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Control of time-periodic systems via symbolic computation with application to chaos control

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Abstract

A general method for the control of linear time-periodic systems employing symbolic computation of Floquet transition matrix is considered in this work. It is shown that this method is applicable to chaos control. Nonlinear chaotic systems can be driven to a desired periodic motion by designing a combination of a feedforward controller and a feedback controller. The design of the feedback controller is achieved through the symbolic computation of fundamental solution matrix of linear time-periodic systems in terms of unknown control gains. Then, the Floquet transition matrix (state transition matrix evaluated at the end of the principal period) can determine the stability of the system owing to classical techniques such as pole placement, Routh–Hurwitz criteria, etc. Thus it is possible to place the Floquet multipliers in the desired locations to determine the control gains. This method can be applied to systems without small parameters. The Duffing's oscillator, the Rössler system and the nonautonomous parametrically forced Lorenz equations are chosen as illustrative examples to demonstrate the application.

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1. Introduction

The modeling of many engineering, economic and biological systems leads to a set of nonlinear ordinary differential equations with time periodic coefficients. Under suitable conditions, these global equations of motion can be expanded about equilibrium solutions to yield equations that describe the local behavior of the system. The stability of the linearized system (about a hyperbolic fixed point or a periodic solution) is completely characterized by Floquet Theory.

The control problems associated with linear time periodic systems can be handled by existing state-space techniques for general time varying systems. The optimal control techniques require the solution of a periodic Riccati equation or for the general dynamic output feedback case, one can solve two time varying differential equations containing periodic coefficients [1,2]. Other methods are based on transformation of the original systems to suitable canonical forms in order to utilize the special properties in controller design [3,4]. However, the canonical transformations are not unique and the implementation is difficult. Stabilization of time-varying systems has also been suggested by the 'pole assignment' technique [5,6] for special class of problems. Recently, Sinha and Joseph [7] have shown that Floquet theory can be used in the control of time periodic systems by implementing Lyapunov–Floquet (L–F) transformation. This technique allows one to design a controller in the time-invariant domain and obtain the desired periodic controller via L–F transformation. Successful applications to rotating mechanical systems with parametric systems were demonstrated by Sinha et al. [8] and Boghiu et al. [9]. Sinha et al. [10] have also demonstrated an application of this method in chaos control to a periodic orbit. Using the same approach, Deshmukh et al. [11] designed state feedback and observer based controllers for large-scale time periodic system. The efficient computation of the L–F transformation matrices was achieved using a 'hybrid formulation' developed by Butcher and Sinha [12]. As pointed out by the authors themselves (also see [13,14]), there is a drawback in Sinha and Joseph method in that the asymptotic stability is not guaranteed because of the presence of a generalized inverse of a rectangular matrix in the expression for the control law. This is because of the fact that the time invariant auxiliary system and the original time periodic system can only be equivalent by minimizing a least square error. The designer may have to go through trial and error or iterations before coming up with a control law that successfully stabilizes the original system asymptotically.

In [15], Sinha and Butcher have shown that the state transition matrix for the linear time-periodic system can be computed in symbolic form in terms of the unknown parameters of the system and by employing the symbolic computation of the state transition matrix of a parametric linear time-periodic system, the linear controller can be designed to guarantee the asymptotic stability of the closed-loop system.

Another possible application of this methodology is in chaos control. Recently, control strategies to suppress bifurcations and chaos in nonlinear systems have been proposed in the literature and most of those studies have dealt with autonomous systems. The nonautonomous systems are often converted to autonomous discrete systems using the Poincaré map where the fixed points of the Poincaré map correspond to the periodic orbits of the initial system and the stabilization problem of orbits is reduced to the stabilization of fixed points. First, it must be pointed out that, except in rare cases, it is very difficult to obtain an analytical expression for the Poincaré map and we need to obtain an approximate expression with numerical or experimental data. Secondly, this method (discrete case) leads to some serious limitations. The most popular among these methods

is the OGY (Ott–Grebogi–Yorke) method [16,17] which uses the Poincaré map of the system. The changes in parameters can only be discrete and this method can be used to stabilize only orbits whose largest Lyapunov exponent is small compared to the reciprocal of time-interval between parameter changes. This can be avoided with the design of continuous time control system.

Because the local stabilization problem of a periodic orbit leads to a linear system with time periodic coefficients, we must use the Floquet theory to guarantee the stability of the system. Sinha et al. [10] have proposed a general approach in the design of active controllers for nonlinear systems exhibiting chaos. In this method, it is shown that a system exhibiting chaos can be driven to a desired periodic motion by designing a combination of a feedforward and a feedback controller. The feedback controller is designed by the method proposed by Sinha and Joseph [7] where a time-invariant auxiliary system is constructed and stabilized with pole placement method. However, since this method uses a least square approximation technique, in certain parameter ranges of the system, it may not be good enough to ensure stability of the periodic orbit.

In the present work, a combination of feedforward and feedback controllers is designed to drive the nonlinear chaotic system to a desired periodic trajectory. The Floquet transition matrix of the closed-loop system is computed symbolically in terms of the unknown control gains of the feedback controller. Then by applying some well-known classical techniques, such as Shur–Cohn or Routh–Hurwitz criteria, we can select the gains to guarantee the asymptotic stability. It is also possible to select the control gains by placing the Floquet multipliers in desired locations. One advantage of this approach is that we can design the controllers in a symbolic form: it can explicitly show how the stability depends on the control gains. Moreover, this technique is applicable to general periodic systems without any restriction on the size of periodic terms.

The Duffing's oscillator, the Rössler system and the nonautonomous parametrically forced Lorenz equations are chosen as illustrative examples to demonstrate the application.

2. Symbolic computation of the Floquet transition matrix

Sinha and Butcher [15] have outlined a technique for computing the fundamental solution matrix for a linear time-periodic dynamical system explicitly as a function of the systems parameters via Picard iteration and expansion in shifted Chebyshev polynomials. When this state transition matrix is evaluated at the end of the principal period, the parameter-dependent Floquet transition matrix (FTM) is obtained, and hence it is possible to express the local stability conditions in a closed form.

Two formulations were outlined: one applicable to general periodic systems and the other for systems in which the periodic system matrix contains constant terms. It was shown that the latter, called the “alternate formulation”, converges much faster than the former “general formulation”.

First, we briefly discuss the “general formulation”.

