Control of General Dynamic Systems With Periodically Varying Parameters Via Liapunov-Floquet Transformation

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A new technique in the design of controllers for linear dynamic systems with periodically varying coefficients is presented. The idea is to utilize the well-known Liapunov-Floquet (L-F) transformation such that the original time-varying system can be reduced to a form suitable for the application of standard time-invariant methods of control theory. For this purpose, a procedure for computing the L-F matrices for general linear periodic systems is outlined. In this procedure, the state transition matrices are expressed in terms of Chebyshev polynomials which permits the computation of L-F matrices as explicit functions of time. Further, it is shown that controllers can be designed in the transformed domain via full state or observer based feedback using principles of pole placement and optimal control theory. The effectiveness of the proposed technique is demonstrated through two examples. The first example belongs to the class of commutative systems while in the second example a triple inverted pendulum subjected to a periodic follower load is considered. It is found that both types of controllers can be successfully designed.

1 Introduction

Linear periodic systems find practical applications in several areas of engineering and pure science such as structures subjected to periodic loading, helicopter rotor blades in forward flight, asymmetric rotor-bearing systems, electrical circuits and quantum mechanics. Mathieu (1868) was the first person who studied a second order differential equation with periodic coefficients while analyzing the wave motion of an elliptical lake. But a complete treatise on the stability analysis of such systems came only a few years later (Floquet, 1883; Poincare, 1899). Floquet's theorems were based on the eigenvalues of the fundamental matrix evaluated at the end of the principal period. An exhaustive study of these results has been provided by Yakubovich and Starzhinskii (1975) and Coddington and Levinson (1955). Based on Floquet's work, Liapunov (1896) introduced a transformation which converted the time periodic system to a time-invariant form. However to date a general method which explicitly computes the Liapunov-Floquet (L-F) transformation matrix does not exist (Bellman 1970, Verhulst 1990).

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The control problems associated with linear periodic systems are also quite challenging due to their time-varying nature. One of the many reasons that could be attributed to this is the fact that the time varying eigenvalues of the periodic matrix do not determine the stability of the systems and one must resort to Floquet analysis. Hence it is difficult to apply both classical and modern control techniques to these problems as opposed to the case of time-invariant systems. Even though several control methodologies for linear periodic systems have been reported (Nishimura, 1972; Hac and Tomizuka, 1990; Bittanti et al., 1986; D'Angelo, 1970, Joseph et al.) in the past, a simple and efficient straight forward method has not yet been suggested. The proposed methods either converted the system to a canonical form or the explicit computation of the state transition matrix was required.

Wu (1978) identified and studied the stability of certain classes of linear time-varying systems by transforming them into time invariant ones without having to use the full information on the state transition matrix. The examples considered included commutative systems, Euler systems and Floquet systems (periodic systems). But a general procedure for finding the L-F matrix explicitly was not indicated. The concept of pole placement for linear time-varying systems was first introduced by Follinger (1978), where he used a phase canonical approach to obtain state feedback control. Later Sommer (1979) extended the design to nonlinear time-varying systems using the same phase canonical approach. But transforming the system into a canonical form is not unique especially for multivariable linear time-varying systems. Calico and Wiesel (1984) discussed the active control problems associated with time-periodic systems where an iterative procedure was suggested to do pole placement and eventually achieve control. The authors admitted that their results were only a subset of possible gain selection techniques and did not represent the most general possibilities. Calise et al. (1992) developed a method for applying output feedback control theory to the design of fixed gain controllers for time-periodic systems. It was found by the authors that the approach was complicated and required two levels of iteration in calculating the optimal feedback gains. Alfhaid and Lee (1988) also addressed the pole assignment problem in linear time-varying systems using state feedback and output feedback. The authors suggested the use of a transformation matrix which converted the problem into a time-invariant form. However, the methods developed to compute these transformations are quite cumbersome and in particular they do not address the computation of such transformations for general periodic systems.

Pandiyan et. al. (1992) presented for the first time a simple and efficient method to compute the L-F transformations for general periodic systems. The approach was based on the computational technique developed by Sinha and Wu (1991) through an application of the well acclaimed Liapunov-Floquet theory. The main advantage of this technique is that the state transition matrix can be evaluated as an explicit function of time t which allows the computation of the L-F transformation matrix in a simple manner. In this paper, first the computation of L-F transformation matrices is briefly discussed through an application of Chebyshev

Nomenclature

$\mathbf{A(t)}$ = $n \times n$ periodic system matrix
$\bar{\mathbf{b}}$ = $\{\mathbf{b}^1\ \mathbf{b}^2\ \mathbf{b}^3 \ldots \mathbf{b}^n\}^T$ vector of Chebyshev coefficients ($nm \times 1$)
$\bar{\mathbf{B}}$ = $[\bar{\mathbf{b}}_1\ \bar{\mathbf{b}}_2\ \bar{\mathbf{b}}_3 \ldots\ldots \bar{\mathbf{b}}_n]$ $nm \times n$ Chebyshev coefficient matrix
$\mathbf{B(t)}$ = $n \times q$ input gain matrix
c = damping parameter
$\bar{c}$ = $\dfrac{c}{ml^2}$
$\bar{\mathbf{B}}$ = constant matrix
$\mathbf{B}^{\#}\mathbf{(t)}$ = generalized inverse of $\mathbf{B(t)}$
$\mathbf{C(t)}$ = $r \times n$ output matrix
$\bar{\mathbf{C}}$ = $n \times n$ complex matrix
$\bar{\mathbf{C}}*$ = conjugate transpose of $\bar{\mathbf{C}}$
$\hat{\mathbf{C}}$ = a constant matrix
$\mathbf{C}^{\#}\mathbf{(t)}$ = generalized inverse of $\mathbf{C}^T\mathbf{(t)}$
$\mathbf{D}$ = $nm \times n$ expansion coefficient matrix
$\bar{\mathbf{e}}\mathbf{(t)}$ = $\mathbf{x(t)} - \hat{\mathbf{x}}\mathbf{(t)}$, error between the actual state and the observed state
$\mathbf{e(t)}$ = error dynamics from Eqs. (28) and (29)
$\mathbf{F(t)}$ = $\mathbf{A(t)} - \mathbf{G}^T\mathbf{(t)C(t)}$, $n \times n$ matrix
$\mathbf{G(t)}$ = $r \times n$ gain matrix
$\bar{\mathbf{G}}\mathbf{(t)}$ = $\mathbf{G}^T\mathbf{(t)}$, $n \times r$ observer gain matrix
$\tilde{\mathbf{G}}$ = gain matrix of the time-invariant system (46)
$\mathbf{H(t)}$ = $\mathbf{B(t)}$
$\mathbf{I}$ = $n \times n$ identity matrix
$\bar{\mathbf{k}}$ = $\dfrac{\mathbf{kl}}{\mathbf{m}^2}$
$\mathbf{k}$ = torsional stiffness of the inverted pendulum
$\mathbf{K(t)}$ = $n \times n$ gain matrix
$\bar{\mathbf{K}}$ = gain matrix of the time-invariant system (29)
$\mathbf{K}$ = constant stiffness matrix
$\mathbf{K^*(t)}$ = periodic stiffness matrix
$\mathbf{l}$ = link length of the inverted pendulum
$\mathbf{L(t)}$ = $2T$-periodic Liapunov-Floquet (L-F) transformation matrix
$\tilde{\mathbf{L}}\mathbf{(t)}$ = $2T$-periodic L-F matrix of the dual system
$\mathbf{m}$ = mass
$\mathbf{M}$ = mass matrix
$\mathbf{P(t)}$ = T-periodic L-F transformation matrix
$\mathbf{P}$ = $\mathbf{P}_1 + \mathbf{P}_2 \cos(\omega t)$, periodic load
$\mathbf{R}$ = real matrix
$\hat{\mathbf{S}}\mathbf{(t)}$ = $n \times nm$ Chebyshev polynomial matrix
$\mathbf{s}_i^*$ = shifted Chebyshev polynomials of the first kind
$\mathbf{T}$ = period
$\mathbf{u(t)}$ = control vector of the time-varying system (Eq. (26))
$\mathbf{v(t)}, \hat{\mathbf{v}}\mathbf{(t)}$ = control vector of systems (29) and (46), respectively
$\mathbf{x(t)}$ = $n \times 1$ state vector from Eq. (1)
$\hat{\mathbf{x}}\mathbf{(t)}$ = $n \times 1$ observer state vector from Eq. (40)
$\tilde{\mathbf{x}}\mathbf{(t)}$ = $n \times 1$ dual system state vector from Eq. (42)
$\mathbf{y(t)}$ = $r \times 1$ output state vector from Eq. (26)
$\tilde{\mathbf{y}}\mathbf{(t)}$ = output state vector of the dual system from Eq. (43)
$\mathbf{z(t)}$ = $n \times 1$ state vector of the time-invariant system (Eq. (28))
$\bar{\mathbf{z}}\mathbf{(t)}, \tilde{\mathbf{z}}\mathbf{(t)}, \hat{\mathbf{z}}\mathbf{(t)}$ = $n \times 1$ state vectors
$\hat{\mathbf{Z}}$ = $nm \times nm$ constant matrix
$\otimes$ = Kronecker product
$\boldsymbol{\alpha}$ = $\mathbf{0} \leq \boldsymbol{\alpha} \leq \mathbf{1}$ for the triple inverted pendulum
γ = $\dfrac{\mathbf{Pl}}{\mathbf{k}}$
$\boldsymbol{\eta}$ = state vector of the triple inverted pendulum from Eq. (54)
$\boldsymbol{\Phi}\mathbf{(t,0)}, \boldsymbol{\Psi}\mathbf{(t,0)}$ = State transition matrices
ω = frequency of the periodic load
Ψ = $0 \leq \Psi \leq \mathbf{T}$
ξ = dummy variable
τ = $\mathbf{0} \leq \tau \leq \mathbf{T}$

