 =$$

$$\int_0^{\eta} \Psi^{-1}(s_1) \Bigg(\int_0^{s_1} \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \tilde{M}^{-1} R^{fo(1)}(s_1 + N, s_2 + N)[\tilde{M}^{-1}]^T \end{bmatrix} [\Psi^{-1}(s_2)]^T ds_2$$

$$+ \int_{s_1}^{\eta} \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \tilde{M}^{-1} R^{fo(2)}(s_1 + N, s_2 + N)[\tilde{M}^{-1}]^T \end{bmatrix} [\Psi^{-1}(s_2)]^T ds_2 \Bigg) ds_1, \tag{28}$$

where $R^{f(2)}(t_1, t_2)$ is valid for $t_2 \geq t_1$, and $R^{f(1)}(t_1, t_2)$ is valid for $t_1 \geq t_2$. The $w(s_2)$, $v(s_2)$, and $p(s_2 + N)$ functions are expanded into Chebeyshev polynomials and the inner integrals are evaluated by using the Chebyshev product and integration matrices, while the v functions are carried along as constants. Next, the $w(s_1)$, $v(s_1)$, and $p(s_1 + N)$ functions are expanded into Chebyshev polynomials, and the outer integral is evaluated in the same manner. Note that the arguments of the v and w functions are always between 0 and 1, and therefore in evaluating $term_4$, these functions do not need to be reexpanded as one steps through time. Next we consider the evaluation of $term_2$ and $term_3$.

To evaluate $term_2$ and $term_3$, equation (20) must first be evaluated. The $p(s+N-1)$ and $v(s-1)$ functions are expanded into Chebyshev polynomials, while the $w(\eta)$ and

$p(\eta+N)$ functions are carried along as constants, and the integral is evaluated as before. This result is then pre-multiplied by $\Psi(1)$ and added to $E[y(N-1)f_0^T(\tilde{t})]$. The resulting expression, for a single-degree-of-freedom system, has the form:

$$
\begin{aligned}
E[y(N)f_0(\tilde{t})] &= \Psi^N(1)\left\{\begin{matrix}(d_1)w(\tilde{t}-1)p(\tilde{t})\\ (d_2)w(\tilde{t}-1)p(\tilde{t})\end{matrix}\right\} + \ldots\ldots \\
&+ \Psi(1)\left\{\begin{matrix}(d_{2N-1})w(\tilde{t}-N)p(\tilde{t})\\ (d_{2N})w(\tilde{t}-N)p(\tilde{t})\end{matrix}\right\}
\end{aligned}
\tag{29}
$$

where the d_k are constants, and η has been replaced with $\tilde{t}-N$. This expression is then inserted into $term_3$ of equation (19), replacing $\tilde{t}$ with s+N. The w and p functions in equation (29) are expanded into Chebyshev polynomials, and the expression is evaluated using the Chebyshev operational matrices. Notice that expansions of the $v(\tau)$ functions are needed over $-1 \le \tau \le 0$, the $w(\tau)$functions must be expanded over the periods $0 \le \tau \le 1$, $1 \le \tau \le 2$, ... , $N-1 \le \tau \le N$, and the p(τ) functions are expanded over $N-1 \le \tau \le N$ and $N \le \tau \le N+1$.

The results of these calculations yield expressions in the form of equation (22). Once again, $R^y(\tilde{t},\tilde{t})$ is valid for $N \le \tilde{t} \le N+1$.

4.2.3. Illustrative examples. Fortran programs have been written that use the *direct method* to calculate the mean square response matrix for any system that can be described in the form of equation (1). The type of excitation is limited to one whose zero-mean process is, (a) a modulated white noise, or (b) has a separable autocorrelation matrix. Although the code has been written to handle multi-degree-of-freedom problems, all of the examples included here are single-degree-of-freedom (SDF) systems. In addition, only the mean square of the displacement response, $E[x(t)x(t)]$, has been included in the graphs, because this is typically the quantity of greatest interest. The initial conditions were set to zero, and it was observed that, at the most, twenty term Chebyshev expansions ($m = 20$) provided excellent convergence.

Example 1

This example has been chosen so that the results can be compared to those obtained by Bucciarelli and Kuo (1970), who applied an approximate analytical technique valid for small ε (see equation [31]). Their paper presents results for an SDF second-order system having constant coefficients and low damping, which is subjected to either white noise or narrow band excitation. The equation of motion is

$$
\begin{gathered}
0.1\ddot{x} + 1.26\dot{x} + 395(1 + a\sin(0.8\omega_n t))x = f(t) \\
\omega_n = 20\pi\ ;\ \omega_d = \omega_n\sqrt{1-\zeta^2}\ ;\ \zeta = 0.1\ \text{(damping ratio)}.
\end{gathered}
\tag{30}
$$

Figure 1 shows the system response to zero-mean narrow band excitation. The autocorrelation of the excitation is taken as:

$$
\begin{gathered}
R^{f_0}(t_1,t_2) = R_0\exp(-\varepsilon\mu\Omega\,|t_1-t_2|)\cos(\Omega\,|t_1-t_2|) \\
R_0 = 6172; \varepsilon = \zeta\left(\frac{\omega_n}{\omega_d}\right) = 0.1005; \Omega = \begin{matrix}\text{frequency about which}\\ \text{the excitation is centered}\end{matrix} = \omega_d(1+\varepsilon\beta)
\end{gathered}
\tag{31}
$$

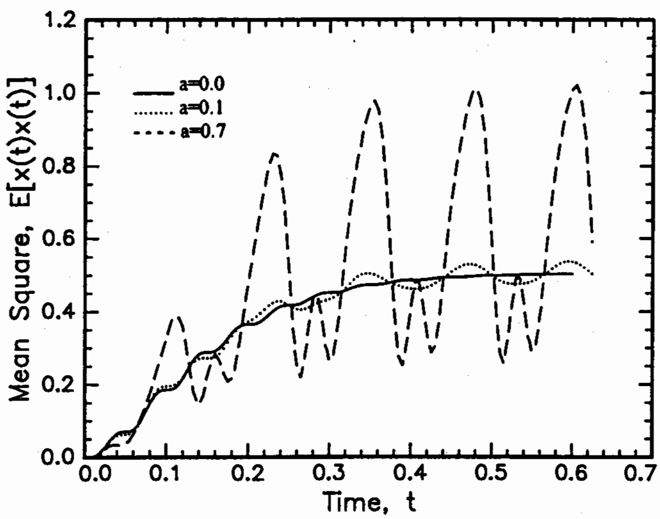

Figure 1. Equation (30) with narrow band excitation centered at ω_d. $\mu = 1.0$, $\beta = 0.0$ (period T=0.125).

For the case $\mu = 1.0, \beta = 0.0, a = 0$, the present analysis agrees very well with the results of Bucciarelli and Kuo (1970). In both analyses, the mean square response reaches a steady state value of 0.5, and this value is approached at the same rate and in the same manner, except that the present analysis shows small oscillations in the transient response, whereas Bucciarelli and Kuo (1970) does not.

Although not included among the figures, the case $\beta = 10, a = 0$, and $\mu = 1.0$ *or* 0.1 have been found to differ substantially from the results of Bucciarelli and Kuo (1970). For $\mu = 1.0$, the steady state predictions are 0.0246 versus 0.019. For $\mu = 0.1$ the steady state values are 0.00663 versus 0.0091. The transient response for $\mu = 1.0$ has been found to differ qualitatively as well. From the present analysis, the first transient peak is below the steady state value, whereas Bucciarelli and Kuo (1970) show the first peak to be substantially greater than the steady state value. For the most part, these differences can be accounted for by the two terms that these authors neglected on the grounds that each is of order ε when compared with the remaining term in their response expression.

Figure 2 shows the system response to modulated white noise with a zero mean. The autocorrelation of the excitation is:

$$\begin{aligned} R^{f_o}(t_1, t_2) &= p(t_1) 491.1\, \delta(t_1 - t_2)\, p(t_2) \\ p(t) &= \sin(\varepsilon\omega_d t), \quad \text{for } 0 \leq \varepsilon\omega_d t \leq \pi \\ p(t) &= 0, \qquad \text{for } \varepsilon\omega_d t \geq \pi \end{aligned} \tag{32}$$

where again $\varepsilon = .1005$. The plot corresponding to $a = 0$ is identical to the one shown in Bucciarelli and Kuo (1970). Both reach a peak value of approximately 0.42, and at the same instant of time.