Consider the linear time-periodic system

$$\dot{x} = A(t, \alpha)x, \quad A(t, \alpha) = A(t + T, \alpha), \quad x(0) = x^0 \quad (1)$$

The equivalent integral form can be written as

$$x(t) = x^0 + \int_0^t A(\tau, \alpha)x(\tau) d\tau \quad (2)$$

where $x(t) \in R^N$ is the state vector, $t \in R^+$ denotes time, and $\alpha \in R^L$ is a parameter vector. The method of Picard iteration is utilized to find the $(k+1)$ th approximation

$$\begin{aligned}
 x^{(k+1)}(t) &= x^0 + \int_0^t A(\tau_k, \alpha)x^{(k)}(\tau_k)d\tau_k \\
 &= \left[I + \int_0^t A(\tau_k, \alpha)d\tau_k + \int_0^t A(\tau_k, \alpha) \int_0^{\tau_k} A(\tau_{k-1}, \alpha)d\tau_{k-1}d\tau_k \right. \\
 &\quad \left. + \dots + \int_0^t A(\tau_k, \alpha) \dots \int_0^{\tau_1} A(\tau_0, \alpha)d\tau_0 \dots d\tau_k \right] x^0
 \end{aligned}
 \tag{3}$$

where τ_0, \dots, τ_k are all dummy variables. The expression in the square brackets is an approximation to the fundamental solution matrix $\Phi(t, \alpha)$ since it is truncated after a finite number of terms (iterations). After the period is normalized to unity via the transformation $t = T\tau$, the normalized 1-periodic system matrix $\bar{A}(\tau, \alpha) = \bar{A}(\tau + 1, \alpha)$ is expanded in m shifted Chebyshev polynomials of the first kind valid in the interval $[0, 1]$ as

$$\bar{A}(\tau, \alpha) = \hat{T}^T(\tau)D(\alpha)
 \tag{4}$$

where $\hat{T}^T(\tau)$ is the $N \times Nm$ Chebyshev polynomial matrix and $D(\alpha)$ is the $Nm \times N$ Chebyshev coefficient matrix. This is inserted into Eq. (3) and the integration and product operational matrices [18] associated with the Chebyshev polynomials are employed to obtain an expression for the expansion of $\Phi(t, \alpha)$ in shifted Chebyshev polynomials as

$$\Phi^{(p,m)}(\tau, \alpha) = \hat{T}^T(\tau) \left[\hat{I} + \left(\sum_{k=1}^p [L(\alpha)]^{k-1} \right) P(\alpha) \right] = \hat{T}^T(\tau)B(\alpha)
 \tag{5}$$

where (p, m) refers to the number of iterations and polynomials employed in the approximation, respectively. The $Nm \times N$ Chebyshev coefficient matrix $B(\alpha)$ is expressed in terms of \hat{I} , $L(\alpha) = \hat{G}^T \hat{Q}_D(\alpha)$, and $P(\alpha) = \hat{G}^T D(\alpha)$ via simple matrix multiplications and additions of the operational matrices. The parameter-dependent FTM $H(\alpha) = \Phi(\tau = 1, \alpha)$ is obtained by evaluating the fundamental solution matrix at the end of the principal period. For additional details on this algorithm (such as convergence issues) as well as an outline of the ‘‘alternate formulation’’ (see [15]).

3. Design of a linear controller in a symbolic form for systems with periodic coefficients

Consider the full state feedback control problem,

$$\dot{x} = A(t)x + B(t)u; \quad A(t) = A(t + T), \quad B(t) = B(t + T)
 \tag{6}$$

with $u = -KX$ such that (6) is controllable. Thus, the closed-loop system is given by

$$\dot{x} = (A - BK)x
 \tag{7}$$

The stability behavior of the origin can be completely characterized in terms of the state transition matrix $\Phi(t, t_0)$ of the system. As $[A(t) - B(t)K]$ is periodic, we can compute the state transition matrix symbolically using the method described in Section 2 with a software like Mathematica.

After selecting a ‘reasonably good’ value for the number of Picard iterations p (so that it converges), we obtain the Floquet transition matrix (FTM) Φ which depends on the control gains

k_1, \dots, k_n . Increasing the number of polynomials m beyond a certain value cannot improve the accuracy if p is too low. Since in general one does not know the exact solution, it was found that the best procedure to achieve a desired accuracy, is, for high value of m , to select an appropriate p for a converged FTM (we compare with a Runge–Kutta method), and then adjust m until the point where desired convergence is retained. This should be done at various locations in the parameter region (the control gains) of interest to ensure uniform convergence before the final analytical solution is obtained.

Generally speaking, Φ contains high degree polynomial expressions of k_1, \dots, k_n and therefore, solving for the eigenvalues in terms of k_1, \dots, k_n is not a trivial task. Instead, we may select them by applying appropriate stability criteria. Thus, we define the characteristic polynomial of Φ as

$$P = \text{Det}(\Phi - \rho I) \tag{8}$$

where ρ (Floquet multiplier) is an eigenvalue of Φ . Then, by introducing the complex fractional transformation

$$\rho = \frac{\lambda + 1}{\lambda - 1} \tag{9}$$

we map the unit circle of the complex plane to the left half plane. It transforms (8) into a polynomial in λ , where coefficients depend on the control gains, and the criteria for continuous-time systems (for example the Routh–Hurwitz criterion) may be applied. Based on the stability condition, the control gains are chosen to guarantee the asymptotic stability.

Moreover, we can also use a technique which is similar to ‘‘Pole Placement Method’’. Indeed, the FTM eigenvalues can be placed at desired locations ρ_i inside the unit circle. Thus, we have:

$$P = \text{Det}(\Phi - \rho I) = \prod_{i=1}^n (\rho - \rho_i) \tag{10}$$

The coefficients of ρ^k in the left-hand side of Eq. (10) are dependent on the control gains k_1, \dots, k_n and coefficients of ρ^k in the right-hand side are known. A term by term comparison provides the values of k_1, \dots, k_n . We can call this method ‘‘Multipliers placement method’’.