polynomials. These transformations then permit the design of controllers for periodic systems via the well known time-invariant techniques. In order to demonstrate the effectiveness of the proposed strategy, two examples are considered. In the first example, the controller design of a commutative system is discussed. For this case, the L-F transformation matrix can be obtained in a closed form. The second example consists of a triple inverted pendulum subjected to a periodic loading. For this problem the L-F transformation matrix is computed through the use of Chebyshev polynomials.

2 Mathematical Background

Consider a linear periodic system

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}(\mathbf{t})\mathbf{x}(\mathbf{t}) \tag{1}$$

where $\mathbf{A}(\mathbf{t}) = \mathbf{A}(\mathbf{t} + \mathbf{T})$, $\mathbf{T}$ being the period of the system. The stability and response of the above equation can be discussed using the well known Floquet theory. In order to compute the L-F transformation matrix for the system (1) some background knowledge of Floquet theory is necessary.

2.1 Results From Floquet Theory

Theorem I: Each $n \times n$ fundamental matrix $\mathbf{\Phi}(\mathbf{t})$ of Eq. (1) can be written as a product of two $n \times n$-matrices as

$$\mathbf{\Phi}(\mathbf{t}) = \mathbf{P}(\mathbf{t})\mathbf{e}^{\overline{C}t} \tag{2}$$

where $\mathbf{P}(\mathbf{t})$ is T-periodic and $\overline{\mathbf{C}}$ is a constant matrix. $\mathbf{P}(\mathbf{t})$ and $\overline{\mathbf{C}}$, in general, are complex.

Remark: $\boldsymbol{\mu}_i$, the eigenvalues of the Floquet Transition Matrix (FTM) $\mathbf{\Phi}(\mathbf{T})$ are called the characteristic multipliers and the stability condition can be expressed as

$$|\boldsymbol{\mu}_i| < 1, \mathbf{i} = 1, 2, \ldots, \mathrm{n}.$$

Corollary - 1: Each fundamental matrix $\boldsymbol{\phi}(\mathbf{t})$ can also be factored as

$$\mathbf{\Phi}(\mathbf{t}) = \mathbf{L}(\mathbf{t})\mathbf{e}^{Rt} \tag{3}$$

where the matrix $\mathbf{L}(\mathbf{t})$ is real and periodic with period $2T$ and $\mathbf{R}$ is an appropriate real matrix.

Corollary - 2: The Liapunov-Floquet transformation

$$\mathbf{x}(\mathbf{t}) = \mathbf{P}(\mathbf{t})\mathbf{z}(\mathbf{t}) \tag{4}$$

reduces the original system (1) to

$$\dot{\mathbf{z}}(\mathbf{t}) = \overline{\mathbf{C}}\mathbf{z}(\mathbf{t}) \tag{5}$$

Moreover, the $2T$-periodic transformation

$$\mathbf{x}(\mathbf{t}) = \mathbf{L}(\mathbf{t})\mathbf{z}(\mathbf{t}) \tag{6}$$

produces a real representation given by

$$\dot{\mathbf{z}}(\mathbf{t}) = \mathbf{R}\mathbf{z}(\mathbf{t}) \tag{7}$$

However, there is no simple way of constructing either $\mathbf{L}(\mathbf{t})$ or $\mathbf{P}(\mathbf{t})$, but for a very special class of systems, called the commutative systems (Lukes, 1982).

In order to determine the L-F transformation matrix $[\mathbf{L}(\mathbf{t})$ or $\mathbf{P}(\mathbf{t})]$, one must compute the state transition matrix $\mathbf{\Phi}(\mathbf{t})$ as an explicit function of time t. An efficient technique for such a computation is briefly outlined below.

2.2 Computation of L-F Transformation Matrix Via Chebyshev Polynomials. Very recently, it has been shown (Sinha and Wu, 1991; Wu 1991; Joseph et al., 1993) that the state transition matrices (STM) of linear periodic systems can be obtained in terms of the shifted Chebyshev polynomials of the first kind. The technique is efficient and since the STM is basically expressed in terms of powers of t, it is suitable for algebraic manipulations as well. In fact, if the dimension is small, the STM and thus the solution can be expressed in a closed form as an explicit function of system parameters. In fact for Mathieu's equation $\mathbf{\Phi}(\mathbf{t})$ has been computed in a symbolic form by Sinha and Juneja (1991).