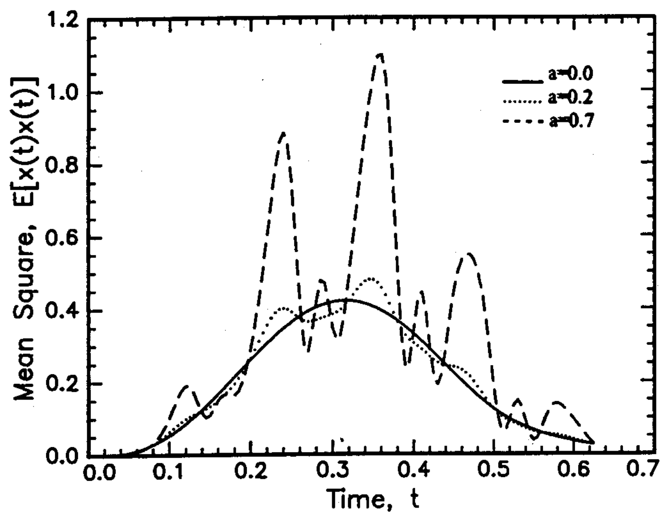

Figure 2. Equation (30) with modulated white noise excitation. The modulation function is a half sine pulse lasting for $0 \leq t \leq 0.5$.

Example 2

The second example is a *teetering rotor* subjected to modulated white noise having, (a) a zero mean, and (b) a non-zero mean. The equation of motion is (Johnson, 1974):

$$\begin{gathered} \ddot{x} + 0.125\pi\gamma\,\dot{x} + \pi^2(\nu^2 + 0.125\gamma\,\mu^2\sin(2\pi t))x = f(t) \\ \gamma = 5.0; \nu = 1.06\,. \end{gathered} \tag{33}$$

μ is known as the *advance ratio*. The excitation statistics and initial conditions are

$$\begin{aligned} R^{f_0}(t_1, t_2) &= p(t_1)\,27.19\,\delta(t_1 - t_2)\,p(t_2) \\ p(t) &= \sin(0.5\pi t) \;\; \text{for } 0 \leq t \leq 2 \\ p(t) &= 0 \;\; \text{for } t \geq 2 \\ y(0) &= \{0, 0\}^T. \end{aligned} \tag{34}$$

For the first case $m_f(t) = 0$, while for the second case $m_f(t) = 10\,p(t)\,\sin(2\pi t)$. Figure 3 shows the effect of $m_f(t)$ for two advance ratios.

Example 3

For this case the equation of motion has no constant stiffness term. It is to be noted that for this case "averaging" or any other small parameter method **cannot be applied**. The

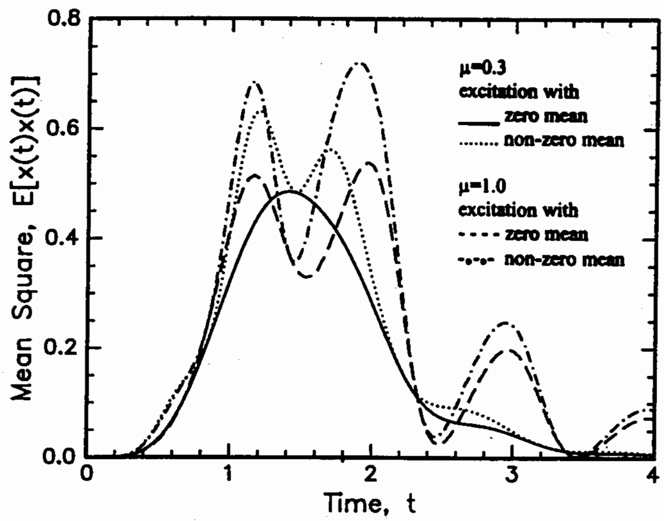

Figure 3. Teetering rotor with white noise excitation, modulated by a half sine pulse lasting for $0 \leq t \leq 2$ (T=1.0).

equation of motion is:

$$0.1\ddot{x} + 0.01257\dot{x} + (0.12\cos(2t))x = f(t). \tag{35}$$

The system response is plotted in Figure 4 for two excitations. The first is a modulated white noise given by:

$$\begin{aligned} R^{fo}(t_1, t_2) &= p(t_1)\, 0.04911\, \delta(t_1 - t_2)\, p(t_2) \\ p(t) &= \sin(0.2\pi t), \quad \text{for } 0 \leq t \leq 5 \\ p(t) &= 0, \quad \text{for } t \geq 5\,. \end{aligned} \tag{36}$$

The second is a narrow-band excitation with autocorrelation:

$$\begin{aligned} R^{fo}(t_1, t_2) = R_0 \exp(-\varepsilon\mu\Omega\, |t_1 - t_2|) \cos(\Omega\, |t_1 - t_2|) \\ R_0 = 0.6172\ ;\ \varepsilon = 0.1005\ ;\ \mu = 1.0\ ;\ \Omega = 6.252. \end{aligned} \tag{37}$$

5. COMPUTATION OF THE L-F TRANSFORMATION MATRIX AND ITS INVERSE

Given the fundamental matrix $\Psi(\tilde{t})$ of the time-periodic system described by equation (3), the real Liapunov-Floquet (L-F) transformation matrix $Q(\tilde{t}) = Q(\tilde{t} + 2)$ is computed as

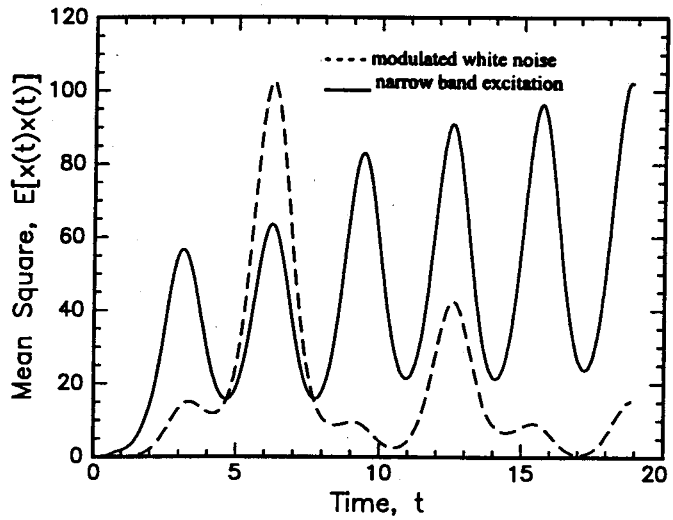

Figure 4. Response from equation (35), with excitations given by equations (36) and (37) (T = π).

follows. By Liapunov-Floquet theory, $\Psi(\tilde{t})$ can be factored as:

$$\Psi(\tilde{t}) = Q(\tilde{t})\exp(\overline{A}\tilde{t}). \tag{38}$$

Because $\Psi(0) = I$, and $Q(\tilde{t})$ has period 2, it can be easily shown that $Q(0) = Q(2) = I$. By Floquet theory, and equation (38),

$$\Psi(2) = \Psi^2(1) = Q(2)\exp(2\overline{A}) = \exp(2\overline{A}). \tag{39}$$

The matrix $\overline{A}$ is computed from equation (39), and $Q(t)$ is then obtained by rearranging equation (38) as

$$Q(\tilde{t}) = \Psi(\tilde{t})\exp(-\overline{A}\tilde{t}), \tag{40}$$

and the inverse is

$$Q^{-1}(\tilde{t}) = \exp(\overline{A}\tilde{t})\Psi^{-1}(\tilde{t}). \tag{41}$$

However, one cannot directly invert $\Psi^{-1}(\tilde{t})$. As discussed in Section 3, $\Psi^{-1}(\tilde{t})$ is obtained by solving the adjoint equation (12). The expressions above are all given in terms of normalized time $\tilde{t}$. This is done to facilitate the use of Chebyshev polynomials. However, when evaluating the response of a system to external excitation, it is more convenient to work in real time. The L-F transformation matrix is converted to real time as follows:

$$Q(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} = \begin{bmatrix} Q_{11}(\tilde{t}) & TQ_{12}(\tilde{t}) \\ \frac{1}{T}Q_{21}(\tilde{t}) & Q_{22}(\tilde{t}) \end{bmatrix}, \tag{42}$$

