Control of Dynamic Systems with Time-Periodic Coefficients via the Lyapunov–Floquet Transformation and Backstepping Technique

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Abstract: We address the problem of designing controllers that guarantee asymptotic stability of a class of linear as well as nonlinear dynamical systems with time-periodic coefficients. Using a repeated procedure consisting of the Lyapunov–Floquet transformation, the backstepping technique, and Floquet theory, the asymptotic stability of the closed-loop linearized system is guaranteed. Further, a Lyapunov matrix for the closed-loop asymptotically stable linearized system is constructed. This Lyapunov function is then used to design a combination of linear and nonlinear controllers in order to guarantee the asymptotic stability of the nonlinear system. The methodology is illustrated by designing linear and nonlinear control laws for a system consisting of two statically coupled pendula, each subjected to a time-periodic force acting in the axial direction.

Key Words: Nonlinear control, Floquet theory, Lyapunov–Floquet transformation, backstepping

1. INTRODUCTION

The modeling of many engineering, economic, and biological systems leads to a set of nonlinear ordinary differential equations with time-periodic coefficients. Under suitable conditions, these global equations of motion can be expanded about equilibrium solutions to yield equations that describe the local behavior of the system. The stability of the linearized system (about a hyperbolic fixed point or a periodic solution) is completely characterized by Floquet theory.

The control problems associated with linear time-periodic systems can be handled by existing state-space techniques for general time-varying systems. The optimal control techniques require the solution of a periodic Riccati equation or, for the general dynamic output feedback case, one can solve two time-varying differential equations containing periodic

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coefficients (see Kwakernaak and Sivan, 1972; Bittanti *et al.*, 1991). Other methods are based on the transformation of the original systems to suitable canonical forms in order to utilize the special properties in controller design (Calico and Wiesel, 1984; Calise *et al.*, 1992). However, the canonical transformations are not unique and the implementation is difficult. Stabilization of time-varying systems has also been suggested by the "pole assignment" technique (Kabamba, 1986; Tornambe and Valigi, 1996) for a special class of problems. Recently, Sinha and Joseph (1994) have shown that Floquet theory can be used in the control of time-periodic systems by implementing Lyapunov–Floquet (L–F) transformation. This technique allows one to design a controller in the time-invariant domain and obtain the desired periodic controller via L–F transformation. Successful applications to rotating mechanical systems with parametric systems have been demonstrated by Sinha *et al.* (1998a) and Boghiu *et al.* (1998). Sinha *et al.* (2000) have also demonstrated an application of this method in chaos control to a periodic orbit. Using the same approach, Deshmukh *et al.* (1999) designed state feedback and observer-based controllers for a large-scale time-periodic system. The efficient computation of the L–F transformation matrices was achieved using a "hybrid formulation" developed by Butcher and Sinha (1996). As pointed out by the authors themselves (also see Lee and Balas, 1998; Montagnier *et al.*, 2001), there is a drawback in the method of Sinha and Joseph in that the asymptotic stability is not guaranteed because of the presence of a generalized inverse of a rectangular matrix in the expression for the control law. This is because of the fact that the time-invariant auxiliary system and the original time-periodic system can only be equivalent by minimizing a least-squares error. The designer may have to go through trial and error or iterations before coming up with a control law that successfully stabilizes the original system asymptotically.

The control problem associated with the nonlinear dynamical systems with time-periodic coefficients has not been addressed as frequently and rigorously as its time-invariant counterpart. The approaches that are being extensively researched, but not limited to, include differential geometric control theory, backstepping and forwarding algorithms, and sliding mode control theory. These methodologies have been successfully applied to design robust and adaptive controllers for nonlinear time-invariant systems. For time-periodic nonlinear systems, Sinha and Pandiyan (1994), Sinha *et al.* (1998b) have demonstrated the use of time-dependent normal forms and center manifold theory in the bifurcation analysis of time-periodic nonlinear dynamical systems. Recently, Pandiyan and Sinha (2001) have presented a method for time-varying controller synthesis for nonlinear systems using the Lyapunov method. On the other hand, David and Sinha (2003) have presented a general method for bifurcation control of nonlinear time-periodic systems through an application of normal forms and center manifold theory. Nevertheless, much of this area of research remains to be explored.

Although not foolproof, the method developed by Sinha and Joseph (1994) for the control of linear time-periodic systems is a significant contribution towards developing a practical tool because it avoids tedious computations involved in solving a Riccati differential equation. In this paper, first an approach is developed to guarantee the asymptotic stability of time-periodic linear dynamical systems using the backstepping technique (Krstic *et al.*, 1995), L–F transformation and Floquet theory. The main objective of the control is the stabilization of an original unstable dynamic system due to strong parametric resonance. Unlike the work reported by Sinha and his associates in the past, and Deshmukh *et al.* (1999), the proposed procedure guarantees the asymptotic stability of the closed-loop linear system. Secondly, the result on the stabilization of linear time-periodic system is then further developed to locally stabilize the nonlinear dynamical systems with time-periodic coefficients. A controller designed with the proposed procedure has a combination of linear and nonlinear control laws. The linear controller asymptotically stabilizes the linearized nonlinear system and the nonlinear part of the controller is designed to guarantee the local asymptotic stability of the nonlinear system based on the Lyapunov function constructed from the linearized system.