In this technique, the solution vector $\mathbf{x}(\mathbf{t})$ and the periodic matrix $\mathbf{A}(\mathbf{t})$ in Eq. (1) are expanded in terms of the shifted Chebyshev polynomials in the interval $[0, T]$ as shown below.

$$\mathbf{x}_i(\mathbf{t}) \approx \sum_{r=0}^{s-1} \mathbf{b}_r^i \mathbf{s}_r^*(\mathbf{t}) \equiv \mathbf{s}^{*T}(\mathbf{t})\mathbf{b}^i, \mathbf{i} = 1, 2 \ldots, \mathbf{n} \tag{8}$$

$$\mathbf{A}(\mathbf{t}) \approx \sum_{r=0}^{s-1} \mathbf{d}_r^{ij} \mathbf{s}_r^*(\mathbf{t}) \equiv \mathbf{s}^{*T}(\mathbf{t})\mathbf{d}^{ij}, \quad \mathbf{i},\mathbf{j} = 1, 2, \ldots, \mathbf{n} \tag{9}$$

where

$$\mathbf{b}^i = \{\mathbf{b}_0^i \ \mathbf{b}_1^i \ldots \mathbf{b}_{m-1}^i\}^T$$

$$\mathbf{d}^{ij} = \{\mathbf{d}_0^{ij} \ \mathbf{d}_1^{ij} \ldots \mathbf{d}_{m-1}^{ij}\}^T$$

and

$$s^{*T}(\mathrm{t}) = \{s_0^*(t) \ s_1^*(\mathrm{t}) \ldots s_{m-1}^*(\mathrm{t})\}.$$

Here $\mathbf{b}_r^i$ are unknown expansion coefficients of $\mathbf{x}_i(\mathbf{t})$, $\mathbf{d}_r^{ij}$ are known expansion coefficients of $\mathbf{A}_{ij}(t)$ and $s_r^*(t)$ are the shifted Chebyshev polynomials of the first kind. Now for convenience in algebraic manipulation an $n \times nm$ Chebyshev polynomial matrix is defined as

$$\hat{\mathbf{S}}(\mathbf{t}) = \mathbf{I} \otimes s^{*T}(t) \tag{10}$$

where $\otimes$ represents the Kronecker product (Bellman 1970, see Appendix), and $\mathbf{I}$ is an $n \times n$ identity matrix. Using the definitions in Eqs. (8), (9) and (10), $\mathbf{x}(\mathbf{t})$ and $\mathbf{A}(\mathbf{t})$ can be rewritten as

$$\mathbf{x}(\mathbf{t}) = \hat{\mathbf{S}}(\mathbf{t})\overline{b} \tag{11}$$

where $\overline{\mathbf{b}} = \{\mathbf{b}^1 \ \mathbf{b}^2 \ \mathbf{b}^3 \ldots \mathbf{b}^n\}^T$ is an $nm \times 1$ vector,

$$\mathbf{A}(\mathbf{t}) = \hat{\mathbf{S}}(\mathbf{t})\mathbf{D} \tag{12}$$

$$\mathbf{A}(\mathbf{t})\mathbf{x}(\mathbf{t}) = \hat{\mathbf{S}}(\mathbf{t})\overline{\mathbf{Q}}\overline{\mathbf{b}} \tag{13}$$

$\mathbf{D} = [\mathbf{d}^{i1} \ \mathbf{d}^{i2} \ \mathbf{d}^{i3} \ldots \mathbf{d}^{ij}]$, i,j = 1,2,3,4,... .n, is an $nm \times n$ matrix and $\overline{\mathbf{Q}}$ is an $nm \times nm$ product operation matrix given in the Appendix (for details see Sinha and Wu, 1991).

The integral form of Eq. (1) is

$$\mathbf{x}(\mathbf{t}) - \mathbf{x}(0) = \int_0^t \mathbf{A}(\xi)\mathbf{x}(\xi)\mathbf{d}\xi \tag{14}$$

where ξ represents a dummy variable. Substituting Eqs. (11), (12) and (13) in (14) and then following the approach of Sinha and Wu (1991), one can obtain a set of linear algebraic equations of the form

$$[\mathbf{I} - \hat{\mathbf{Z}}]\overline{\mathbf{b}} = \mathbf{x}(0) \tag{15}$$

where $\hat{\mathbf{Z}}$ is an $nm \times nm$ constant matrix defined in the Appendix by Eq. (A2) and $\overline{\mathbf{b}}$ is the vector of unknown Chebyshev coefficients. Once the b^i are obtained from Eq. (15), the solution for $\mathbf{x}(\mathbf{t})$ is given by equation (8) which simply represents a super convergent power series in t.

In order to compute the L-F transformation matrix, one needs to find the state transition matrix $\mathbf{\Phi}(\mathbf{t},\mathbf{0})$ associated with the linear system given by Eq. (1). This requires a set of solutions with n initial conditions: $\mathbf{x}_i(\mathbf{0}) = (1,0,0..0)$, $(0,1,0..0)$, $(0,0,1,0....0)$, ..., $(0,0,....,1)$. It is to be noted that all $\overline{\mathbf{b}}_i$'s corresponding to the above set of initial conditions can be determined simultaneously by defining the right hand of Eq. (15) in the matrix form. Then the state transition matrix is given by

$$\mathbf{\Phi}(\mathbf{t},\mathbf{0}) = \hat{\mathbf{S}}(\mathbf{t})\overline{\mathbf{B}} \tag{16}$$

where $\overline{\mathbf{B}} = [\overline{\mathbf{b}}_1\ \overline{\mathbf{b}}_2\ \overline{\mathbf{b}}_3 \ldots\ldots \overline{\mathbf{b}}_n]$ and $\mathbf{\Phi}(\mathbf{0},\mathbf{0}) = \mathbf{I}$. It has to be noted that this STM is valid only for $\mathbf{0} \le \mathbf{t} \le \mathbf{T}$ since the shifted Chebyshev polynomials of the first kind are defined over the interval $[0,T]$. When $t > T$, the STM can be evaluated using Floquet theory (Coddington and Levinson 1955) as

$$\mathbf{\Phi}(\mathbf{t},\mathbf{0}) = \mathbf{\Phi}(\psi,\mathbf{0})\mathbf{\Phi}^n(\mathbf{T},\mathbf{0}) \tag{17}$$

where $t = nT + \psi$, $\psi \in [\mathbf{0},\mathbf{T}]$, and $n = 1,2,3,\ldots\ldots$ Since $\mathbf{\Phi}(\mathbf{0},\mathbf{0}) = \mathbf{I}$, Eq. (2) yields $\mathbf{P}(\mathbf{0}) = \mathbf{P}(\mathbf{T}) = \mathbf{I}$ and hence the FTM $\mathbf{\Phi}(\mathbf{T})$ can be written as

$$\mathbf{\Phi}(\mathbf{T}) = \mathbf{e}^{\overline{C}T} \tag{18}$$

By performing an eigen-analysis on the above equation, $\overline{\mathbf{C}}$ can be computed easily. Then the T-periodic L-F transformation matrix is

$$\mathbf{P}(\mathbf{t}) = \mathbf{\Phi}(\mathbf{t})\mathbf{e}^{-\overline{C}t} \tag{19}$$

In order to evaluate the 2T-periodic real L-F transformation matrix $\mathbf{L}(t)$, first we note that (Coddington and Levinson 1955)

$$\begin{aligned}\mathbf{\Phi}(\mathbf{2T}) &= \mathbf{\Phi}(\mathbf{T})\mathbf{\Phi}(\mathbf{T}) \\ &= \mathbf{e}^{\overline{C}T}\mathbf{e}^{\overline{C}^*T} \\ &= \mathbf{e}^{2RT}\end{aligned} \tag{20}$$

where $\overline{\mathbf{C}}^*$ is the conjugate matrix of $\overline{\mathbf{C}}$, $\mathbf{R} = \dfrac{\overline{\mathbf{C}} + \overline{\mathbf{C}}^*}{2}$ and the 2-T L-F matrix can be represented as

$$\begin{aligned}&\mathbf{L}(\mathbf{t}) = \mathbf{\Phi}(\mathbf{t})\mathbf{e}^{-Rt} && ;\ 0 \le t \le T \\ &\mathbf{L}(\tau+\mathbf{T}) = \mathbf{\Phi}(\tau)\mathbf{L}(\mathbf{T})\mathbf{e}^{-R\tau} && ;\ T \le (T+\tau) \le 2T\ ;\ 0 \le \tau \le T\end{aligned} \tag{21}$$

It should be noted that $\mathbf{L}(\mathbf{t}) = \mathbf{L}(\mathbf{t} + \mathbf{2T})$.