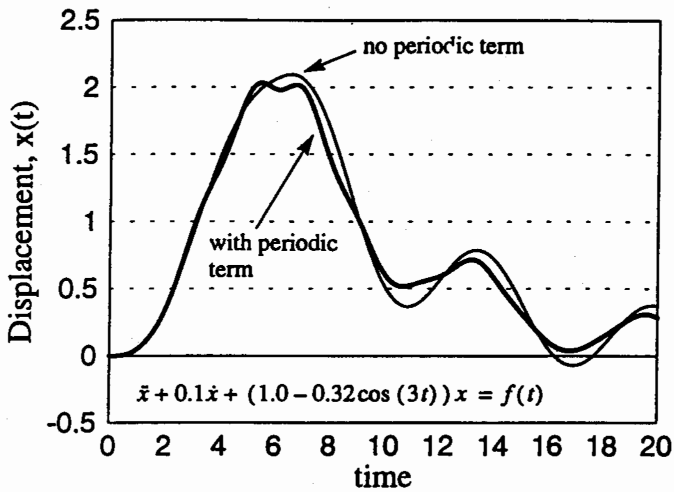

Figure 5. Response from equation (52), with excitation f_1 (equation 53).

where the $Q_{ij}, ij = 1, 2$, are the $n \times n$ submatrices of Q. The inverse of the L-F transformation matrix is converted to real time in exactly the same way. Now the time-periodic system described by equation (1) can be converted to a time-invariant form, in real time.

Following Sinha and Pandiyan (1994), because $Q(t)$ and $Q^{-1}(t)$ are periodic, the elements of these matrices are expressed in a finite Fourier series. The transformation $y(t) = Q(t)z(t)$ in the state-space form of equation (1) yields:

$$\dot{z} = \overline{A}z + Q^{-1}(t) \begin{Bmatrix} 0_n \\ M^{-1}f(t) \end{Bmatrix}^T, \tag{43}$$

where $\overline{A}$ is a real constant matrix.

6. RESPONSE BY LIAPUNOV-FLOQUET TRANSFORMATION METHOD

In the L-F transformation method, response expressions are formulated for the time-invariant system. The fundamental matrix of the linear system given by equation (43) is simply $\exp(\overline{A}t)$, and its inverse is $\exp(-\overline{A}t)$. Unlike the Chebyshev polynomial representation of the fundamental matrix of the periodic system, these matrices are valid for all time.

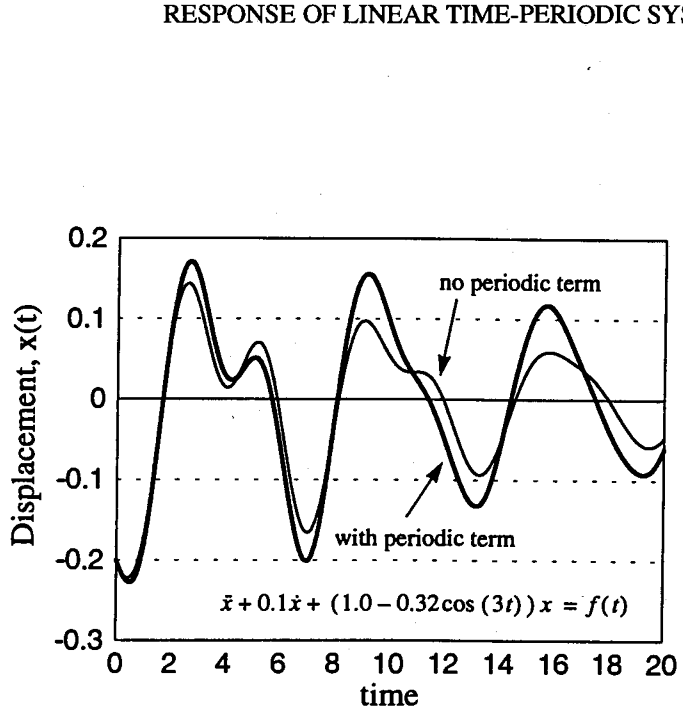

Figure 6. Response from equation (52), with excitation f_2 (equation 53).

6.1. Deterministic Response

The convolution integral is once again used to formulate the response $z(t)$ in equation (43) to an arbitrary deterministic excitation $f(t)$. Note that because $\mathcal{Q}(0) = I$, $z(0) = y(0)$, and the response is given by

$$z(t) = \exp[\overline{A}t]z(0) + \exp[\overline{A}t] \int_0^t \exp[-\overline{A}s]\mathcal{Q}^{-1}(s) \left\{ \begin{array}{c} 0 \\ \tilde{M}^{-1}f(s) \end{array} \right\} ds. \tag{44}$$

The response y(t) is then obtained from the transformation $y = \mathcal{Q}(t)z$. Considering the analytical solvability of equation (44), we note the following. The fundamental matrix of the constant system, $\exp[\overline{A}t]$, and its inverse $\exp[-\overline{A}t]$, are made up of terms involving *sine, cosine,* and *exponential* functions. The L-F transformation, $\mathcal{Q}(t)$, and its inverse are written in terms of Fourier series. Therefore, equation (44) can be evaluated analytically for many types of forcing functions, including sinusoids, exponential pulses, and polynomials. The expressions do become lengthy as the dimensions of the system increase, and as the force becomes more complex, so a symbolic mathematical software, such as *Macsyma, Mathematica, Maple,* and so on proves to be very helpful.

6.1.1. Illustrative examples.

Example 1

To provide a simple example of the symbolic analysis of equation (44), the linear part of a non-linear system considered by Sinha and Pandiyan (1994) has been selected. The equation of motion is

$$\dot{x} = \begin{bmatrix} -1+\alpha\cos^2 t & 1-\alpha \sin t \cos t \\ -1-\alpha \sin t \cos t & -1+\alpha \sin^2 t \end{bmatrix} x + \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}. \tag{45}$$

This is a so-called commutative system and the L-F transformation is of a very simple form (see Lukes, 1982) :

$$Q(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{46}$$

and, in this case, $Q^{-1}(t) = Q^T(t)$. Applying the transformation $x = Q(t)z$ yields:

$$\dot{z} = \overline{A}z + Q^{-1}(t)\begin{Bmatrix} 0 \\ f(t) \end{Bmatrix} \quad \text{where } \overline{A} = \begin{bmatrix} \alpha-1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{47}$$

If the excitation is taken as

$$f(t) = t, \tag{48}$$

a unit ramp input, then the integral in equation (44) becomes:

$$\int_0^t \begin{bmatrix} e^{-(\alpha-1)s} & 0 \\ 0 & e^s \end{bmatrix} \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix} \begin{Bmatrix} 0 \\ s \end{Bmatrix} ds = \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}, \tag{49}$$

where

$$\begin{aligned} U_1 &= \left[\frac{-se^{-(\alpha-1)s}}{(\alpha-1)^2+1}(-(\alpha-1)\sin(s) - \cos(s)) \right. \\ &+ \left. \frac{e^{-(\alpha-1)s}}{((\alpha-1)^2+1)^2}(((\alpha-1)^2-1)\sin(s) + 2(\alpha-1)\cos(s))\right]_0^t \\ U_2 &= \left[\frac{se^s}{2}(-(\alpha-1)\cos(s) - \sin(s)) - \frac{e^s}{2^2} 2\sin(s)\right]_0^t. \end{aligned} \tag{50}$$

Therefore, the response $x(t)$ of the system is:

$$x(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} e^{(\alpha-1)t} & 0 \\ 0 & e^{-t} \end{bmatrix} \left(\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} + \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}\right). \tag{51}$$

Example 2

As a more practical example, the following system is investigated:

$$\ddot{x} + 0.1\dot{x} + (1.0 - 0.32\cos(3t))x = f(t). \tag{52}$$

Results are presented for each of the following deterministic forcing functions:

$$\begin{aligned} f_1(t) &= \begin{cases} F_0 \frac{t}{t_d} \; ; \; 0 \le t \le t_d \\ F_0 \exp(-at^*) \quad ; \; t = t^* + t_d > t_d \end{cases} \\ f_2(t) &= F_0(\exp(-at) - \exp(-bt))\sin(\Omega t) . \end{aligned} \tag{53}$$

The response of the system is computed and compared with that of the constant system obtained by dropping the periodic term. Using *Mathematica*, equation (44) has been evaluated symbolically for each of these forces, and then the various parameters replaced with numerical values to obtain the plots. Figure 5 shows the response to f_1 with $F_0 = 2, t_d = 6, a = 0.2$, and $y(0) = 0$. Figure 6 shows the response to f_2 with $F_0 = 4, a = 0.25, b = 0.3, \Omega = 2$, and $y(0) = \{-0.2, -0.1\}^T$.