For example, we can apply the method to the Mathieu equation: $\ddot{y} + (a + b \cos t)y = 0$, where a and b are the systems parameters. Thus, we have $\dot{X} = AX$ with

$$X = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

and

$$A = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -b & 0 \end{bmatrix} \cos(t).$$

For the parameter values $a=b=1.5$, the system is unstable as shown in Fig. 1. By introducing the control vector $u(t)$, we have $\dot{X} = AX + Bu$ with $u = -KX$; $K = (k_1 \ k_2)$. B is such that the system is controllable and we select $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, the closed-loop system is given by $\dot{X} = (A - BK)X$. k_1 and k_2 must satisfy Routh–Hurwitz inequalities in order to guarantee asymptotic stability. Fig. 2 plotted with Mathematica shows the area that guarantees asymptotic stability. Twenty-two Picard iterations and 18 polynomials are used in the computation and the controlled dynamics

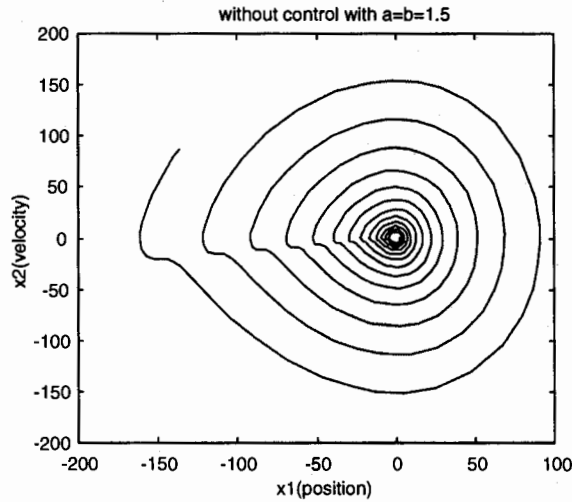


Fig. 1. Unstable case of the Mathieu equation.

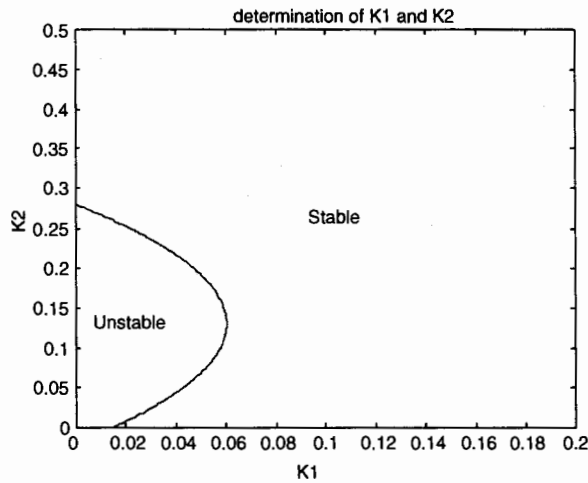


Fig. 2. Stable–unstable areas of asymptotic stability for the Mathieu equation.

is shown in Fig. 3. By using the method described earlier as the “Multipliers placement method”, we may place the desired Floquet multipliers at, say, $\rho_1=0.3$; $\rho_2=0.1$ to find $k_1=0.175$; $k_2=0.381$.

4. Design of controllers for nonlinear systems exhibiting chaos

We consider a general nonlinear, nonautonomous system

$$\dot{x} = f[t, x(t)] \tag{11}$$

which has a chaotic attractor for a given set of parameter values.

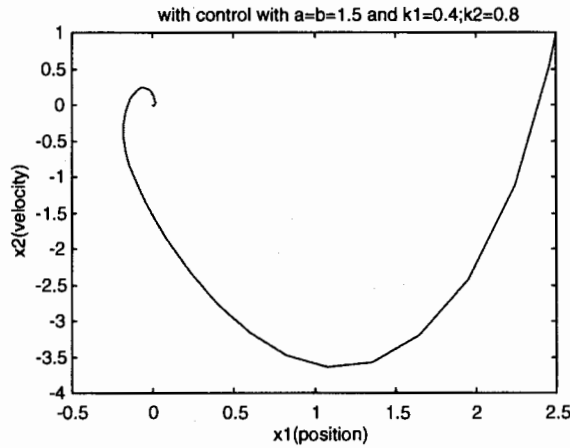


Fig. 3. Phase plane plot of the controlled Mathieu equation.

As the chaotic response is not desirable, the objective of this method is to choose a control law to drive the response to a desired periodic orbit. Let $y(t)$ be the desired orbit which may be an unstable periodic orbit of Eq. (11). The advantage of this method is that the desired periodic orbits need not to be a solution of the uncontrolled system.

Now we consider the system with the control law $u(t)$ (with two control parts) given by

$$\dot{x} = f[t, x(t)] + u(t); \quad u(t) = u_f + u_t \tag{12}$$

where u_f is the feedforward control and u_t is the time-varying feedback control. u_f and u_t are taken as

$$u_f = \dot{y} - f[t, y(t)], \quad u_t = F(t)(x - y) \tag{13}$$

where $F(t)$, in general, is the time-varying linear state feedback matrix.

As shown in [10], by defining the error between the actual and desired trajectories

$$e = x - y \tag{14}$$

and using the form of u_f given in Eq. (13), Eq. (12) can be written as

$$\dot{e} = g[t, e(t)] + u_t \tag{15}$$

Assuming that

$$g(t, 0) = 0 \quad \forall t \geq 0 \tag{16}$$

where g is a C^1 function, and defining

$$A(t) = \left[\frac{\partial g(t, e)}{\partial e} \right]_{e=0} \tag{17}$$

and

$$h(t, e) = g(t, e) - A(t)e \tag{18}$$

if the condition

$$\lim_{\|e\| \rightarrow 0} \sup \frac{\|h(t, e)\|}{\|e\|} = 0 \quad (19)$$

holds, then the system

$$\dot{e} = A(t)e(t) + u_t \quad (20)$$

is called the linearization of Eq. (15) around the origin.

The local stability of Eq. (12) is guaranteed by the global asymptotic stability of the linearized system given by Eq. (20). Since $A(t)$ is periodic, we can design constant control gains by computing the state transition matrix symbolically using the method outlined in Section 3. The FTM is obtained in a symbolic form as a function of both the system parameters and the control gains. The FTM eigenvalues can be placed at desired locations to yield appropriate values of the control gains as a function of the parameters in some domain \mathcal{D} . Since the control gains are functions of the system parameter and guarantee stability for all parameters in \mathcal{D} , we can eliminate chaos from the entire parameter range we are interested in. The desired periodic orbit may be independent of the parameters, which means that the system is driven to the same orbit in the entire parameter range, or it may be a function of the parameters in the given range. With this method, it is possible to drive the system to a periodic orbit which does not have the same period as the original system. However, in order to compute the FTM of the linearized system, we must assume that the frequencies (ω_1 : frequency of the original system; ω_2 : frequency of the desired orbit) are commensurate ($n_1\omega_1 = n_2\omega_2$; n_1, n_2 , integers).