If one is interested in finding $\mathbf{\Phi}^{-1}(\mathbf{t})$, then there are two avenues. $\mathbf{\Phi}(\mathbf{t})$ can possibly be inverted through a symbolic software like MACSYMA/MATHEMATICA/MAPLE, however, it is not an easy task. A feasible approach is to first find the state transition matrix $\mathbf{\Psi}(\mathbf{t})$ of the adjoint system

$$\dot{\mathbf{w}}(\mathbf{t}) = -\mathbf{A}^T(\mathbf{t})\mathbf{w}(\mathbf{t}) \tag{22}$$

and use the following relationship (Yakubovich and Starzhinskii 1975),

$$\mathbf{\Phi}^{-1}(\mathbf{t}) = \mathbf{\Psi}^T(\mathbf{t}) \tag{23}$$

The computation of $\mathbf{\Phi}^{-1}(\mathbf{t})$ is essential in determining $\mathbf{L}^{-1}(\mathbf{t})$ or $\mathbf{P}^{-1}(\mathbf{t})$. For example, the inverse T-periodic L-F transformation matrix can be evaluated utilizing the properties of the adjoint system as shown below.

$$\begin{aligned}\mathbf{P}^{-1}(\mathbf{t}) &= [\mathbf{\Phi}(\mathbf{t})\mathbf{e}^{-\overline{C}t}]^{-1} \\ &= \mathbf{e}^{\overline{C}t}\mathbf{\Phi}^{-1}(\mathbf{t}) \\ &= \mathbf{e}^{\overline{C}t}\mathbf{\Psi}^T(\mathbf{t})\end{aligned} \tag{24}$$

The technique described here has been successfully employed to compute the STM's for relatively large systems (Sinha and Wu, 1991; Wu, 1991; Joseph et al., 1993).

3 Controller Designs

3.1 Full State Feedback Controller. The control problem associated with a periodic system can be written as

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}(\mathbf{t})\mathbf{x}(\mathbf{t}) + \mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}) \tag{25}$$

$$\mathbf{y}(\mathbf{t}) = \mathbf{C}(\mathbf{t})\,\mathbf{x}(\mathbf{t}) \tag{26}$$

where $[\mathbf{A}(\mathbf{t}),\mathbf{B}(\mathbf{t})]$ and $[\mathbf{A}(\mathbf{t}),\mathbf{C}(\mathbf{t})]$ form controllable and observable pairs, respectively. Applying the Liapunov-Floquet transformation

$$\mathbf{x}(\mathbf{t}) = \mathbf{L}(\mathbf{t})\mathbf{z}(\mathbf{t}) \tag{27}$$

to equation (25) yields

$$\dot{\mathbf{z}}(\mathbf{t}) = \mathbf{R}\mathbf{z}(\mathbf{t}) + \mathbf{L}^{-1}(\mathbf{t})\mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}) \tag{28}$$

where $\mathbf{R}$ is a real matrix as defined in Section 2.1. Now consider an auxiliary system of the form

$$\dot{\overline{\mathbf{z}}}(\mathbf{t}) = \mathbf{R}\overline{\mathbf{z}}(\mathbf{t}) + \tilde{\mathbf{B}}\mathbf{v}(\mathbf{t}) \tag{29}$$

where $\tilde{\mathbf{B}}$ is a constant matrix such that $(\mathbf{R},\tilde{\mathbf{B}})$ is a controllable pair. The control law for Eq. (29) can be written as,

$$\mathbf{v}(\mathbf{t}) = -\overline{\mathbf{K}}\overline{\mathbf{z}}(\mathbf{t}) \tag{30}$$

Defining $\mathbf{e}(\mathbf{t}) = \mathbf{z}(\mathbf{t}) - \overline{\mathbf{z}}(\mathbf{t})$, the error dynamics from Eqs. (28) and (29) can be represented as

$$\dot{\mathbf{e}}(\mathbf{t}) = \mathbf{R}\mathbf{e}(\mathbf{t}) + \mathbf{L}^{-1}(\mathbf{t})\mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}) - \tilde{\mathbf{B}}\mathbf{v}(\mathbf{t}) \tag{31}$$

or

$$\begin{aligned}\dot{\mathbf{e}}(\mathbf{t}) = (\mathbf{R} - \tilde{\mathbf{B}}\overline{\mathbf{K}})\mathbf{e}(\mathbf{t}) + \mathbf{L}^{-1}(\mathbf{t})\mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}) \\ - \tilde{\mathbf{B}}\mathbf{v}(\mathbf{t}) + \tilde{\mathbf{B}}\overline{\mathbf{K}}\mathbf{e}(\mathbf{t})\end{aligned} \tag{32}$$

Substituting $\mathbf{v}(\mathbf{t})$ from (30) and $\mathbf{e}(\mathbf{t}) = \mathbf{z}(\mathbf{t}) - \overline{\mathbf{z}}(\mathbf{t})$ in Eq. (32) we obtain

$$\begin{aligned}\dot{\mathbf{e}}(\mathbf{t}) = (\mathbf{R} - \tilde{\mathbf{B}}\overline{\mathbf{K}})\mathbf{e}(\mathbf{t}) + \mathbf{L}^{-1}(\mathbf{t})\mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}) \\ + \tilde{\mathbf{B}}\overline{\mathbf{K}}\mathbf{z}(\mathbf{t})\end{aligned} \tag{33}$$