6.2. Stochastic Response

We consider again the case in which the forcing vector in equation (1) is stochastic and can be written in the form of equation (17). As mentioned in Section 4.2, the system response to the two components of the force can be considered separately. The mean of the response, $m_y(t)$, can be determined from the mean of the excitation, $m_f(t)$, using the approach described above for deterministic excitation. The autocorrelation response due to $m_f(t)$ can be easily computed by taking the outer product of $m_y(t_1)$ and $m_y(t_2)$. Throughout the rest of this section, only the autocorrelation response due to $f_0(t)$ is considered.

Using equation (44), and taking the initial conditions to be uncorrelated with the excitation, the autocorrelation response $R^z(t_1, t_2) = E[z(t_1)z(t_2)^T]$ can be formulated as (Nigam, 1983):

$$R^z(t_1,t_2) = \exp[\overline{A}t_1]\left[\sum_{i=1}^{2} term_i\right]\exp[\overline{A}t_2]^T \; ; \quad term_1 = R^z(0,0)$$

$$term_2 = \int_0^{t_1} \exp[-\overline{A}s_1][Q^{-1}(s_1)] \tag{54}$$

$$\int_0^{t_2} \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & M^{-1}R^{f_0}(s_1,s_2)[M^{-1}]^T \end{bmatrix} [Q^{-1}(s_2)]^T \exp[-\overline{A}s_2]^T ds_2 ds_1 \, .$$

This expression can be evaluated numerically as long as the functions that make up $R^{f_0}(s_1, s_2)$ are well-behaved. Alternatively, the expression can be evaluated analytically in some cases. Considering the analytical evaluation of equation (54), we note the following. The elements of $Q(t)$ and $Q^{-1}(t)$ are expressed as Fourier series, typically involving six or seven terms. The terms within exp $[-\overline{A}t]$ are exponentials multiplying sinusoids. $Term_1$ is a known initial condition. $Term_2$ depends on the autocorrelation of the forcing vector, $R^{f_0}(t_1, t_2)$, and consequently must be evaluated according to the type of excitation being considered.

If the excitation is modulated white noise, or modulated band-limited excitation (as described by equation [27]), the ability to analytically evaluate $term_2$ depends on the form of the modulation functions $p_i(t)$ (see equation [23] and equation [27]). If these functions are made up of sinusoids, exponentials, and/or polynomials, the integrations that are required can be found in standard integration tables. The expressions become very long, however, so the assistance of a symbolic mathematical software, such as *Mathematica, Maple*, or *Macsyma*, is necessary to work almost any practical problem.

6.2.1. Modulated white noise. For the case of modulated white noise excitation, $R^{f_0}(t_1, t_2)$ is taken in the form of equation (23). The observation regarding shifts in the delta function of equation (23) applies to the *L-F transformation method* also. $Term_2$ in equation (54) becomes:

$$term_2 = \int_0^t \exp[-\overline{A}s][Q^{-1}(s)] \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \tilde{M}^{-1}P(s_1)[\tilde{M}^{-1}]^T \end{bmatrix} [Q^{-1}(s)]^T \exp[-\overline{A}s]^T ds_1, \tag{55}$$

where $P_{ij}(s_1) = R_{ij}p_i(s_1)p_j(s_1)$, and $t = t_1$ *or* t_2, whichever is smaller.

6.2.2. Modulated band-limited excitation. For the case of modulated band-limited excitation, the elements of $R^{f_0}(t_1, t_2)$ are taken to be in the form of equations (25), and $R^g_{ij}(t_1, t_2)$ is in the form of equation (27), that is,

$$R^g_{ij}(t_1,t_2) = \begin{cases} R^{g(1)}_{ij} = R_{ij}\exp(-\mu_{ij}\Omega_{ij}(t_1-t_2))\cos(\Omega_{ij}(t_1-t_2)); & t_1 \geq t_2 \\ R^{g(2)}_{ij} = R_{ij}\exp(-\mu_{ij}\Omega_{ij}(t_2-t_1))\cos(\Omega_{ij}(t_2-t_1)); & t_2 \geq t_1 \end{cases} \tag{56}$$

Then $term_2$ in equation (54) becomes:

$$\begin{aligned} term_2 = &\int_0^{t_1} \exp[-\overline{A}s_1][Q^{-1}(s_1)] \left(\int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & \tilde{M}^{-1}R^{f_0(1)}(s_1,s_2)[\tilde{M}^{-1}]^T \end{bmatrix} \right. \\ &[Q^{-1}(s_2)]^T \exp[-\overline{A}s_2]^T ds_2 \\ &+ \int_{s_1}^{t_2} \begin{bmatrix} 0 & 0 \\ 0 & \tilde{M}^{-1}R^{f_0(2)}(s_1,s_2)[\tilde{M}^{-1}]^T \end{bmatrix} \\ &\left. [Q^{-1}(s_2)]^T \exp[-\overline{A}s_2]^T ds_2 \right) ds_1, \end{aligned} \tag{57}$$

which is valid for $t_2 \geq t_1$. Similarly, an expression can be formed that is valid for $t_1 \geq t_2$, but it is simpler to recognize that $E[y_i(t_1)y_j(t_2)] = E[y_j(t_2)y_i(t_1)]$, and so only the case $t_2 \geq t_1$ need be computed.

6.2.3. Illustrative examples.

Example 1

Again consider the commutative system given by equation (45). $R^x(t_1, t_2)$ can be expressed as:

$$R^x(t_1,t_2) = Q(t_1)R^z(t_1,t_2)Q^T(t_2), \tag{58}$$

where Q is defined in equation (46). The forcing function is taken to be zero-mean white noise, modulated by a sine wave:

$$R^f(t_1,t_2) = \sin(\Omega t_1)\,\delta(t_2-t_1)\,\sin(\Omega t_2). \tag{59}$$

Setting initial conditions to *zero*, only $term_2$ in equation (54) remains. Representing this term by matrix U, and inserting the expression into equation (58), yields:

$$\mathrm{E}[x(t_1)x^T(t_2)] = \begin{bmatrix} \cos t_1 & \sin t_1 \\ -\sin t_1 & \cos t_1 \end{bmatrix} \begin{bmatrix} e^{(\alpha-1)t_1} & 0 \\ 0 & e^{-t_1} \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} e^{(\alpha-1)t_2} & 0 \\ 0 & e^{-t_2} \end{bmatrix} \begin{bmatrix} \cos t_2 & -\sin t_2 \\ \sin t_2 & \cos t_2 \end{bmatrix}. \tag{60}$$

The expressions for the elements of the U matrix are given below. The notation $[\Phi(s_1)]_0^t$ means $\Phi(t) - \Phi(0)$, where $t = t_1$ *or* t_2, whichever is smaller.