2. BACKGROUND

Consider an n-dimensional state-space system with time-periodic system matrix given by

$$\dot{x} = A(t)x;\ A(t+T) = A(t). \tag{1}$$

$\Phi(t)$, the state transition matrix (STM) of equation (1), can be factored as (Floquet theory)

$$\Phi(t) = Q(t)\mathrm{e}^{Rt} \tag{2}$$

where $Q(t)$ is a $2T$ periodic matrix and R is a constant matrix. Matrix $Q(t)$ is called the L–F transformation matrix and application of the state transformation $x(t) = Q(t)z(t)$ transforms equation (1) to a completely time-invariant differential equation $\dot{z} = Rz$ (Yakubovich and Starzhinskii, 1975). This time-invariant differential equation completely characterizes the stability of the original time-periodic equation. The eigenvalues of matrix R are called *"Floquet exponents"* of equation (1). For asymptotic stability of equation (1), all the Floquet exponents must have negative real parts. Equivalently, all the eigenvalues of the *Floquet transition matrix* (FTM) (or STM evaluated at the end of principal period, T) must lie within the unit circle of the complex plane.

A practical scheme for the computation of the L–F transformation matrix for general time-periodic systems has been developed by Sinha *et al.* (1996) using Chebyshev polynomials. Butcher and Sinha (1996) refined the algorithm to develop "Hybrid formulation" which is extremely efficient in computing STMs for large-scale systems. In the present paper, we use the hybrid formulation for the computation of L–F transformation matrices.

3. PROBLEM FORMULATION AND METHODOLOGY

Consider the control problem associated with a general linear dynamical system with time-periodic coefficients given by

$$\dot{x} = A(t)x + B(t)u \tag{3}$$

where $x(t)$ is an n-dimensional state vector and $u(t)$ is an m-dimensional control input vector. $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ dimensional system and control input gain matrices, respectively, such that $A(t) = A(t+T)$ and $B(t) = B(t+T)$. It is desirable to find a feedback control vector of the form

$$u(t) = -K(t)x(t) \tag{4}$$

with $K(t) = K(t+T)$ of an appropriate dimension. As mentioned in the introduction, it is possible to design a feedback control law of the form (4) using optimal control techniques by solving a time-periodic Riccati equation. However, we would like to present a new approach that involves the backstepping technique and L–F transformation. More precisely, we formulate successive normal coordinate transformations that reduce system (3) to a subsystem for which it is convenient to design a virtual control law that asymptotically stabilizes the subsystem. The reduced subsystem is obtained through a series of such transformations creating successive stages of application. Then working backwards, through each of the stages, the virtual control law for that particular stage and the backstepping transformation for the previous stage are computed. Eventually, culmination of such an iterative procedure is the expression for the time-periodic gain matrix and a control law (4) that asymptotically stabilizes the original system (3).

3.1. A Stabilization Procedure

The procedure described in this section is applicable to all the stages of successive reduction of system (3) to an appropriate subsystem. So we start with the original system and derive conditions under which asymptotic stability is guaranteed. This very condition forms the basis for a finite number of stages of application of the normal coordinate transformation as explained in detail in the following.

Applying a bounded and invertible transformation, in this paper referred to as a "normal coordinate" transformation given by $x(t) = L(t)\hat{x}(t)$, where $L(t)$ is an $n \times n$ matrix with period T, equation (3) transforms to

$$\dot{\hat{x}} = [L^{-1}(t)A(t)L(t) - L^{-1}(t)\dot{L}(t)]\hat{x} + L^{-1}(t)B(t)u. \tag{5}$$

$L(t)$ is chosen so that

$$L^{-1}(t)B(t) = \begin{bmatrix} 0_{(n-m)\times n} \\ I_{m\times m} \end{bmatrix} = \overline{B}.$$

Hence, without the loss of generality, any system of the form (3) can be transformed to

$$\dot{\hat{x}} = \overline{A}(t)\hat{x} + \overline{B}u \tag{6}$$

where the matrices are identified by comparison of equations (5) and (6). The system given by equation (6) is written in block-partitioned form as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} \overline{A}_{11}(t) & \overline{A}_{12}(t) \\ \overline{A}_{21}(t) & \overline{A}_{22}(t) \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \tag{7}$$

where subscripts 11 and 22 correspond to the blocks of dimension $(n-m) \times (n-m)$ and $m \times m$, respectively. Now, applying yet another bounded, invertible transformation called the "backstepping" transformation (Krstic *et al.*, 1995) to system (7) given by