Since $(\mathbf{R} - \tilde{\mathbf{B}}\overline{\mathbf{K}})$ is a stability matrix, the two systems described by equations (28) and (29) can be made equivalent in the least square sense if

$$\mathbf{L}^{-1}(\mathbf{t})\mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}) = -\tilde{\mathbf{B}}\overline{\mathbf{K}}\mathbf{z}(\mathbf{t}) \tag{34}$$

i.e.,

$$\mathbf{u}(\mathbf{t}) = -\mathbf{B}^{\#}(\mathbf{t})\mathbf{L}(\mathbf{t})\tilde{\mathbf{B}}\overline{\mathbf{K}}\mathbf{z}(\mathbf{t}) \tag{35}$$

where $\mathbf{B}^{\#}(\mathbf{t})$ is the generalized inverse (Nashed 1976, Lovass-Nagy et al. 1978) of $\mathbf{B}(\mathbf{t})$ which satisfies

$$\mathbf{B}^{\#}(\mathbf{t})\,\mathbf{B}(\mathbf{t}) = \mathbf{I}_n \tag{36}$$

Substituting for $\mathbf{z}(\mathbf{t})$ in Eq. (35) from Eq. (27), we obtain

$$\mathbf{u}(\mathbf{t}) = -\mathbf{B}^{\#}(\mathbf{t})\mathbf{L}(\mathbf{t})\tilde{\mathbf{B}}\overline{\mathbf{K}}\mathbf{L}^{-1}(\mathbf{t})\mathbf{x}(\mathbf{t}) \tag{37}$$

which provides the control law for the time-varying system (25) and further, if

$$\mathbf{u}(\mathbf{t}) = -\mathbf{K}(\mathbf{t})\mathbf{x}(\mathbf{t}) \tag{38}$$

then the time-varying gain matrix $\mathbf{K}(\mathbf{t})$ is given by

$$K(t) = B^{\#}(t)L(t)\tilde{B}\overline{K}L^{-1}(t) \tag{39}$$

The time-invariant gain $\overline{K}$ is chosen by applying the pole placement technique or the optimal control theory on the auxiliary system (29) such that it is asymptotically stable.

3.2 Observer Based Controller. Observer designs for general linear time-varying systems have been attempted mostly by using canonical transformations where the system matrices are converted to a companion form (O'Reilly 1983). But with the advent of the new approach developed in this paper, a time-varying periodic system can be converted to a time-invariant one and the observer problem becomes much simpler.

The observer equation for the system defined by Eq. (26) can be written as

$$\dot{\hat{x}}(t) = F(t)\hat{x}(y) + \overline{G}(t)y(t) + H(t)u(t) \tag{40}$$

where $\hat{x}(t)$ is the $n \times 1$ observer state vector. Matrices $F(t)$ and $H(t)$ are chosen such that $F(t) = A(t) - \overline{G}(t)C(t)$ and $H(t) = B(t)$. An error state vector can be defined as $\bar{e}(t) = x(t) - \hat{x}(t)$, such that the error dynamics is given by

$$\dot{\bar{e}}(t) = [A(t) - \overline{G}(t)C(t)]e(t) \tag{41}$$

The gain matrix $\overline{G}(t)$ has to be selected such that the error vector asymptotically goes to zero as time tends to infinity. To compute this gain matrix, the well-known dual system approach is considered. The dual system of Eq. (26) is given by (Kwakernaak and Sivan, 1972)

$$\dot{\bar{x}}(t) = A^T(t)\bar{x}(t) + C^T(t)u(t) \tag{42}$$

$$\bar{y} = B^T(t)\bar{z}(t) \tag{43}$$

An L-F transformation can be applied to the above system and thus the dual system can also be transformed to a time-invariant one as shown in Section 3.1. With the L-F transformation

$$\bar{x}(t) = \tilde{L}(t)\bar{z}(t) \tag{44}$$

the dual system gets transformed to

$$\dot{\bar{z}}(t) = \tilde{R}\bar{z}(t) + \tilde{L}^{-1}(t)C^T(t)u(t) \tag{45}$$

Once again a time-invariant auxiliary system is taken as

$$\dot{\hat{z}}(t) = \hat{R}\hat{z}(t) + \hat{C}\hat{v}(t) \tag{46}$$

Similar to the case of controller design, an error dynamics of equations (45) and (46) yields

$$u(t) = -C^{\#}(t)\tilde{L}(t)\hat{C}\tilde{G}\bar{z}(t) \tag{47}$$

where $C^{\#}(t)$ is the generalized inverse of $C^T(t)$. With $u(t) = -G(t)\bar{x}(t)$ and using Eq. (44) in (47) provides the time-varying gain matrix $G(t)$ of the dual system and is computed as

$$G(t) = -C^{\#}(t)\tilde{L}(t)\hat{C}\tilde{G}\tilde{L}^{-1}(t) \tag{48}$$

Therefore, by theorem 4.8 of Kwakernaak and Sivan (1972) the observer gain matrix is $\overline{G}(t) = G^T(t)$. The time invariant gain matrix $\tilde{G}$ is obtained by applying pole placement technique or optimal control theory on system (47) such that the error dynamics given by equation (41) goes asymptotically to zero. Using the relations $y(t) = C(t)x(t)$ and $u(t) = -K(t)\hat{x}(t)$ in equation (40), the observer based closed-loop system can be written as

$$\begin{Bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{Bmatrix} = \begin{bmatrix} A(t) & -B(t)K(t) \\ \overline{G}(t)C(t) & A(t) - \overline{G}(t)C(t) - B(t)K(t) \end{bmatrix} \begin{Bmatrix} x(t) \\ \hat{x}(t) \end{Bmatrix} \tag{49}$$

4 Applications

Example 1: A Commutative System. Consider the commutative system described by the set of equations as shown below:

$$\dot{x}(t) = \omega\begin{bmatrix} -1 + \alpha\cos^2(2\pi t) & 1 - \alpha\sin(2\pi t)\cos(2\pi t) \\ -1 - \alpha\sin(2\pi t)\cos(2\pi t) & -1 + \alpha\sin^2(2\pi t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ t \end{bmatrix} u(t) \tag{50}$$

where α is a parameter. The fundamental solution (STM), $\Phi(t)$ of this system is known in a closed form as (Mohler 1991)

$$\Phi(t) = \begin{bmatrix} e^{(\alpha-1)2\pi t}\cos(2\pi t) & e^{-2\pi t}\sin(2\pi t) \\ -e^{(\alpha-1)2\pi t}\sin(2\pi t) & e^{-2\pi t}\cos(2\pi t) \end{bmatrix} = P(t)e^{\bar{C}t} \tag{51}$$

Factoring the state transition matrix as shown above, the T-periodic Liapunov-Floquet (L-F) transformation matrix $P(t)$ is

$$P(t) = \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{bmatrix} \tag{52}$$

and

$$e^{\bar{C}t} = \begin{bmatrix} e^{(\alpha-1)2\pi t} & 0 \\ 0 & e^{-2\pi t} \end{bmatrix} \tag{53}$$

From Eq. (53) it is evident that the commutative system (50) is unstable for all $\alpha > 1$. This system can be controlled by designing a full state feedback controller described in Section 3.1. The response characteristics are shown in Figs. 2, 3(*a*), and 3(*b*).