5. Applications

5.1. The Duffing's oscillator

Consider the Duffing's oscillator with the control law $u(t)$ given by

$$\ddot{x} = -\alpha x - x^3 - 2\zeta\dot{x} + f \cos \omega t + u \quad (21)$$

with $u(t) = u_f + u_t$, where α is the stiffness parameter, ζ ($\zeta > 0$) is the viscous damping coefficient, f and ω are the amplitude and frequency of the external input, respectively. It is well-known that in the absence of the control term, for certain values of system parameters, Eq. (21) possesses a chaotic attractor. The objective is to choose the feedback control law such that the response of the controlled Duffing's equation results in an asymptotically stable desired periodic orbit or a fixed point. For $\alpha = -1$, $\zeta = 0.125$, $f = 1$, $\omega = 1$, the system undergoes chaotic motion, as shown in Fig. 4. We choose the desired simple periodic orbit as $y(t) = 2 \cos t + \sin t$, and from Eq. (13), the feedforward control u_f is

$$u_f = \ddot{y} - y + y^3 + 0.25y - \cos t. \quad (22)$$

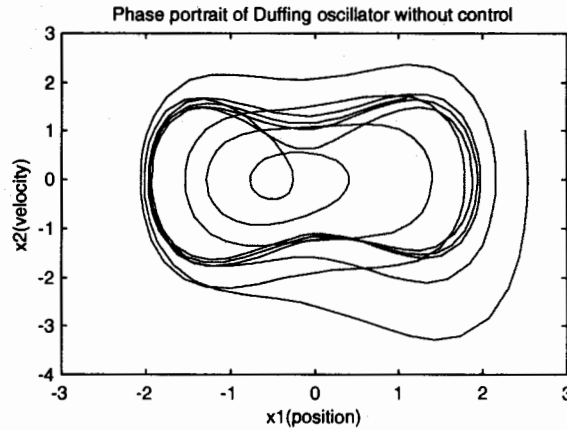


Fig. 4. Chaotic motion of Duffing’s oscillator.

and the linearized equation in the error state $e(t)$ is given by

$$\dot{e}(t) = \left(\begin{bmatrix} 0 & 1 \\ -6.5 & -0.25 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -4.5 & 0 \end{bmatrix} \cos 2t + \begin{bmatrix} 0 & 0 \\ -6 & 0 \end{bmatrix} \sin 2t \right) e(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t \quad (23)$$

Taking $u_t = -Ke$; $K = (k_1 \ k_2)$, we can obtain the Floquet transition matrix (FTM) Φ for the closed-loop system which depends on k_1 and k_2 . Fig. 5 shows the area which guarantees asymptotic stability in the k_1k_2 plane. If k_1 and k_2 are selected from the stable region then $e \rightarrow 0$ as $t \rightarrow \infty$ and consequently $x \rightarrow y$. Eighteen Chebyshev polynomials and 22 Picard iterations were used in the computation. Therefore, the closed-loop system given by Eq. (21) takes the form $\ddot{x} = x - x^3 - 0.25\dot{x} + \ddot{y} - y + y^3 + 0.25\dot{y} - k_1(x - y) - k_2(\dot{x} - \dot{y})$ where $y(t) = 2 \cos t + \sin t$, $k_1 = 0.25$ and $k_2 = 0.4$. The oscillator is controlled to the periodic orbit $y(t)$ as shown on Fig. 6. In the “Multipliers placement method”, if we place the Floquet multipliers at $\rho_1 = 0.8$; $\rho_2 = 0.3$ then using

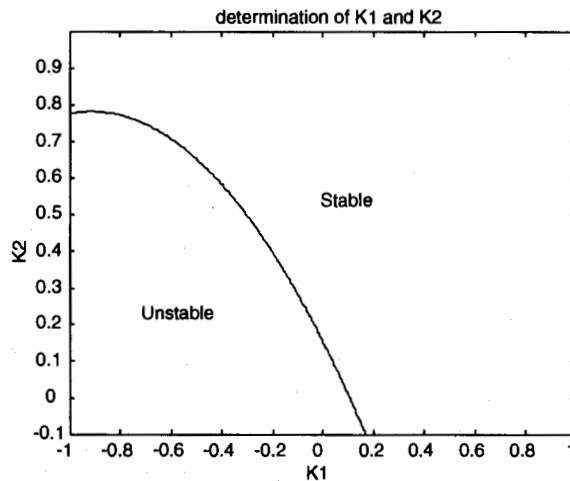


Fig. 5. Stable–unstable areas of asymptotic stability for the linearized system for the Duffing’s oscillator.

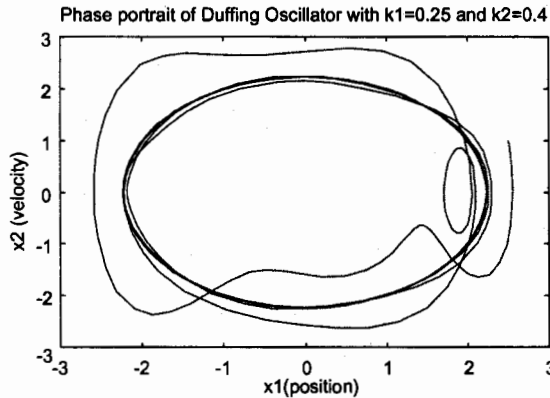


Fig. 6. Trajectory of the Duffing’s oscillator controlled to periodic $y(t)=2\cos t + \sin t$.

Mathematica, we find $k_1=0.102$; $k_2=0.196$ which yields similar results. The same method can be applied for control to a fixed point. For this case, let the desired trajectory be $y(t)=y_0$ and thus from Eq. (13)

$$u_f = y_0 + y_0^3 - \cos t. \tag{24}$$

It should be noticed that in this case matrix $A(t)$ of Eq. (20) has constant coefficients and we can directly employ Routh–Hurwitz criterion to generate the feedback portion of the control law.