$$\begin{aligned} \hat{z}_1 &= \hat{x}_1 \\ \hat{z}_2 &= \hat{x}_2 - \breve{K}(t)\hat{x}_1 \end{aligned} \tag{8}$$

and assuming that $\breve{K}(t)$ is known at this point, we obtain

$$\dot{\hat{z}}_1 = (\overline{A}_{11} - \overline{A}_{12}\breve{K})\hat{z}_1 + \overline{A}_{12}\hat{z}_2 \tag{9a}$$

$$\dot{\hat{z}}_2 = (\overline{A}_{21} - \overline{A}_{22}\breve{K} + \dot{\breve{K}} + \breve{K}(\overline{A}_{11} - \overline{A}_{12}\breve{K}))\hat{z}_1 + (\overline{A}_{22} + \breve{K}\overline{A}_{12})\hat{z}_2 + u \tag{9b}$$

where explicit time dependence is suppressed for brevity. It is important to note that transformation (8) depends on the existence of a periodic matrix, $\breve{K}(t)$. This is an unknown matrix and is actually found in the second last stage, but to go ahead with the proceedings in the stabilization procedure, we assume that it is computed and known.

The necessary condition that $\breve{K}(t)$ should satisfy is that the system

$$\dot{\hat{x}}_1 = \overline{A}_{11}(t)\hat{x}_1 + \overline{A}_{12}(t)\hat{x}_2 \tag{10}$$

with $\hat{x}_2(t) = \breve{K}(t)\hat{x}_1$ as a virtual control input is asymptotically stable. Assuming that this condition is satisfied, we now move towards the stabilization of the interconnected system (9a) and (9b). In equation (9b), the control input is split into $u(t) = u_1(t) + u_2(t)$ with $u_2(t)$ chosen so that system

$$\dot{\hat{z}}_2 = (\overline{A}_{22}(t) + \breve{K}(t)\overline{A}_{12}(t))\hat{z}_2 + u_2(t) \tag{11}$$

is asymptotically stable. Stabilization of system (11) is achieved by using Floquet theory and L–F transformation as follows. Applying L–F transformation $\overline{z}_2 = \hat{Q}_{22}\hat{z}_{22}$ (Sinha *et al.*, 1996) to system (11), we obtain

$$\dot{\bar{z}}_2 = \hat{R}_{22}\bar{z}_2 + \hat{Q}_{22}^{-1}(t)u_2(t) \tag{12}$$

where $\hat{R}_{22}$ is a real constant matrix. Choosing $u_2(t) = -\hat{Q}_{22}\hat{K}_{22}\bar{z}_2 = -\hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1}\hat{z}_2$, with $\hat{K}_{22}$ constant so that $\dot{\bar{z}}_2 = (\hat{R}_{22} - \hat{K}_{22})\bar{z}_2$ is asymptotically stable, implies that equation (12) is asymptotically stable.

Remark. Another possible method of stabilizing (12) is to choose $u_2(t) = -(\overline{A}_{22}(t) + \breve{K}(t)\overline{A}_{12}(t) + \bar{K}_2)\hat{z}_2$ so that the closed-loop system $\dot{\hat{z}}_2 = \bar{K}_{22}\hat{z}_2$ is asymptotically stable, i.e. all the eigenvalues of $\bar{K}_{22}$ have negative real parts. The system represented by equations (9a) and (9b), with $u_2(t)$ in its place, is viewed as an interconnection of two subsystems given by

$$\dot{\hat{z}}_1 = (\overline{A}_{11} - \overline{A}_{12}\breve{K})\hat{z}_1 \tag{13a}$$

$$\dot{\hat{z}}_2 = (\overline{A}_{22} + \breve{K}\overline{A}_{12} - \hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1})\hat{z}_2 + u_1 \tag{13b}$$

with $\overline{A}_{12}(t)\hat{z}_2$ in equation (9a) and $(\overline{A}_{21}-\overline{A}_{22}\breve{K}+\dot{\breve{K}}+\breve{K}(\overline{A}_{11}-\overline{A}_{12}\breve{K}))\hat{z}_1$ in equation (9b) as interconnection terms. Note that system (13a) and (13b) without u_1 is asymptotically stable. Therefore, for system (9a) and (9b), u_1 is now chosen exactly to cancel the $(\overline{A}_{11}-\overline{A}_{22}\breve{K}+\dot{\breve{K}}+\breve{K}(\overline{A}_{11}-\overline{A}_{12}\breve{K}))\hat{z}_1$ term. With

$$\begin{aligned} u_1(t) &= (\overline{A}_{11}-\overline{A}_{22}\breve{K}+\dot{\breve{K}}+\breve{K}(\overline{A}_{11}-\overline{A}_{12}\breve{K}))\hat{z}_1 \\ u_2(t) &= -\hat{Q}_{22}\hat{K}_{22}Q_{22}^{-1}\hat{z}_2 \end{aligned} \tag{14}$$