Example 2: A Triple Inverted Pendulum. As another example, consider the triple inverted pendulum subjected to a periodically varying load as shown in figure (1). Following Lagrange's formulation, the linearized equations of motion of the system are obtained as

$$M\ddot{\eta} + C\dot{\eta} + [K + K^*(t)]\eta = u(t) \tag{54}$$

where,

$$M = ml^2\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \quad C = c\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$[K + K^*(t)] = k\begin{bmatrix} (2-\gamma) & -1 & \alpha\gamma \\ -1 & (2-\gamma) & -(1+\alpha\gamma) \\ 0 & -1 & \{1 - \gamma(1-\alpha)\} \end{bmatrix}$$

Also $\gamma = \frac{Pl}{k}$, $P = P_1 + P_2\cos(\omega t)$ and $u(t) = \{u_1(t)u_2(t)u_3(t)\}^T$

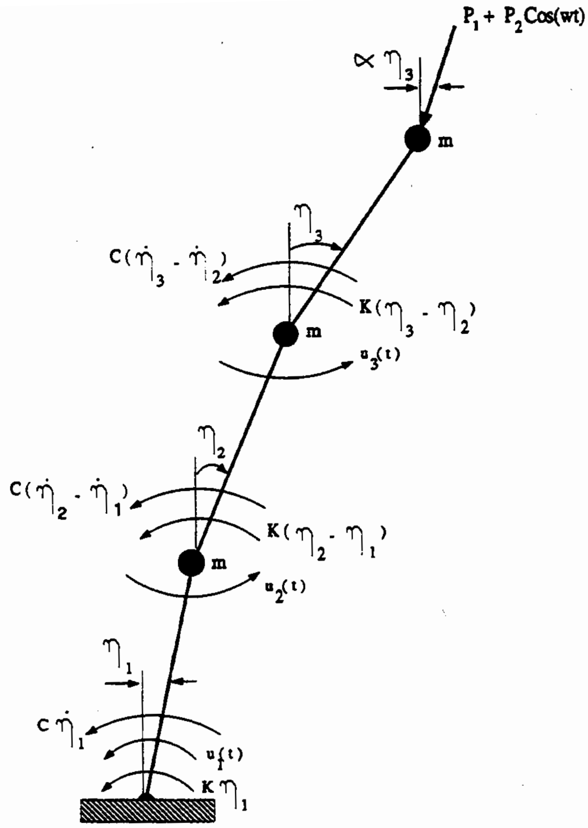

Fig. 1 A triple inverted pendulum subjected to a periodic load

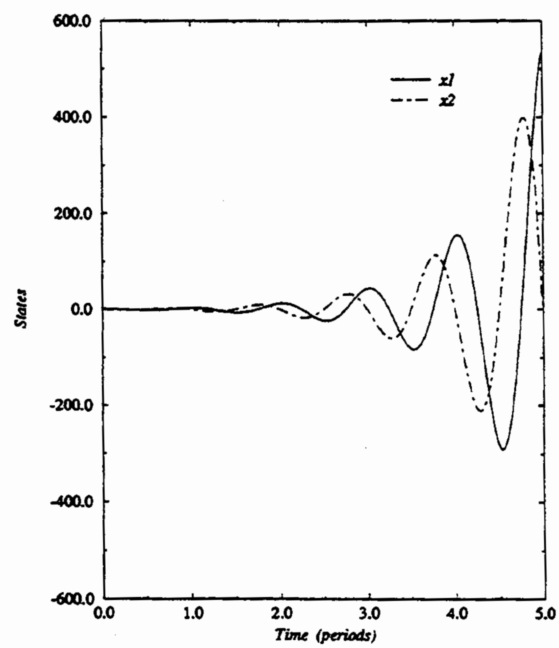

Fig. 2 Uncontrolled states of the commutative system (alpha = 1.2)

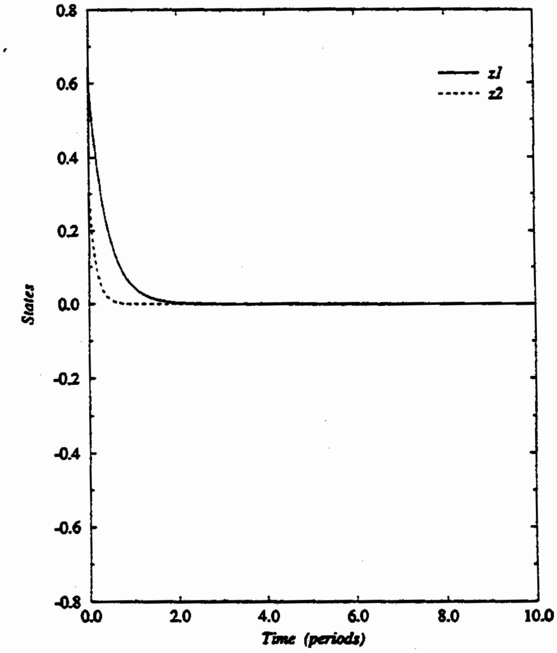

Fig. 3(*a*) Full state feedback control of the time-invariant commutative system (alpha = 1.2)

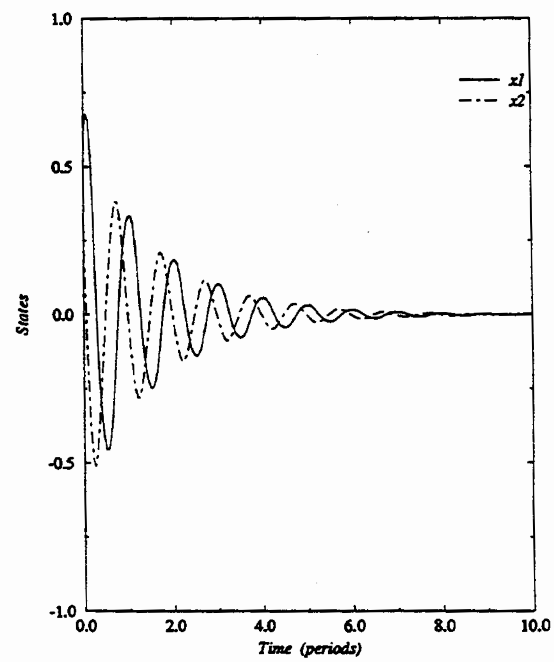

Fig. 3(*b*) Full state feedback control of the time-varying commutative system (alpha = 1.2)

is the control torque vector. Note that in the above equations the stiffness matrix has time-varying periodic terms with period $\mathbf{T} = \frac{2\pi}{\omega}$. The set of second order equations (54) can be rewritten in the state-space form as

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}(\mathbf{t})\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}) \tag{55}$$

where $\mathbf{x}(\mathbf{t}) = [\eta_1(t)\ \eta_2(t)\ \eta_3(t)\ \dot{\eta}_1(t)\ \dot{\eta}_2(t)\ \dot{\eta}_3(t)]^T$,

$$\mathbf{A}(\mathbf{t}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \bar{\mathbf{k}}(\gamma-3) & \bar{\mathbf{k}}(3-\gamma) & -\bar{\mathbf{k}} & -\bar{c} & 0 & 0 \\ \bar{\mathbf{k}}(4-\gamma) & 2\bar{\mathbf{k}}(\gamma-3) & \bar{\mathbf{k}}(3-\gamma) & \bar{c} & -\bar{c} & 0 \\ -\bar{\mathbf{k}} & \bar{\mathbf{k}}(4-\gamma) & -\bar{\mathbf{k}}(3+\gamma(\alpha-2)) & 0 & \bar{c} & -\bar{c} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

with $\bar{\mathbf{k}} = \dfrac{\mathbf{k}}{\mathbf{ml}^2}$ and $\bar{\mathbf{c}} = \dfrac{\mathbf{c}}{\mathbf{ml}^2}$.

Using the procedure described in Section 3.1, Eq. (54) can now be transformed into a time-invariant form and the corresponding full state gain matrix can be easily computed. Under the assumption that only the state variables $[\eta_1\ \eta_2\ \eta_3]$ are measured, the output equation is given by,

$$\mathbf{y(t)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \mathbf{x} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{Bmatrix} \qquad (56)$$

Based on the above equation an observer based controller is also designed as per the procedure described in Section 3.2.