For example, if the desired fixed point is $(y_0, \dot{y}_0) = (-0.4, 0)$, then the error equation can be shown to be

$$\begin{Bmatrix} \dot{e} \\ \ddot{e} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.52 & -0.25 \end{bmatrix} \begin{Bmatrix} e \\ \dot{e} \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t$$

Once again taking $K = \{k_1 \ k_2\}$, the closed-loop system in error e is given by

$$\begin{Bmatrix} \dot{e} \\ \ddot{e} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.52 - k_1 & -0.25 - k_2 \end{bmatrix} \begin{Bmatrix} e \\ \dot{e} \end{Bmatrix} \tag{25}$$

By using the Routh–Hurwitz criterion, we obtain the conditions:

$$\begin{cases} k_1 > 0.52 \\ k_2 > -0.25 \end{cases}$$

Using $k_1=0.9$ and $k_2=0$, the controlled equation is given as $\ddot{x} = (1 - k_1)x - x^3 - (0.25 + k_2)\dot{x} - 0.4(k_1 + k_2) + 0.4 - 0.4^3$. The dynamics of the oscillator controlled to the fixed point $(y_0, \dot{y}_0) = (-0.4, 0)$ is shown in Fig. 7.

5.2. The Rössler’s system

The Rössler equations form a nonlinear system notable for its recognizable strange attractor (the “Rössler bands”) and the example it presents of the low degree of nonlinearity necessary to induce chaos. The Rössler equations are given by

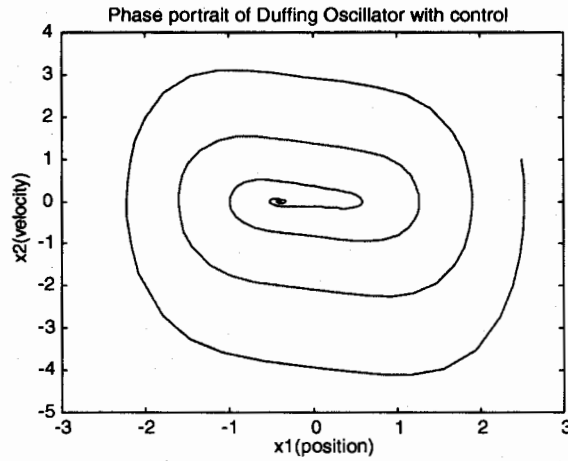


Fig. 7. Control of the Duffing's oscillator to the fixed point $(y_0, \dot{y}_0) = (-0.4, 0)$.

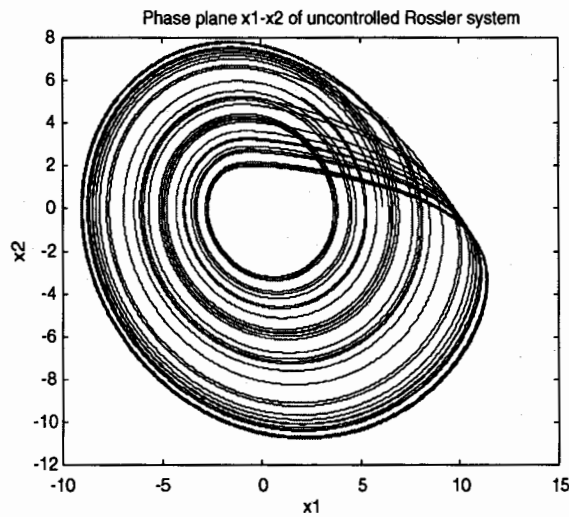


Fig. 8. The chaotic attractor of Rössler system for $a=5.7$.

$$\begin{aligned}
 \dot{x}_1 &= -x_2 - x_3 \\
 \dot{x}_2 &= x_1 + 0.2x_2 \\
 \dot{x}_3 &= 0.2 + x_3(x_1 - a)
 \end{aligned}
 \tag{26}$$

where a is the critical parameter. There is only a single mild nonlinearity, located in the third state equation. Nonetheless, for $a=5.7$, Eq. (26) possesses a familiar chaotic attractor shown in Figs. 8 and 9. Consider an arbitrary desired periodic trajectory

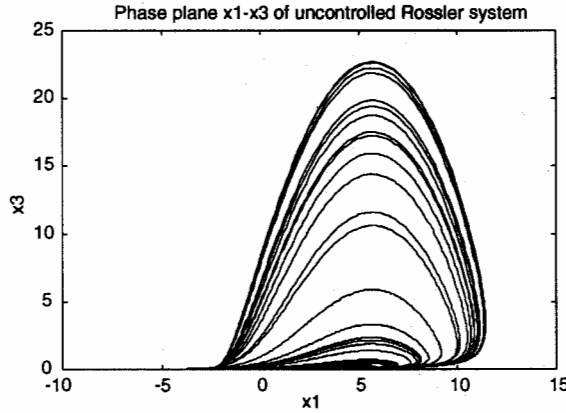


Fig. 9. The chaotic attractor of the Rössler system for $a=5.7$.

$$\begin{cases} y_1 \\ y_2 \\ y_3 \end{cases} = \begin{cases} 5 + \cos 2\pi t \\ \sin 2\pi t \\ \sin 2\pi t \end{cases} \tag{27}$$

With the feedforward control u_f and the feedback control u_t Eq. (26) takes the form

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 + u_{f,1} + u_t \\ \dot{x}_2 &= x_1 + 0.2x_2 + u_{f,2} \\ \dot{x}_3 &= 0.2 + x_3(x_1 - 5.7) + u_{f,3} \end{aligned} \tag{28}$$

The feed forward control u_f is chosen as

$$\begin{cases} u_{f,1} = \dot{y}_1 - f_1(y_1, t) \\ u_{f,2} = \dot{y}_2 - f_2(y_2, t) \\ u_{f,3} = \dot{y}_3 - f_3(y_3, t) \end{cases} \tag{29}$$

Including this feedforward control and linearizing about the goal trajectory given by Eq. (27), we can express the linearized system in $e (= x - y)$ as

$$\begin{cases} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{cases} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ \sin 2\pi t & 0 & -0.7 + \cos 2\pi t \end{bmatrix} \begin{cases} e_1 \\ e_2 \\ e_3 \end{cases} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t \tag{30}$$

As the control is to a periodic orbit, periodic terms appear in the matrix. In general, we may assume $u_t = -K\{e_1 \ e_2 \ e_3\}^T$ with $K = \{k_1 \ k_2 \ k_3\}$. However, for simplicity we set $k_3=0$ and with symbolic computation, we obtain conditions on k_1 and k_2 to guarantee asymptotic stability. Fig. 10 shows the area that guarantees asymptotic stability (18 Chebyshev polynomials and 22 Picard iterations are used). Fig. 11 shows the controlled dynamics to the periodic orbit for a typical set of k_1 and k_2 (control gains) selected from the stable area. Using the ‘‘Multipliers place-

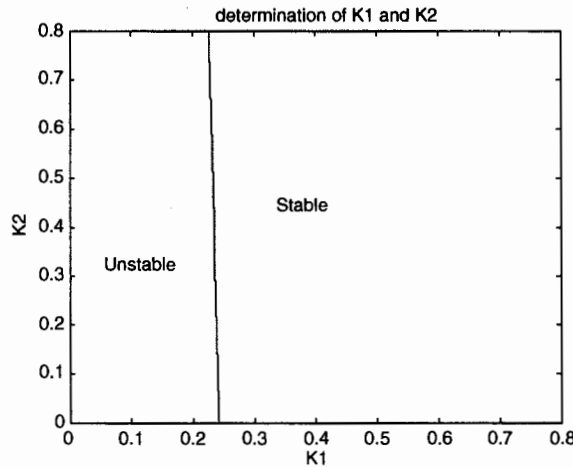


Fig. 10. Stable area guarantees asymptotic stability ($k_3=0$) for the Rössler’s system.