system (9a) and (9b) is given by

$$\begin{bmatrix} \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \end{bmatrix} = \begin{bmatrix} \overline{A}_{11}-\overline{A}_{12}\breve{K} & \overline{A}_{12} \\ 0 & \overline{A}_{22}+\breve{K}\overline{A}_{12}-\hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} \tag{15}$$

and is asymptotically stable, as both of its diagonal subsystems are asymptotically stable and $\overline{A}_{12}(t)$ is a bounded matrix periodic in time. The control law that has rendered the asymptotic stability is given as

$$u(t) = -(\overline{A}_{21}-\overline{A}_{22}\breve{K}+\dot{\breve{K}}+\breve{K}(\overline{A}_{11}-\overline{A}_{12}\breve{K}))\hat{z}_1-\hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1}\hat{z}_2 \tag{16}$$

and, using transformation (8), we obtain

$$u(t) = -(\overline{A}_{21}-\overline{A}_{22}\breve{K}+\dot{\breve{K}}+\breve{K}(\overline{A}_{11}-\overline{A}_{12}\breve{K}))\hat{x}_1-\hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1}(\hat{x}_2+\breve{K}\hat{x}_1) \tag{17}$$

Using $x(t) = L(t)\hat{x}(t)$, the control law in the original coordinates

$$u(t) = [-M_1(t) \quad -M_2(t)]L^{-1}(t)x(t) \tag{18}$$

with

$$\begin{aligned} M_1(t) &= \overline{A}_{21}-\overline{A}_{22}\breve{K}+\dot{\breve{K}}+\breve{K}(\overline{A}_{11}-\overline{A}_{12}\breve{K})+\hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1}\breve{K} \\ M_2(t) &= \hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1}. \end{aligned} \tag{19}$$

3.2. *A Backstepping Scheme for General Systems*

For general time-periodic systems (3), the stabilization procedure has to be applied repeatedly in order to compute the feedback control necessary to ensure the asymptotic stability of the closed-loop system. For integer values of $i = 1, 2, ..., k$, consider a general system of the form

$$\dot{x}^i = A^i(t)x^i + B^i(t)u \tag{20}$$

with $A^1(t) = A(t)$, $B^1(t) = B(t)$ and a superscript denotes the original system for $i = 1$ and subsequently, the i^{th} subsystem for $i > 1$. Applying a normal coordinate transformation ${}^{i}L(t)$ to system (20) satisfying.

$$ {}^{i}L^{-1}B^{i}(t) = \begin{bmatrix} 0_{(n-im)\times m} \\ I_{m\times m} \end{bmatrix} = \bar{B}^{i} \tag{21}$$

we obtain

$$\begin{bmatrix} \dot{\hat{x}}_1^i \\ \dot{\hat{x}}_2^i \end{bmatrix} = \begin{bmatrix} \overline{A}_{11}^{i}(t) & \overline{A}_{12}^{i}(t) \\ \overline{A}_{21}^{i}(t) & \overline{A}_{22}^{i}(t) \end{bmatrix} \begin{bmatrix} \hat{x}_1^i \\ \hat{x}_2^i \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{x}_{i-1} \tag{22}$$

where $\hat{x}_2^0 = u(t)$. This transformation process is carried out "k" times with $A^{i+1}(t) = \overline{A}_{11}^{i}(t)$, $B^{i+1}(t) = \overline{A}_{12}^{i}(t)$ and

$$ {}^{i+1}L^{-1}B^{i+1}(t) = \begin{bmatrix} 0_{(n-(i+1)m)\times m} \\ I_{m\times m} \end{bmatrix} = \bar{B}^{i+1} \tag{23}$$

and "k" is an integer such that the final virtual gain matrix $B^{k+1}(t)$ has a number of columns greater than or equal to the number of rows. The number of times the normal coordinate transformation is applied gives the number of "stages" of application of a stabilization procedure that was referred to earlier. The resulting system is given as

$$\dot{x}_1^{k+1} = A^{k+1}x_1^i + B^{k+1}x_2^{k+1} \tag{24}$$

with explicit time dependence suppressed for brevity and $x_l^{k+1} = \hat{x}_l^k$ for $l = 1, 2$. The system given by equation (23) is asymptotically stabilized by a virtual control law

$$\hat{x}_2^k = -(A^{k+1} + (B^{k+1})^{\#}\bar{K})\hat{x}_1^k = -\breve{K}^{k+1}(t)\hat{x}_1^k \tag{25}$$

with a positive definite $\bar{K}$ and

$$(B^{k+1}(t))^{\#} = (B^{k+1}(t))^{\mathrm{T}}((B^{k+1}(t)(B^{k+1}(t))^{\mathrm{T}})^{-1} \tag{26}$$