The above designed controllers should cater to both the failure modes, namely, the divergence and the flutter instabilities of the inverted pendulum (Leipholz 1972) common to this kind of structure. The divergence instability is a buckling failure and the flutter instability is a dynamic failure. The type of instability is usually determined from the characteristic exponents of the FTM $\mathbf{\Phi(t)}$. If one of the real exponents is positive then the system fails in buckling. On the other hand if the real part of at least one pair of the complex exponents becomes positive then the system fails in flutter. It is to be noted that the type of failure depends on the parameters $\mathbf{P}_1$, $\mathbf{P}_2$, α, and ω. For the system considered, the values of $\mathbf{m}$, $\mathbf{l}$ and $\mathbf{k}$ are assumed to be constants.

The controllers are tested for a typical parameter set of $\dfrac{\mathbf{P}_1\mathbf{l}}{\mathbf{k}} = 1.0$, $\dfrac{\mathbf{P}_2\mathbf{l}}{\mathbf{k}} = 0.7$, $\bar{\mathbf{k}} = 1.0$, $\alpha = 1.0$, $\bar{\mathbf{c}} = 0.5$ and $\omega = 1.0$, for which the system fails in buckling. The corresponding response characteristics of the system with and without the controllers are shown in Figs. 4, 5(*a*), 5(*b*), 6(*a*), 6(*b*), and 6(*c*). It is found that the dual system approach has been very successful in designing observers for periodic mechanical systems.

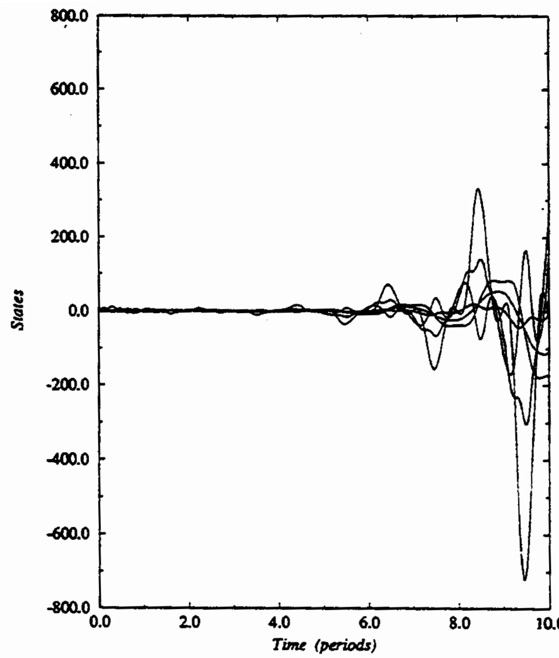

Fig. 4 Uncontrolled states of the triple inverted pendulum

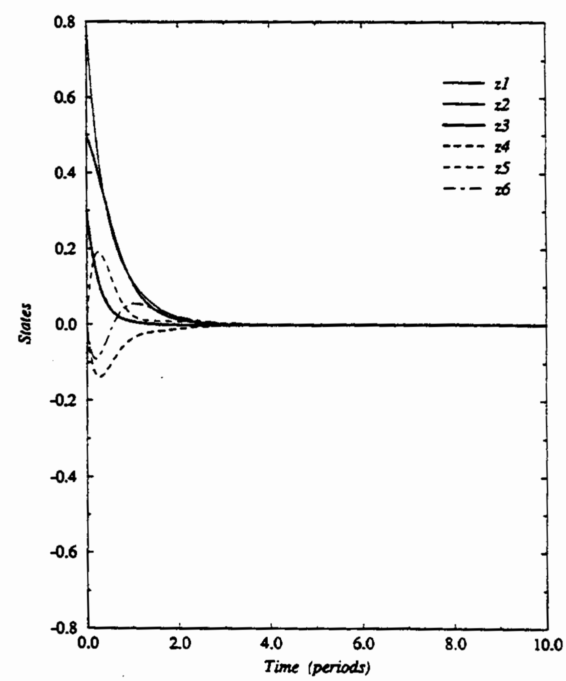

Fig. 5(*a*) Full state feedback control of the time-invariant triple inverted pendulum

5 Discussion of Results and Conclusions

For the first time, a technique for computing the Liapunov-Floquet (L-F) transformation matrix for general linear systems with periodically varying parameters is being presented. The method exploits the advantage of evaluating the state transition matrix as an explicit function of time in terms of Chebyshev polynomials. Hence the L-F matrix can also be expressed as an explicit function of time in a closed form for a given set of parameters. This L-F matrix can then be used to transform the periodically varying control problem to a form which is suitable for application of time-invariant techniques of control theory. The new methodology opens more avenues in designing control strategies for linear periodic systems.

Controller designs for the commutative system as well as the triple inverted pendulum have been successfully achieved using the L-F transformation matrix. Figs. 2 and 4 show the open-loop unstable behavior of the two examples considered. The closed-loop response of the examples considered with the designed controllers are described in Figs. 3(*a*), 3(*b*), 5(*a*), 5(*b*), 6(*a*), 6(*b*), and 6(*c*). The results of full state feedback control are depicted in Figs. 3(*a*), 3(*b*), 5(*a*), and 5(*b*). Figures 3(*a*) and 5(*a*) portray the closed-loop behavior of the time-invariant system which was obtained by applying the L-F transformation to the examples which are originally

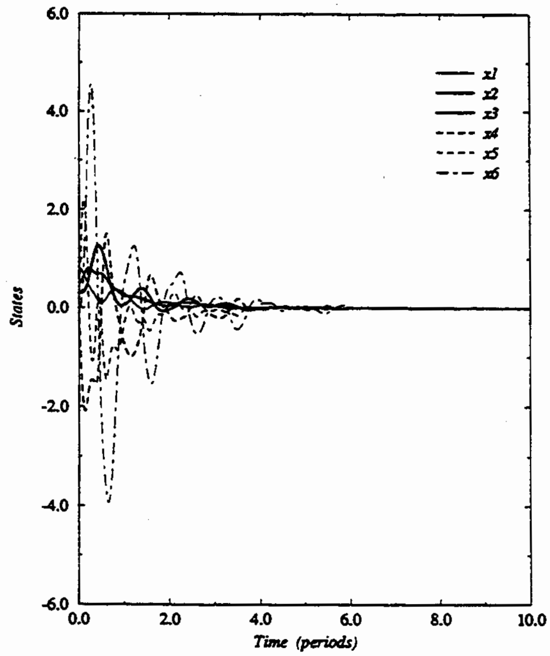

Fig. 5(*b*) Full state feedback control of the time-varying triple inverted pendulum

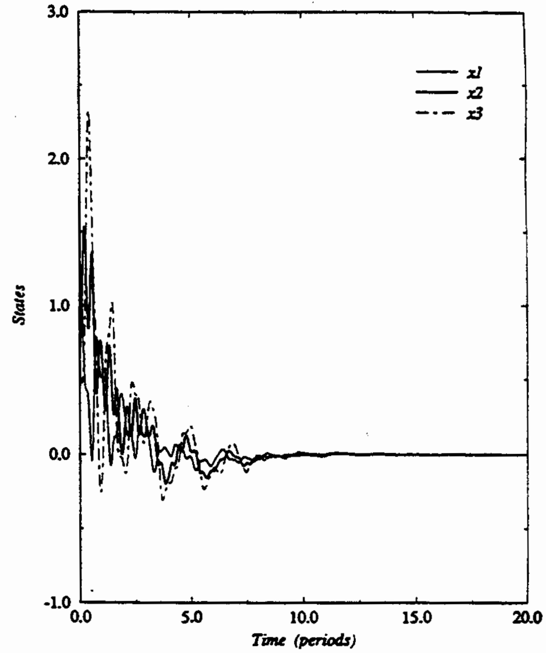

Fig. 6(*b*) Observer based control of the time-varying triple inverted pendulum (states: x1, x2, x3)