$$\begin{aligned} U_1 = & \left[\frac{\exp(-2(\alpha-1)s_1)}{8}\left(-\frac{1}{\alpha-1} - \frac{-2(\alpha-1)\cos 2\Omega s_1 + 2\Omega \sin 2\Omega s_1}{2((\alpha-1)^2+\Omega^2)}\right.\right. \\ & + \frac{2(\alpha-1)\cos 2s_1 - 2\sin 2s_1}{2((\alpha-1)^2+1)} \\ & + \frac{-2(\alpha-1)\cos(2-2\Omega)s_1 + (2-2\Omega)\sin(2-2\Omega)s_1}{4((\alpha-1)^2+(1-\Omega)^2)} \\ & \left.\left. + \frac{-2(\alpha-1)\cos(2+2\Omega)s_1 + (2+2\Omega)\sin(2+2\Omega)s_1}{4((\alpha-1)^2+(1+\Omega)^2)}\right)\right]_0^t \end{aligned} \tag{61a}$$

$$\begin{aligned} U_2 = U_3 = & \left[\frac{\exp(-(\alpha-2)s_1)}{4}\left(\frac{(\alpha-2)\sin 2s_1 + 2\cos 2s_1}{(\alpha-2)^2+4}\right.\right. \\ & + \frac{(\alpha-2)\sin 2(1+\Omega)s_1 + 2(1+\Omega)\cos 2(1+\Omega)s_1}{2((\alpha-2)^2+2(1+\Omega)^2)} \\ & \left.\left. + \frac{(\alpha-2)\sin 2(1-\Omega)s_1 + 2(1-\Omega)\cos 2(1-\Omega)s_1}{2((\alpha-2)^2+2(1-\Omega)^2)}\right)\right]_0^t \end{aligned} \tag{61b}$$

$$\begin{aligned} U_4 = & \left[\frac{\exp(2s_1)}{8}\left(1 - \frac{2\cos 2\Omega s_1 + 2\Omega \sin 2\Omega s_1}{2(1+\Omega^2)} + \frac{2\cos 2s_1 + 2\sin 2s_1}{4}\right.\right. \\ & - \frac{2\cos 2(1-\Omega)s_1 + 2(1-\Omega)\sin 2(1-\Omega)s_1}{4(1+(1-\Omega)^2)} \\ & \left.\left. - \frac{2\cos 2(1+\Omega)s_1 + 2(1+\Omega)\sin 2(1+\Omega)s_1}{4(1+(1+\Omega)^2)}\right)\right]_0^t. \end{aligned} \tag{61c}$$

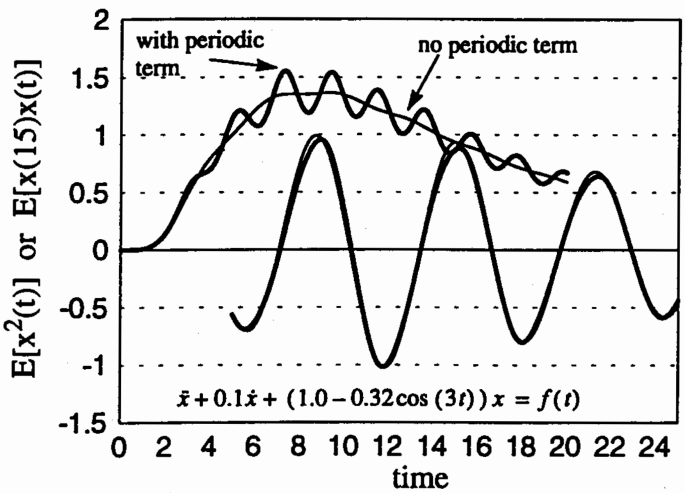

Figure 7. E[$x^2(t)$] and E[$x(15)x(t)$] from equation (52) for the modulated white noise excitation f_3 (equation 62).

Example 2

Again consider the system described by equation (52). Response statistics are presented for the following white noise excitation and initial conditions.

$$\begin{array}{c} R^{f_3} = 2\pi p(t_1)\delta(t_1 - t_2)p(t_2), \quad p(t) = 5(\exp(-0.25t) - \exp(-0.3t)), \\ m_f = 0.0, \quad y(0) = \{0, 0\}^T. \end{array} \tag{62}$$

Using *Mathematica*, equation (54) has been evaluated analytically for the above case. Figure 7 shows the system's response due to f_3.

Response statistics are also presented for the following band-limited excitation. The excitation f is defined according to equations (17) and (18), and the autocorrelation is given by

$$\begin{array}{c} R_{ij}^{f_0}(t_1, t_2) = p_i(t_1)R_{ij}^{g_4}(t_1, t_2)p_j(t_2), \\ R^{g_4} = \exp(-(0.3)(4)\,|t_1 - t_2|)\cos(4\,|t_1 - t_2|), \quad p(t) = 1.0, \\ m_f = \sin(3t), \quad y(0) = \{0.4, -0.3\}^T. \end{array} \tag{63}$$

Using *Mathematica*, equation (54) has been evaluated analytically for the above case. Figures 8 and 9 are surface plots of the displacement and velocity autocorrelation responses due to f_4.

Example 3

In this example the equation of motion has no constant stiffness term. It is to be noted that for this case “averaging” or any other small parameter method **cannot be applied.**

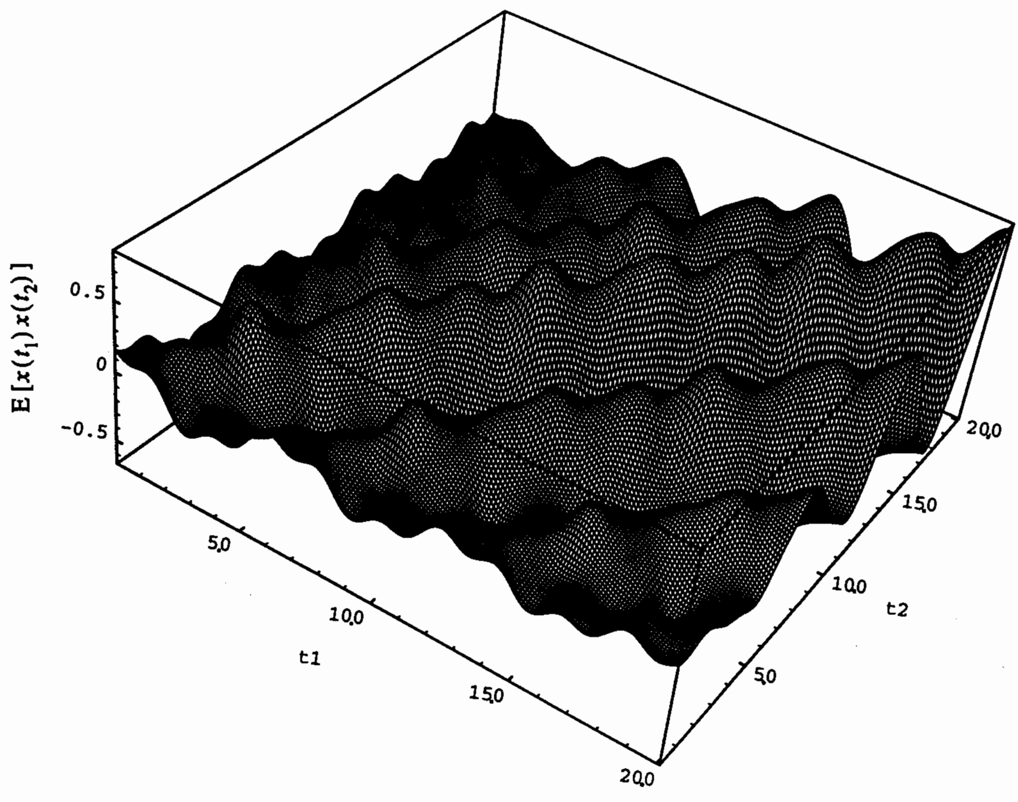

Figure 8. Displacement autocorrelation response from equation (52) due to excitation given by equation (63).