is the generalized inverse of $B^{k+1}(t)$ in the minimum norm sense. Now, it is important to note here that since $B^{k+1}(t)$ has a number of rows less than the number of columns, the generalized inverse in the minimum norm sense enables the virtual control $\hat{x}_2^k$ to stabilize system (24) asymptotically. This is quite different from the generalized inverse in Deshmukh *et al.* (1999), which is obtained by minimizing a least-squares error between the two quantities and should not be confused with the minimum norm generalized inverse used here. The stabilization of equation (24) by virtual control triggers a backward tracking process of finding backstepping transformations and virtual control laws for all the intermediate stages leading back to the actual control law. System (24) is in the normal coordinates and is transformed into original coordinates by applying the inverse of the transformation given by equation (20) for $i = k$. Thus tracing backwards from k to the original system (3) for $i = 1, 2, \ldots., k-1, k$, the backstepping transformation in each stage of application of a stabilization procedure can be computed as

$$\begin{aligned} \hat{z}_1^{k+1-i} &= \hat{x}_1^{k+1-i} \\ \hat{z}_2^{k+1-i} &= \hat{x}_2^{k+1-i} - \breve{K}^{k+2-i}(t)\hat{x}_1^{k+1-i}. \end{aligned} \tag{27}$$

The virtual control input for the $(k-i)$th stage is given by

$$\hat{x}_2^{k-i}(t) = \{[-M_1(t) \; - M_2(t)]L^{-1}(t)x(t)\}^{k+1-i} \tag{28}$$

with

$$\begin{aligned} M_1(t) &= \overline{A}_{21} - \overline{A}_{22}\breve{K} + \dot{\breve{K}} + \breve{K}(\overline{A}_{11} - \overline{A}_{12}\breve{K}) + \hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1}\breve{K} \\ M_2(t) &= \hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1} \end{aligned} \tag{29a}$$

where the symbol $\{\ \}^j$ denotes the quantities in the curly brackets in the stage j. If Floquet theory is used with equation (12) in that stage, equation (29a) gives the desired virtual and actual control gains. If an alternate method in Remark 1 is used in a particular stage, then the expression (29a) changes to

$$\begin{aligned} M_1(t) &= \overline{A}_{21} - \overline{A}_{22}\breve{K} + \dot{\breve{K}} + \breve{K}(\overline{A}_{11} - \overline{A}_{12}\breve{K}) + \hat{Q}_{22}\hat{K}_{22}\hat{Q}_{22}^{-1}\breve{K} \\ M_2(t) &= (\overline{A}_{22} + \breve{K}\overline{A}_{12} + \bar{K}_2). \end{aligned} \tag{29b}$$

The virtual control law is in the normal coordinates and is transformed to the original coordinates by normal coordinate transformation in each of the stages. A proper perspective to understand the successive process is to realize that, at every stage, the proceedings happen exactly as per the details outlined in Section 3.1 except for the fact that the actual control takes the form of virtual control for all the intermediate stages.

4. LOCAL STABILIZATION OF NONLINEAR SYSTEMS

The results obtained in the previous section for linear systems can be used to design local controllers for nonlinear systems. The time-periodic nonlinear systems considered here form a particular class of nonlinear systems referred to as "non-critical systems". For such systems including the Hamiltonian systems, the Floquet multipliers of the linearized time-periodic system are not critical. If the multipliers are critical, then one has to compute the center manifold dynamics and design the control law, which is an entirely different and exhaustive problem called "bifurcation control", and no attempts are made here to solve this problem.

Consider a local description of a nonlinear dynamical system with time-periodic coefficients given by

$$\dot{x} = A(t)x + f(x,t) + (B(t) + g(x,t))u \tag{30}$$

where $A(t)$, $B(t)$ are as described in Section 3 and $f(x,t)$ and $g(x,t)$ are $n \times 1$ and $n \times m$ dimensional vector and matrix, respectively, of smooth nonlinearities (monomials) in x, and T periodic in time. All elements of $f(x,t)$ and $g(x,t)$ are 0 when $x = 0$. The control vector u comprises of a linear as well as a nonlinear control, i.e. $u = u_l + u_{nl}$. From the results obtained in Section 3, the linearized system of equation (30) given by equation (3) can be stabilized asymptotically using backstepping and L–F transformation by a feedback control law of the form (4). The asymptotically stable closed-loop linear system is written as

$$\dot{x} = (A(t) - B(t)K(t))x(t) \tag{31}$$

where we have used $u_l = -K(t)x$.

At this point, using Floquet theory and L–F transformation, a symmetric positive definite Lyapunov function matrix can be obtained for system (31). Applying L–F transformation $x(t) = Q(t)z(t)$ to system (31), we obtain a completely time-invariant system $\dot{z} = Rz$ where R is a constant, real matrix and, since system (31) is asymptotically stable, R has all its eigenvalues with negative real parts. A Lyapunov function matrix is found for system $\dot{z} = Rz$ so that $R^{\mathrm{T}}P + PR = -C$, C being symmetric and positive definite. Since $V(t) = z^{\mathrm{T}}(t)Pz(t)$ is a Lyapunov function for $\dot{z} = Rz$, the Lyapunov function for system (31) can be computed as

$$\hat{V}(t) = x^{\mathrm{T}}(t)\hat{P}(t)x(t), \ \hat{P}(t) = Q^{-\mathrm{T}}(t)PQ^{-1}(t). \tag{32}$$

Simple calculations show that $\hat{P}(t)$ satisfies

$$\dot{\hat{P}} + (A(t) - B(t)K(t))^{\mathrm{T}}\hat{P}(t) + \hat{P}(t)(A(t) - B(t)K(t)) = -\hat{C}(t) \tag{33}$$

where $\hat{C}(t) = Q^{-\mathrm{T}}(t)CQ^{-1}(t)$ is a positive definite function.