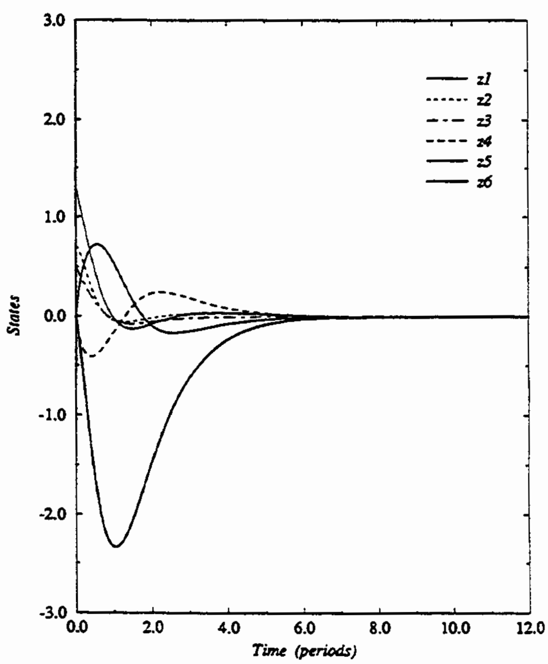

Fig. 6(*a*) Observer based control of the time-invariant triple inverted pendulum

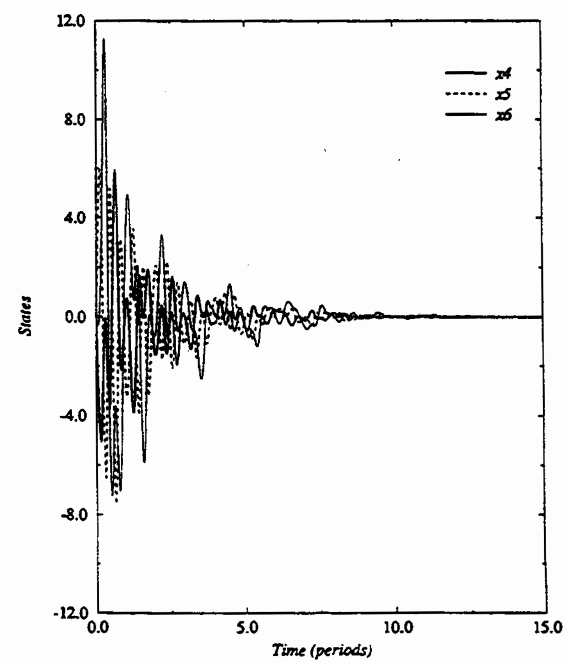

Fig. 6(*c*) Observer based control of the time-varying triple inverted pendulum (states: x4, x5, x6)

time-varying in nature. Figures 3(*b*) and 5(*b*) depict the closed-loop behavior of the corresponding time-varying systems. From Figs. 6(*a*), 6(*b*), and 6(*c*) it is quite clear that the dual system approach of designing observers for periodically time varying systems is an accurate and efficient procedure. Although the results presented are for the buckling failure, similar results were also obtained for the flutter failure. But, for the sake of brevity those results are omitted.

In conclusion, a novel technique in designing control strategies for periodic time-varying linear systems has been suggested. The approach is based on the formulation of a general procedure for computing the Liapunov-Floquet transformation matrices for such systems. The procedure incorporates the expansion of the state vector and the periodic coefficients of the time-varying system in terms of Chebyshev polynomials such that the state transition and the L-F matrices can be computed as explicit functions of time. For small-scale systems it is even possible to compute the L-F matrix in a symbolic form through the use of MATHEMATICA/MACSYMA/MAPLE. The application of Liapunov-Floquet transformation allows the designer to work in a time-invariant domain for developing control strategies for the time periodic systems. It is needless to say that in this approach one never has to worry about canonical forms. It is concluded that the technique is applicable to general linear periodic systems and will hopefully serve as a viable alternate tool in the design of controllers for this class of problems in future.

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APPENDIX

In Eq. (1) let

$$\mathbf{A}(t) = \mathbf{A}_0 + \overline{\mathbf{A}}_0(t) \tag{A1}$$

where $\mathbf{A}_0$ and $\overline{\mathbf{A}}_0(t)$ are $n \times n$ matrices.

The $\hat{\mathbf{Z}}$ matrix appearing in equation (15) can be written as

$$\hat{\mathbf{Z}}_{nm \times nm} = [\mathbf{A}_0 \otimes \mathbf{G}^T + \mathbf{C}_A \otimes \mathbf{G}^T \overline{\mathbf{Q}}] \tag{A2}$$

where $\mathbf{C}_A$ is the coefficient matrix of $\overline{\mathbf{A}}_0(t)$ and from Sinha and Wu (1991), $\mathbf{G}^T$ the $m \times m$ integration operational matrix and $\overline{\mathbf{Q}}(\mathbf{d}_j)$ the $m \times m$ product operational matrix are given as

$$\mathbf{G}^T = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \cdots & \cdot & 0 \\ -1/8 & 0 & 1/8 & 0 & \cdots & \cdot & 0 \\ -1/6 & -1/4 & 0 & 1/12 & \cdots & \cdot & 0 \\ 1/16 & 0 & -1/8 & 0 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \dfrac{1}{4(m-1)} \\ \dfrac{(-1)^m}{2m(m-2)} & \cdot & \cdot & \cdot & \cdots & \dfrac{-1}{4(m-2)} & 0 \end{bmatrix} \tag{A3}$$

$$\mathbf{Q}(\mathbf{d}_j) = \begin{bmatrix} \mathbf{d}_0 & \mathbf{d}_1/2 & \mathbf{d}_2/2 & \cdots & \mathbf{d}_{m-1} \\ \mathbf{d}_1 & \mathbf{d}_0+\mathbf{d}_2/2 & (\mathbf{d}_1+\mathbf{d}_3)/2 & \cdots & (\mathbf{d}_{m-2}+\mathbf{d}_m)/2 \\ \mathbf{d}_2 & (\mathbf{d}_1+\mathbf{d}_3)/2 & \mathbf{d}_0+\mathbf{d}_4/2 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{d}_{m-1} & (\mathbf{d}_{m-2}+\mathbf{d}_m)/2 & \cdot & \cdots & \mathbf{d}_0+\mathbf{d}_{2m-2}/2 \end{bmatrix} \tag{A4}$$

Kronecker Product:

Consider a 2×2 square matrix $\mathbf{A}$ and an $n \times m$ matrix $\mathbf{B}$. The Kronecker product of the two matrices is defined by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{a}_{11}\mathbf{B} & \mathbf{a}_{12}\mathbf{B} \\ \mathbf{a}_{21}\mathbf{B} & \mathbf{a}_{22}\mathbf{B} \end{bmatrix} \tag{A5}$$

The resulting matrix is of size $2n \times 2m$.