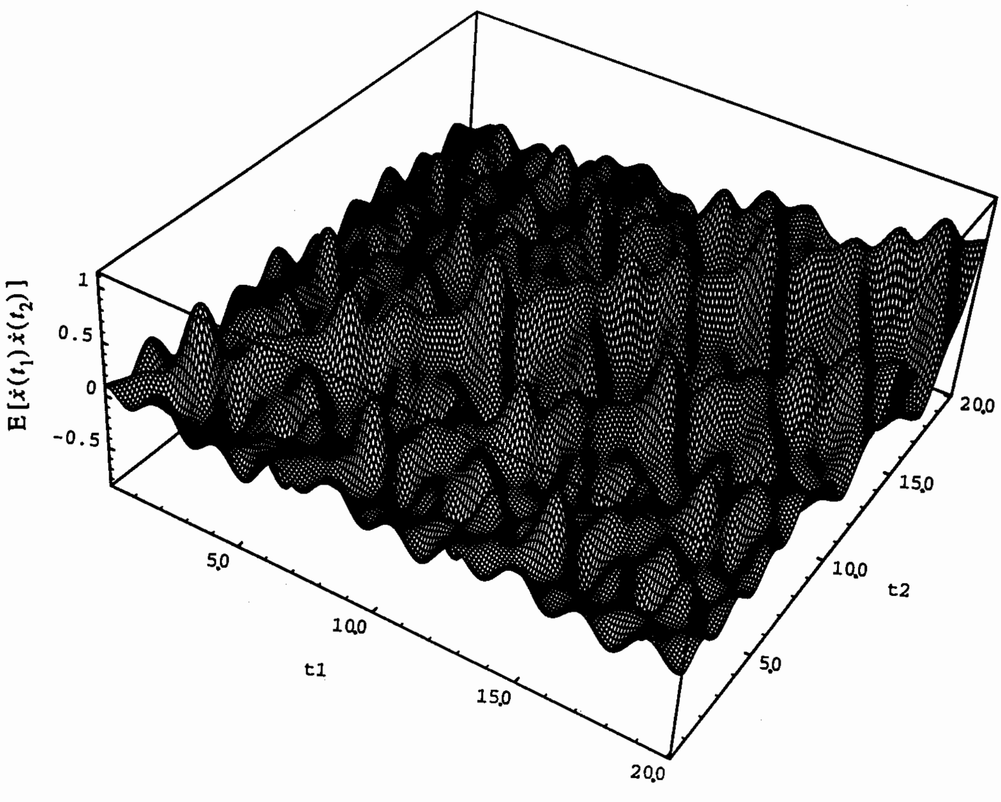

Figure 9. Velocity autocorrelation response from equation (52) due to excitation given by equation (63).

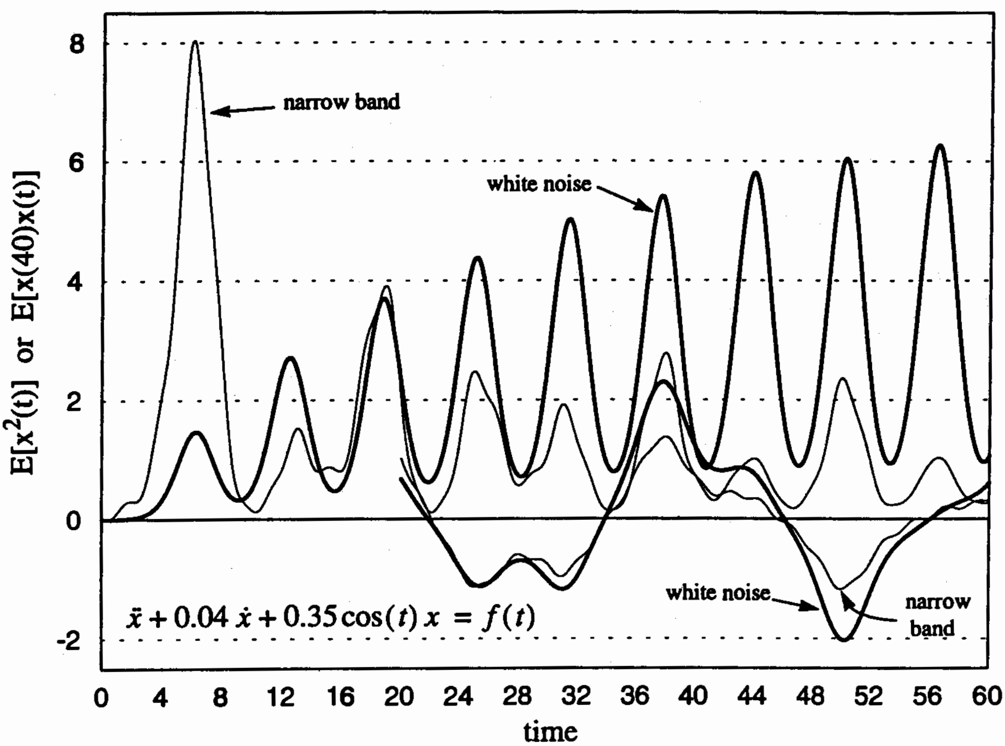

Figure 10. E[$x^2(t)$] and E[$x(40)x(t)$] from equation (64), due to the noise excitation f_5 (equation [65]) and due to the narrow band excitation f_6 (equation [66]).

The equation of motion is given by

$$\ddot{x} + 0.04\dot{x} + 0.35\cos(t)x = f(t). \tag{64}$$

The system response is plotted in Figure 10 for two excitations. The first is zero-mean white noise modulated by the unit step function:

$$R^{f_5}(t_1, t_2) = 0.01\delta(t_1 - t_2); \quad t_1, t_2 \geq 0. \tag{65}$$

The second is narrow band:

$$R^{f_6}(t_1, t_2) = 0.75e^{-0.003(3.0)|t_1 - t_2|}\cos(3.0\,|t_1 - t_2|); \quad t_1, t_2 \geq 0 \tag{66}$$

The initial conditions have been set to *zero* in both cases. In Figure 10 the oscillations in the mean square response due to f_5 appear to be approaching a steady state amplitude. This is in fact the case; the peaks were found to approach a steady state magnitude of around 7.

Example 4

Consider the double pendulum, with base excitation, shown in Figure 11. The linearized equations of motion, for small q_1 and q_2, are

$$\begin{bmatrix} 2mL^2 & mL^2 \\ mL^2 & mL^2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}$$
$$+ \begin{bmatrix} 2k + 2mLL_1\omega^2\sin(\omega t) & -k \\ -k & k + mLL_1\omega^2\sin(\omega t) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} \tag{67}$$

where $u_1(t)$ and $u_2(t)$ are random torques applied at *joints 1* and 2, respectively. Defining

$$\tau = \left(\sqrt{\frac{k}{mL^2}}\right) t \tag{68}$$

and substituting this into equation (67), and dividing through by k yields a nondimensional form for equation (67):

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1'' \\ q_2'' \end{Bmatrix} + 2\zeta \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix}$$
$$+ \begin{bmatrix} 2 + 2\varepsilon\gamma^2\sin(\gamma\tau) & -1 \\ -1 & 1 + \varepsilon\gamma^2\sin(\gamma\tau) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$
$$= \frac{1}{k} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{Bmatrix}. \tag{69}$$

where $\zeta = \frac{c}{2\sqrt{mL^2k}}$, $\varepsilon = \frac{L_1}{L}$, and $\gamma = \omega\sqrt{\frac{mL^2}{k}}$. Let the excitation be band-limited, and

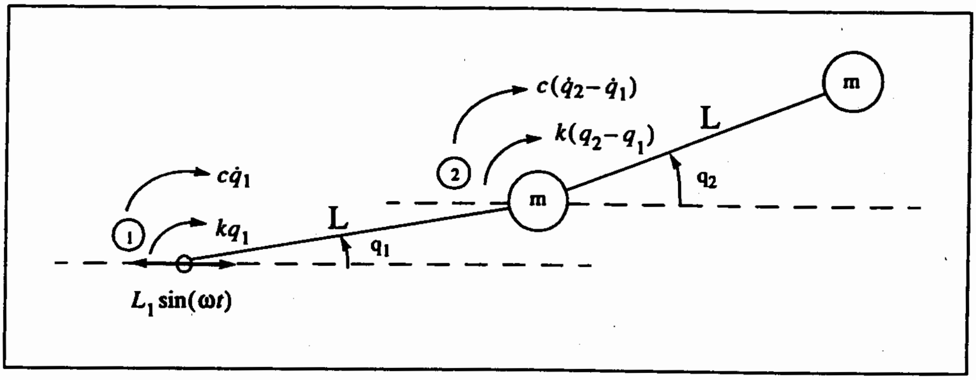

Figure 11. A double pendulum, with periodic base excitation, moving in horizontal plane.