The nonlinear control vector $u_{nl}(t)$ is designed on the basis of the Lyapunov function (32) constructed for the linearized system. With the linear control vector $u_l(t)$ in its place, system (30) can be written as

$$\begin{aligned} \dot{x} &= (A(t) - B(t)K(t))x + f(x,t) + (B(t) + g(x,t))u_{nl}(t) \\ &- g(x,t)K(t)x \end{aligned} \tag{34}$$

or

$$\begin{aligned} \dot{x} &= (A(t) - B(t)K(t))x + \bar{f}(x,t) + (B(t) + g(x,t))u_{nl}(t) \\ \bar{f}(x,t) &= f(x,t) - g(x,t)K(t)x. \end{aligned} \tag{35}$$

The time derivative of the Lyapunov function (32) along the trajectories of system (35) is given by

$$\begin{aligned}\dot{\hat{V}}(t) &= \{x^{\mathrm{T}}[(A(t)-B(t)K(t))^{\mathrm{T}}\hat{P}(t)+\hat{P}(t)(A(t)-B(t)K(t))+\dot{\hat{P}}(t)]x\}\\ &+ [\bar{f}^{\mathrm{T}}(x,t)\hat{P}(t)+u_{nl}^{\mathrm{T}}[B(t)+g(x,t)]^{\mathrm{T}}\hat{P}(t)]x\\ &+ x^{\mathrm{T}}[\hat{P}(t)\bar{f}(x,t)+\hat{P}(t)[B(t)+g(x,t)]u_{nl}]. \end{aligned}\tag{36}$$

Following Freeman and Kokotovic (1996), the nonlinear control vector is chosen as

$$\begin{aligned} u_{nl}(t) &= -\frac{[B+g]^{\mathrm{T}}\hat{P}x}{x^{\mathrm{T}}\hat{P}[B+g][B+g]^{\mathrm{T}}\hat{P}x}(x^{\mathrm{T}}\hat{P}\hat{P}x+\bar{f}^{\mathrm{T}}\bar{f}+3x^{\mathrm{T}}\hat{P}\bar{f})\\ &= 0 \qquad \begin{array}{l}\forall x \ni [x^{\mathrm{T}}\hat{P}[B+g][B+g]^{\mathrm{T}}\hat{P}x]\neq 0\\ \forall x \ni [x^{\mathrm{T}}\hat{P}[B+g][B+g]^{\mathrm{T}}\hat{P}x]=0\end{array}\end{aligned}\tag{37}$$

where the arguments of variables have been dropped for brevity.

Therefore, using equations (33) and (37), equation (36) yields

$$\dot{\hat{V}} = x^{\mathrm{T}}[-\hat{C}(t)-2(\hat{P}x+\bar{f})^{\mathrm{T}}(\hat{P}x+\bar{f})]x<0 \tag{38}$$

for all x and t such that $[x^{\mathrm{T}}\hat{P}[B+g][B+g]^{\mathrm{T}}\hat{P}x]\neq 0$. Another simple choice for the nonlinear control vector is

$$\begin{aligned} u_{nl}(t) &= -\frac{[B+g]^{\mathrm{T}}\hat{P}x}{x^{\mathrm{T}}\hat{P}[B+g][B+g]^{\mathrm{T}}\hat{P}x}(x^{\mathrm{T}}\hat{P}\bar{f})\\ &= 0 \qquad \begin{array}{l}\forall x \ni [x^{\mathrm{T}}\hat{P}[B+g][B+g]^{\mathrm{T}}\hat{P}x]\neq 0\\ \forall x \ni [x^{\mathrm{T}}\hat{P}[B+g][B+g]^{\mathrm{T}}\hat{P}x]=0\end{array}\end{aligned}\tag{39}$$

in which case the directional derivative (36) is given by

$$\dot{\hat{V}} = -x^{\mathrm{T}}\hat{C}(t)x \tag{40}$$

The actual control vector is obtained as $u(t)=u_l(t)+u_{nl}(t)$.