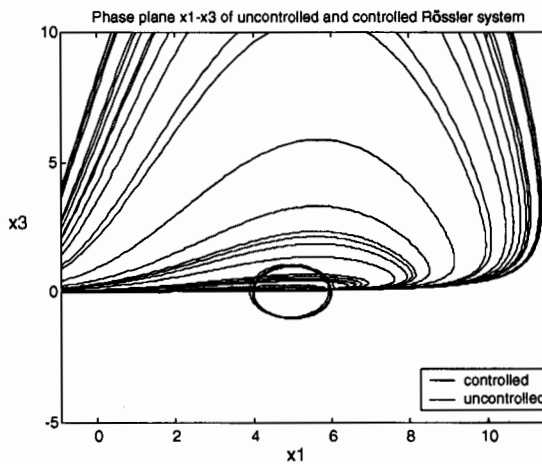


Fig. 11. The motion of the Rössler system is driven to periodic orbit: $(y_1, y_2, y_3) = (5 + \cos 2\pi t, \sin 2\pi t, \sin 2\pi t)$.

ment method”, we place the desired Floquet multipliers at $\rho_1=0.4+0.8j$, $\rho_2=0.4-0.8j$ and $\rho_3=0.5$ and find $k_1=0.421$; $k_2=0.324$; $k_3=-1.145$. Fig. 12 shows the controlled motion with these gains.

5.3. Controlling the dynamics of a chaotic convective flow

The dynamics of this example has been studied by Dorning et al. [19]. We consider a horizontal thermal convection layer subjected to a high temperature at the bottom boundary and a low temperature at the top boundary. Frequently, this simple geometric model provides a relevant representation of the dynamics of the natural cooling of a heat-generating component whose top

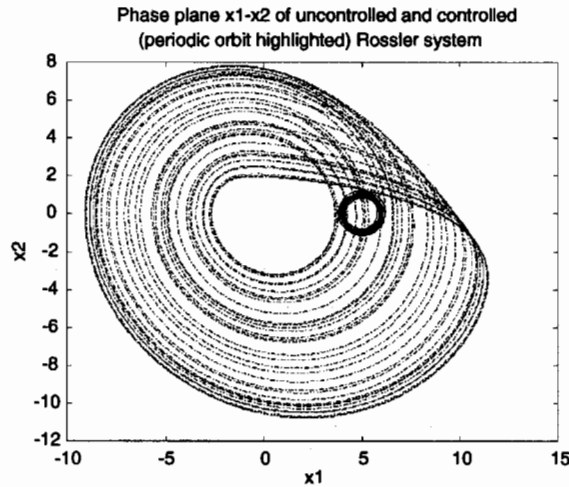


Fig. 12. Control of the Rössler system with “Multipliers placement method” to periodic orbit: $(y_1, y_2, y_3) = (5 + \cos 2\pi t, \sin 2\pi t, \sin 2\pi t)$.

supplies heat to the lower boundary of the fluid layer. In many cases of technological interest, this lower surface is not retained at a constant temperature, rather varies in some time-dependant way which is generally periodic. Hence, in order to ensure the safety of those components in electrical power generating and distribution systems that are dependent upon convective cooling, it is important to be able to control those systems. We use Navier–Stokes equations for mass and momentum conversation. Then, following Lorenz method and making Fourier expansions of the stream function and temperature field and truncating them at the same low order as Lorenz did, it is straightforward to arrive at the nonautonomous parametrically forced Lorenz equations. These are given by

$$\begin{cases} \dot{x}(t) = -\sigma x(t) + \sigma y(t) \\ \dot{y}(t) = [\rho_0 + \rho_1 \cos \omega_1 t]x(t) - y(t) - x(t)z(t) \\ \dot{z}(t) = x(t)y(t) - bz(t) \end{cases} \quad (31)$$

where $x(t)$, $y(t)$, $z(t)$ are, respectively, the time-dependent Fourier coefficients of the first term in the expansion of the stream function and the first two terms in the expansion of the temperature field. For $\rho_0 = 26.5$; $\rho_1 = 5$; $\omega_1 = 2\pi$; $\sigma = 2$; $b = 0.6$, we obtain the “Shaken butterfly”, a chaotic attractor as shown in Figs. 13 and 14 in x – z projection.

Introducing the error e between the actual and the desired trajectories, we obtain the linearized system in e and design the control gains of u_t . Thus, for the arbitrary desired periodic orbit

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} \sin \omega_2 t \\ \sin \omega_2 t \\ 20 + \cos \omega_2 t \end{Bmatrix} \quad (32)$$

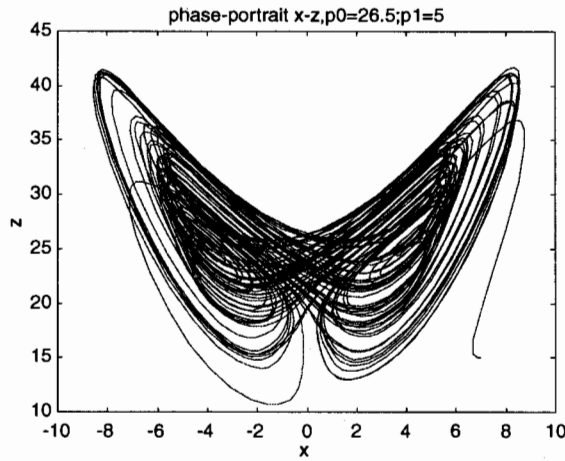


Fig. 13. The “Shaken butterfly” chaotic attractor of the nonautonomous parametrically forced Lorenz equations.