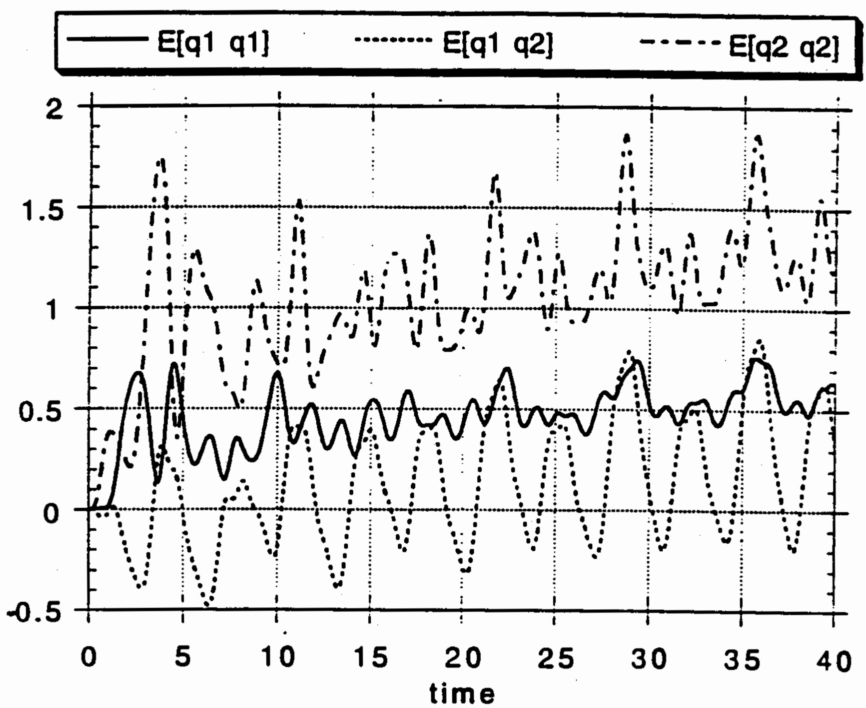

Figure 12. Some typical mean-square responses for the double pendulum example.

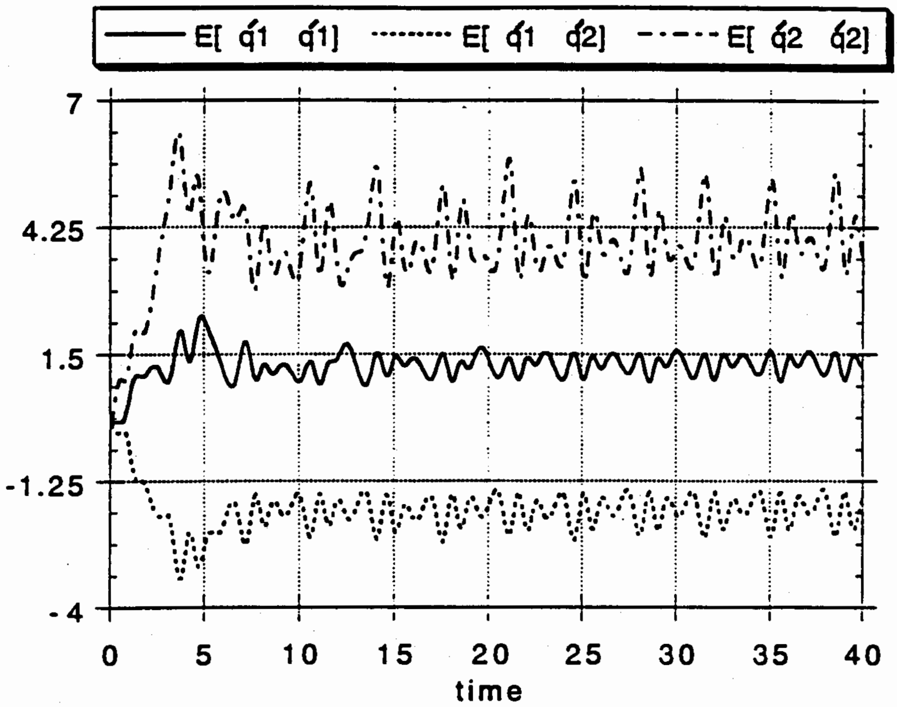

Figure 13. Some typical mean-square responses for the double pendulum example.

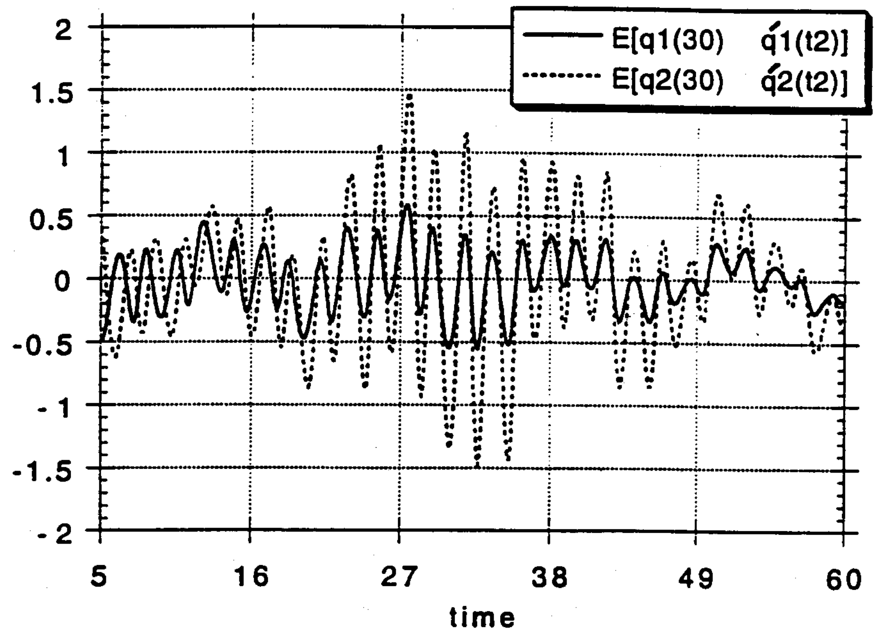

Figure 14. Some typical autocorrelation responses for the double pendulum example.

further let $\tilde{u}_2(t) = 2\tilde{u}_1(t)$. The autocorrelation of $\tilde{u}_1(t)$ is taken to be

$$R^{\tilde{u}_1}(\tau_1, \tau_2) = R\ \exp(-\mu\Omega\,|\tau_1 - \tau_2|)\cos(\Omega\,|\tau_1 - \tau_2|). \tag{70}$$

The parameter values that have been selected for this example are: $\zeta = 0.05$, $\varepsilon = 0.3$, $\gamma = 1.8$, $R = 1.0$, $\mu = 0.02$, and $\Omega = 3.0$. Initial conditions have been set to *zero*, and the system response is given in Figures 12 through 14.

7. DISCUSSION AND CONCLUSIONS

In this paper, two general techniques for finding the response of linear systems with periodic coefficients, subjected to deterministic and/or stochastic excitation, have been presented. In the *direct method* the response is directly calculated using a Chebyshev polynomial representation of the fundamental matrix. The other approach is based on the Liapunov-Floquet transformation matrix, which permits the system response to be calculated using the time-invariant methods.

The *direct method*, where the forcing vector autocorrelation matrix is expanded into Chebyshev polynomials and Chebyshev operational matrices are used to evaluate the response statistics, has been shown to be applicable to a wide range of excitations. The restrictions are, first, that the correlation terms, $R^g_{ij}(t_1, t_2)$, of the pre-modulation zero-mean excitation, $g(t)$, must not have discontinuity in their definitions except at $t_1 = t_2$. Second, they must either be scaled delta functions, or be represented in the form of equation (26). These requirements are consistent with the representation of a white noise excitation, and with the common method of describing a band-limited excitation. The number of terms that must be used in the Chebyshev polynomial expansions depends on the complexity of the functions and responses that one wishes to approximate. If a system has a natural frequency that is large compared with the primary frequency of the periodic coefficients, a large number of Chebyshev polynomials must be used. The same is true if the excitation terms have a complex shape over the length of the system period. For instance, the single-degree-of-freedom examples in the previous section have natural frequencies that are smaller than the frequencies of their periodic terms. These systems were solved using 20-term expansions, which provided at least 7 digits of accuracy, and required around 5 minutes to solve for 8 periods in the case of band-limited excitation. In contrast, the torsional natural frequency of a two-degree-of-freedom helicopter blade example discussed in Spires (1995) is 8 times the primary frequency of the periodic coefficients. This system was solved using 90-term expansions, which provided 4 to 5 digits of accuracy, and required a little under 1.5 hours to solve for 5 periods.