5. SIMULATION EXAMPLE

The methodology proposed in Sections 3 and 4 can be effectively applied to practical engineering structures modeled by nonlinear differential systems with time-periodic coefficients. As an example, we design a nonlinear controller for a system consisting of two inverted pendula moving in the horizontal plane with time-dependent load acting on each of the pendula. Each pendulum is supported at the base by a torsional spring. The loading consists of a constant and a time-periodic part. The control force is applied at only one of the joints. The structural diagram of the system considered is shown in Figure 1. The equations of motion can be shown to be

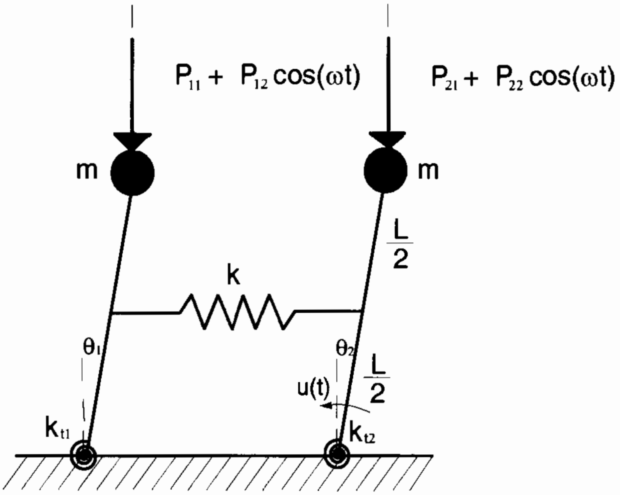

Figure 1. Parametrically excited inverted pendula

$$
\begin{aligned}
ml^2\ddot{\theta}_1 + k_{t1}\theta_1 + k\frac{l^2}{4}q_1(\theta_1,\theta_2) - P_1(t)l\sin\theta_1 &= 0\\
ml^2\ddot{\theta}_2 + k_{t2}\theta_2 + k\frac{l^2}{4}q_2(\theta_1,\theta_2) - P_2(t)l\sin\theta_2 &= u(t)
\end{aligned}
\qquad (41)
$$

where $q_1()$and $q_2()$ are nonlinear functions of $(\theta_2-\theta_1)$, $P_1(t) = P_{11} + P_{12}\cos(\omega t)$ and $P_2 = P_{21} + P_{22}\cos(\omega t)$.

The local dynamics can be obtained by expanding these global equations of motion about the fixed point $(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) = (0,0,0,0)$ and $u(t) = 0$. Keeping terms up to the cubic order, these equations may be approximated as

$$
\begin{aligned}
&\ddot{\theta}_1 + \frac{k_{t1}}{ml^2}\theta_1 + \frac{k}{4m}[c_1(\theta_1-\theta_2) + c_2(\theta_1-\theta_2)^2 + c_3(\theta_1-\theta_2)^3]\\
-\;&\left[\frac{P_{11}l}{ml^2} + \frac{P_{12}l}{ml^2}\cos(\omega t)\right]\left(\theta_1 - \frac{\theta_1^3}{6}\right) = 0
\end{aligned}
\qquad (42a)
$$

$$
\begin{aligned}
&\ddot{\theta}_2 + \frac{k_{t2}}{ml^2}\theta_2 + \frac{k}{4m}[c_1(\theta_2-\theta_1) + c_2(\theta_2-\theta_1)^2 + c_3(\theta_2-\theta_1)^3]\\
-\;&\left[\frac{P_{21}l}{ml^2} + \frac{P_{22}l}{ml^2}\cos(\omega t)\right]\left(\theta_2 - \frac{\theta_2^3}{6}\right) = u(t)
\end{aligned}
\qquad (42b)
$$

where $c_1 \sim 1$, $c_2 \sim \frac{1}{2}$, and $c_3 \sim \frac{1}{6}$.

The state-space form of these equations is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ X_1 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 1 \\ X_3 & 0 & X_4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ f_1(x_1, x_3, t) \\ 0 \\ f_2(x_1, x_3, t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad (43)$$

where

$$\begin{aligned} f_1(x_1, x_3, t) &= \left[\frac{P_{11}}{ml} + \frac{P_{12}}{ml}\cos(\omega t)\right]\left(-\frac{x_1^3}{6}\right) - \frac{k}{4m}[c_2(x_1 - x_3)^2 \\ &+ c_3(x_1 - x_3)^3] \qquad (44a) \\ f_2(x_1, x_3, t) &= \left[\frac{P_{21}}{ml} + \frac{P_{22}}{ml}\cos(\omega t)\right]\left(-\frac{x_3^3}{6}\right) - \frac{k}{4m}[c_2(x_3 - x_1)^2 \\ &+ c_3(x_3 - x_1)^3] \qquad (44b) \end{aligned}$$

and

$$\begin{aligned} X_1 &= \left[\frac{P_{11}}{ml} - \left(\frac{kc_1}{4m}\right) - \left(\frac{k_{t1}}{ml^2}\right)\right] + \left[\frac{P_{12}}{ml}\right]\cos(\omega t) \\ X_2 &= X_3 = \frac{kc_1}{4m} \\ X_4 &= \left[\frac{P_{21}}{ml} - \left(\frac{kc_1}{4m}\right) - \left(\frac{k_{t2}}{ml^2}\right)\right] + \left[\frac{P_{22}}{ml}\right]\cos(\omega t) \qquad (45) \end{aligned}$$

$$(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = (x_1, x_2, x_3, x_4)$$

In the following we consider two typical set of parameters for numerical simulation.