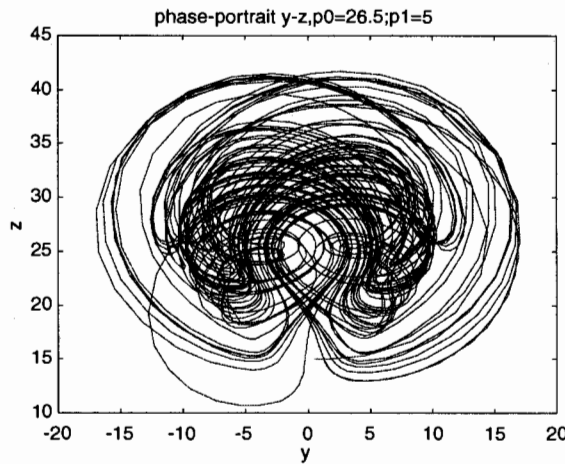


Fig. 14. The uncontrolled trajectory of the nonautonomous parametrically forced Lorenz equations.

and with $u_t = -KX$, where $K = \{k_1, k_2, k_3\}$, $X = \{x \ y \ z\}^T$, and

$$B = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Eq. (31) takes the form

$$\begin{cases} \dot{x}(t) = -\sigma x(t) + \sigma y(t) + u_{f,1} \\ \dot{y}(t) = [\rho_0 + \rho_1 \cos \omega_1 t]x(t) - y(t) - x(t)z(t) + u_{f,2} + u_t \\ \dot{z}(t) = x(t)y(t) - bz(t) + u_{f,3} \end{cases} \quad (33)$$

The feedforward control is chosen as

$$\begin{cases} u_{f,1} = \dot{y}_1 - f_1(y_1, t) \\ u_{f,2} = \dot{y}_2 - f_2(y_2, t) \\ u_{f,3} = \dot{y}_3 - f_3(y_3, t) \end{cases} \quad (34)$$

where $f = \{f_1 \ f_2 \ f_3\}^T$ is the right-hand side of Eq. (31). For the desired periodic orbit, Eq. (34) becomes

$$\begin{cases} u_{f,1} = \omega_2 \cos \omega_2 t \\ u_{f,2} = \omega_2 \cos \omega_2 t + (21 - \rho_0) \sin \omega_2 t + \frac{\sin 2\omega_2 t}{2} + \frac{\rho_1}{2} [\sin(\omega_1 - \omega_2)t - \sin(\omega_1 + \omega_2)t] \\ u_{f,3} = (20b - \frac{1}{2}) + b \cos \omega_2 t - \omega_2 \sin \omega_2 t + \frac{\cos 2\omega_2 t}{2} \end{cases} \quad (35)$$

By including this feedforward control and linearizing about the goal trajectory, we can express the linearized system in e as

$$\begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ [(\rho_0 + \rho_1 \cos \omega_1 t) - 20 - \cos \omega_2 t] & -1 & -\sin \omega_2 t \\ \sin \omega_2 t & \sin \omega_2 t & -b \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_t \quad (36)$$

We select $u_t = -K\{e_1 \ e_2 \ e_3\}^T$ with $K = \{k_1 \ k_2 \ k_3\}$. We can compute the FTM for the closed-loop system if ω_1 and ω_2 are commensurate. Let $\omega_2 = 2\pi = \omega_1$ and, for simplicity, take $k_2 = k_3 = 0$, then with symbolic computation we obtain $k_1 \geq 5.18$ to guarantee the asymptotic stability. The uncontrolled as well as controlled trajectories are shown in Figs. 15 and 16. With 2 control gains k_1 and k_2 ($k_3 = 0$), we obtain conditions on k_1 and k_2 to yield asymptotic stability. Fig. 17 shows the area which guarantees the asymptotic stability using 18 Chebyshev polynomials and 22 Picard iterations. A controller with three gains for the desired periodic orbit

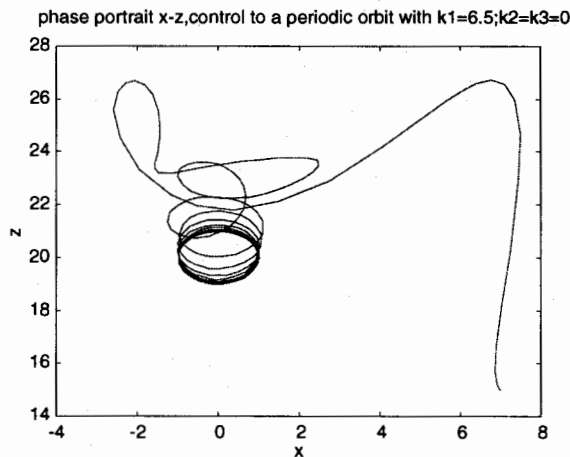


Fig. 15. Control of the nonautonomous parametrically forced Lorenz equations to the periodic orbit: $(y_1, y_2, y_3) = (\sin \omega_2 t, \sin \omega_2 t, 20 + \cos \omega_2 t)$ with $\omega_1 = \omega_2 = 2\pi$.

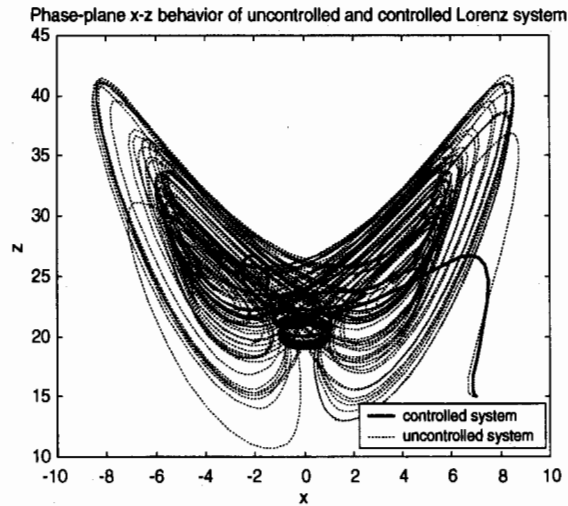


Fig. 16. Controlled motion of the nonautonomous parametrically forced Lorenz equations to the periodic orbit: $(y_1, y_2, y_3) = (\sin \omega_2 t, \sin \omega_2 t, 20 + \cos \omega_2 t)$ with $\omega_1 = \omega_2 = 2\pi$.