A key advantage to this method is that once the expansions of the correlation terms $R^g_{ij}(t_1, t_2)$ and the modulation functions $p_k(t)$ have been obtained, they can be used repeatedly as system parameters are varied, as long as the period of the system remains the same. Approximately 90% of the time spent stepping through periods is devoted to computing the Chebyshev coefficients of the excitations terms. Therefore, if these expansions are stored in a file the first time they are computed, the effect of altering system parameters can be quickly determined. This can be done by computing the new fundamental matrix and its inverse and then reevaluating the response expression using the stored coefficients of the excitation terms.

The method of converting the system to a time-invariant form, using the L-F transformation matrix, and then evaluating the autocorrelation expression analytically has been shown to be another viable approach. The restriction regarding discontinuities in the definition of the excitation cross-correlation terms applies to this method also. The restriction that the excitation's autocorrelation terms be separable does not apply, however. Using the *Liapunov-Floquet (L-F) transformation method*, it has been found that the deterministic and stochastic response of linear time-periodic systems can be obtained symbolically in many cases. Symbolic evaluation of response expressions can be very time consuming, however, and is not practical for systems larger than 2 DOF (band-limited excitation) or 4 DOF (white noise). If symbolic evaluation of the system response is found to be unreasonable, the *L-F transformation method* provides a useful formulation for numerical evaluation. If the analytical evaluation of a problem turns out to be unreasonable, the *L-F transformation method* provides a useful formulation for numerical analysis.

The Liapunov-Floquet transformation approach has the potential of being extended to the nonlinear response problem. Because the L-F transformation preserves the full dynamics of the linear part of the periodic system, the application of small parameter techniques (such as perturbations, averaging, and statistical linearization) after the transformation should yield a more accurate solution than if these methods had been applied directly.

It is concluded that calculation of the fundamental matrix via Chebyshev polynomials, and of the Liapunov-Floquet transformation matrix, provide the basis for development of practical methods for finding the autocorrelation and mean square response of linear time-periodic systems subjected to stochastic excitations. These methods are straightforward and free from small parameter limitations.

8. APPENDIX

Appendix A: Chebyshev Polynomials and their Operational Matrices

The shifted Chebyshev polynomials of the first kind may be obtained using:

$$s_r(\eta) = \cos(r\theta)\ ; \quad \cos\theta = 2\eta - 1\ , \quad 0 \leq \theta \leq \pi\ , \quad r = 0, 1, 2, 3, ... \tag{A1}$$

A function $\Theta(t)$, defined over $t \in [a, b]$, can be expanded into a series of shifted Chebyshev polynomials, up to m terms, which is valid over $\eta = (t-a)/(b-a) \in [0,1]$. The function can then be approximated as $\Theta(t) = \{s^*(\eta)\}^T\{b\}$, where $\{s^*(\eta)\}$ is the m-long Chebyshev polynomial vector, and $\{b\}$ an m-long coefficient vector. The coefficient of the r^{th} shifted Chebyshev polynomial of the first kind can be calculated using:

$$b_r = \frac{1}{\delta}\int_0^{\pi} \Theta\left(a + \frac{(b-a)}{2}(\cos\theta + 1)\right)\cos(r\theta)d\theta\ ; \delta = \begin{cases} \pi ......(r=0) \\ \pi/2......(r \neq 0) \end{cases} \tag{A2}$$

Relationships similar to (A1) and (A2) exist for Chebyshev polynomials of the second kind.

Chebyshev integration and product matrices have been developed that allow convenient manipulation of certain expressions. Noting that, because time has been normalized to $[0,1]$,

$$\frac{d(\Theta(t))}{dt} = T^{-1} \times \frac{d(\Theta(\eta))}{d\eta}, \tag{A3}$$

where $T = b - a$ is the length of time over which $\Theta(t)$ has been expanded, the integral of $\Theta(t)$ from a to t can be expressed as:

$$\int_a^t \Theta(\tau)d\tau = T\int_0^{\eta}\{s^*(\tau)\}^T\{b\}d\tau = T\{s^*(\eta)\}^T G^T\{b\}, \tag{A4}$$

where G is an $m \times m$ Chebyshev integration matrix. For shifted Chebyshev polynomials of the first kind, the integration matrix is given by

$$G=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & \cdots & 0 \\ -\frac{1}{8} & 0 & \frac{1}{8} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{6} & -\frac{1}{4} & 0 & \frac{1}{12} & 0 & \cdots & 0 \\ \frac{1}{16} & 0 & -\frac{1}{8} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \frac{1}{4(j-1)} \\ \frac{-(-1)^{i-1}}{2i(i-2)} & 0 & 0 & \cdots & 0 & \frac{-1}{4(i-2)} & 0 \end{bmatrix}. \tag{A5}$$

If two functions $\Theta(t) = \{s^*(\eta)\}^T\{b\}$ and $\Phi(t) = \{s^*(\eta)\}^T\{a\}$, are multiplied the result is:

$$\Theta(t)\Phi(t) = \{b\}^T\{s^*(\eta)\}\{s^*(\eta)\}^T\{a\} = \{s^*(\eta)\}^T \mathcal{Q}_b\{a\} \tag{A6}$$

where $\mathcal{Q}$ is an $m \times m$ Chebyshev product matrix. For shifted Chebyshev polynomials of the first kind, the product matrix is given by:

$$\mathcal{Q}_b=\begin{bmatrix} b_0 & \frac{b_1}{2} & \cdots & \cdots & \cdots & \frac{b_{j-1}}{2} \\ b_1 & \ddots & & \frac{b_{j-i}+b_{j+i-2}}{2} & & \\ \vdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ \vdots & \frac{b_{i-j}+b_{i+j-2}}{2} & & & \ddots & \\ b_{i-1} & & & & & b_0+\frac{b_{2i-2}}{2} \end{bmatrix}. \tag{A7}$$

Let $\Theta(t)$ represent an $n \times n$ matrix of functions, which can be expressed as

$$\begin{aligned} \Theta(t) &= \begin{bmatrix} \{b_{1,1}\}^T & \cdots & \{b_{1,n}\}^T \\ \vdots & \ddots & \vdots \\ \{b_{n,1}\}^T & \cdots & \{b_{n,n}\}^T \end{bmatrix} [\hat{s}^*(\eta)]^T = \Phi'[\hat{s}^*(\eta)]^T \\ &= [\hat{s}^*(\eta)] \begin{bmatrix} \{b_{1,1}\} & \cdots & \{b_{1,n}\} \\ \vdots & \ddots & \vdots \\ \{b_{n,1}\} & \cdots & \{b_{n,n}\} \end{bmatrix} = [\hat{s}^*(\eta)]\Phi, \end{aligned} \tag{A8}$$

where Φ' consists of $n \times n$ m-long row vectors, Φ consists of $n \times n$ m-long column vectors, and $[\hat{s}^*(\eta)]$ is defined by equation (6). Then the product $\Phi'[\hat{s}^*(\eta)]^T[\hat{s}^*(\eta)]$ can be expressed as

$$\Phi'[\hat{s}^*(\eta)]^T[\hat{s}^*(\eta)] = [\hat{s}^*(\eta)] \begin{bmatrix} \mathcal{Q}_{b_{1,1}} & \cdots & \mathcal{Q}_{b_{1,n}} \\ \vdots & \ddots & \vdots \\ \mathcal{Q}_{b_{n,1}} & \cdots & \mathcal{Q}_{b_{n,n}} \end{bmatrix} = [\hat{s}^*(\eta)]\mathcal{Q}_{\Phi'} = [\hat{s}^*(\eta)]\mathcal{Q}_{\Phi}. \tag{A9}$$

Appendix B: Kronecker Product

Consider a 2×2 square matrix A and an $n \times m$ matrix B. The Kronecker product of the two matrices is defined by:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}. \tag{A10}$$

The resulting matrix is of size $2n \times 2m$

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