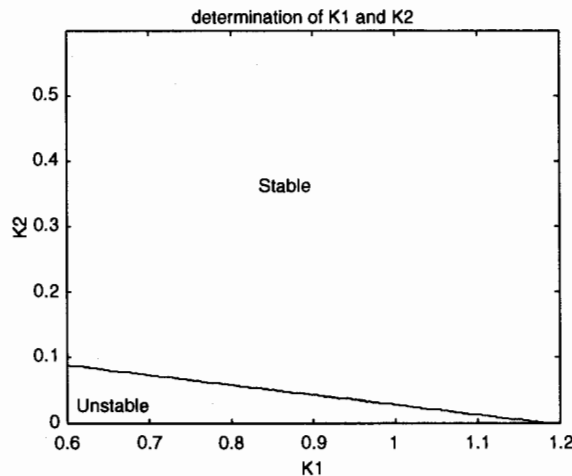


Fig. 17. Stable area guarantees asymptotic stability for the nonautonomous parametrically forced Lorenz equations ($\omega_1 = \omega_2 = 2\pi$) and $k_3 = 0$.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \sin \omega_2 t \\ \sin \omega_2 t \\ 24 + \cos \omega_2 t \end{pmatrix}$$

was also designed and the simulation results are shown in Fig. 18. We can also drive the system to a periodic orbit which does not have the same period as the original system. For example, we can choose $2\omega_2 = \omega_1 = 2\pi$ (period 2) and the desired periodic orbit as

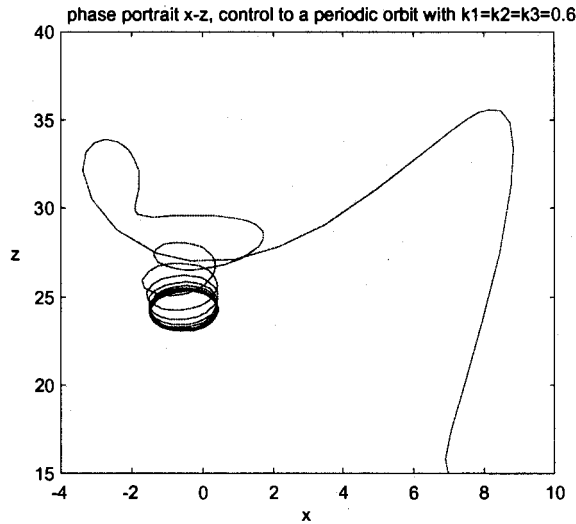


Fig. 18. Controlled trajectory of the nonautonomous parametrically forced Lorenz equations to the periodic orbit: $(y_1, y_2, y_3) = (\sin \omega_2 t, \sin \omega_2 t, 24 + \cos \omega_2 t)$; $\omega_1 = \omega_2 = 2\pi$ with three control gains.

$$\begin{cases} y_1 \\ y_2 \\ y_3 \end{cases} = \begin{cases} \sin \omega_2 t \\ \sin \omega_2 t \\ 20 + \cos \omega_2 t \end{cases}, \quad \text{with } u_t = -KX$$

$$\left(K = \{k_1, k_2, k_3\}, \quad X = \begin{cases} x \\ y \\ z \end{cases}, \quad B = \begin{cases} 0 \\ 1 \\ 0 \end{cases} \right)$$

a controller is designed following the same methodology as before and the response is shown in Fig. 19.

6. Conclusions

A symbolic computational procedure for designing controllers for linear systems with time periodic coefficients has been suggested. Apart from its applicability to the linear systems, the method is found to be useful in the local control of general nonlinear, nonautonomous systems to a desired periodic orbit. In particular, it has been shown that the method can be successfully applied to design active controllers for nonlinear systems exhibiting chaos where the chaotic motion needs to be driven to a periodic orbit. The desired periodic motion need not be embedded in the chaotic attractor and may have any commensurate period as compared to the original time periodic system.

The control vector consists of two parts, viz., a feedforward component and a feedback component. The linearization about the desired periodic orbit leads to a set of linear equations with periodic coefficients and the symbolic computation of the control gains can be achieved using the

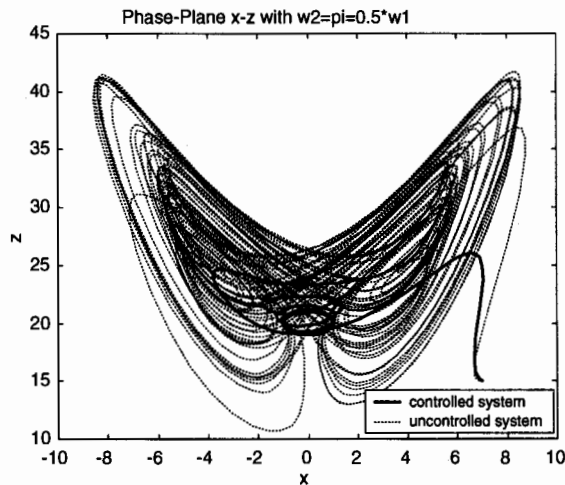


Fig. 19. Controlled motion of the nonautonomous parametrically forced Lorenz equations to the periodic orbit: $(y_1, y_2, y_3) = (\sin \omega_2 t, \sin \omega_2 t, 20 + \cos \omega_2 t)$; $2T_2 = T_1 = 2B$ (period 2).

Chebyshev polynomials and Picard Iteration. Classical techniques, such as Routh–Hurwitz criterion, can be used to stabilize the system once the Floquet transition matrix for the closed-loop system is computed in the symbolic form using a commercially available software such as Mathematica, Maple, etc. Similar to the idea of “Pole placement technique”, a “Multiplier placement method” has been suggested to place the Floquet multipliers inside the unit circle at the desired locations in order to determine the control gains. Numerical results are presented for the Duffing’s oscillator, the Rössler system and the nonautonomous parametrically forced Lorenz equations to show the effectiveness of the technique.

It must be pointed out that this method does not require a lot of computation effort since the algorithm for the symbolic computation, introduced by Sinha and Butcher [15] is based on only simple matrix multiplications and additions. Moreover, this technique guarantees the desired stability which is not the case with the least square minimization method used by Sinha and Joseph [7]. Although this approach provides an approximate symbolic form of the FTM, it is found to be very accurate when compared to the numerical (Runge–Kutta) results assuming that sufficient number of polynomials and iterations are used in the computation. If necessary, one can compute the solution to any desired accuracy. However, the symbolic computation can become very time-consuming for a large number of parameters (a large number of control gains) and for high dimensional systems. Even though we have not used optimal control and we do not design controllers by minimizing the effort, the results are quite good. The convergence appears to be slow as compared to the numerical techniques but it is expected since the approach presented here is symbolic in nature.

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