For parameter set I, we let

$$\begin{aligned} X_2 &= X_3 = 1 \\ X_1 &= X_4 = -1 + \cos(\omega t) \\ \omega &= 1. \qquad (46) \end{aligned}$$

For this set the system is unstable due to parametric resonance and one of the four Floquet multipliers lies outside the unit circle in the complex plane. For parameter set II, $\omega = 2$ in equation (46) and in this case the system is Lyapunov stable as all the four Floquet multipliers lie on the unit circle in the complex plane. The unstable (set I), or the marginally stable (set II), system is stabilized using the proposed methodology. In order to obtain a linear control law $u_l(t)$ for the linearized system of equation (43) given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ X_1 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 1 \\ X_3 & 0 & X_4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_l(t) \tag{47}$$

a condition of stabilizability is obtained for a subsystem of system (47). This system is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ X_1 & 0 & X_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_4 \tag{48}$$

The above system should be asymptotically stable with x_4 as the virtual input. It is to be noted that, for this example, $k = 3$. In order to stabilize equation (48), a condition of stabilizability is obtained for a subsystem of system (48) given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ X_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ X_2 \end{bmatrix} x_3 \tag{49}$$

and system (49) should be asymptotically stable with x_3 as a virtual input. Finally, in order to stabilize equation (49), a condition of stabilizability is obtained for a subsystem of system (49) given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1 = x_2 \tag{50}$$

This should be asymptotically stable with x_2 as a virtual input. Thus, we have applied normal coordinate transformation three times to arrive at system (50) from equation (47). This is the termination stage. For system (50), the virtual control x_2 can be simply set to $-x_1$ so that it is asymptotically stable. This virtual control input determines the backstepping transformation for system (49) and the virtual control x_3 for system (49) is obtained. This specifies the backstepping transformation required for system (48) and hence the virtual control input x_4 for system (48). Finally, the virtual control for system (48) gives the backstepping transformation for the original system (47) and the actual linear control vector $u_l(t)$ is computed. For the system considered, the linear control gain matrix $K(t)$ corresponding to equation (4) is given as

$$\begin{aligned} K(t) &= [K_{11}\ K_{12}\ K_{13}\ K_{14}], \text{where } K_{11} = X_3 + \frac{1}{X_2}\left[\dot{Y}_1 + Y_2 X_1 - \frac{3}{X_2}Y_1\right] \\ K_{12} &= \frac{1}{X_2}\left[Y_1 + \dot{Y}_2 - \frac{3}{X_2}Y_2\right], K_{13} = X_4 + Y_2 - \frac{9}{X_2}, K_{14} = 1 + \frac{3}{X_2}, \end{aligned} \tag{51a}$$

$$Y_1 = 3X_1 + \dot{X}_1 + 1,\ Y_2 = 3 + X_1 \tag{51b}$$

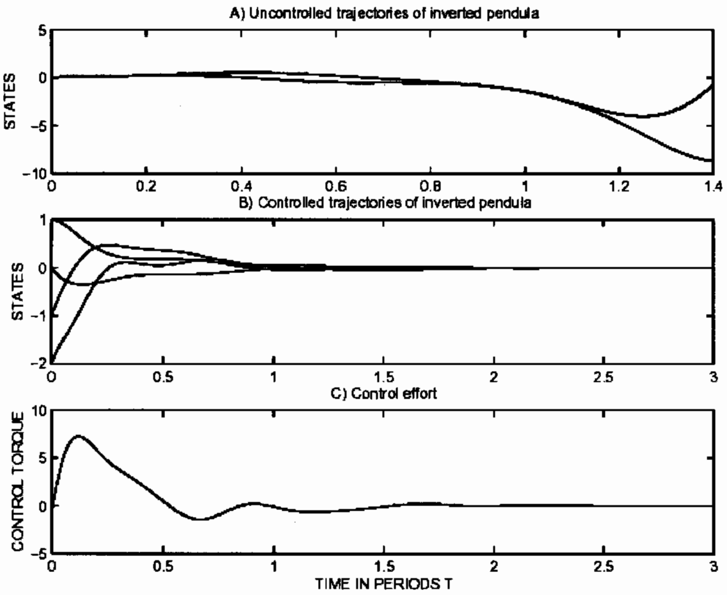

Figure 2. Simulation results for the linear system (case I).

where X_1, X_2, X_3, X_4 are given by equation (46). At this stage the nonlinear control law as given by equation (39) is designed. The total control effort is given by $u(t) = u_l(t) + u_{nl}(t)$. Figure 2(A) shows the uncontrolled motion of the linear system for parameter set I. The states of the system are plotted against the normalized time (period of parametric excitation). Figure 2(B) shows the controlled states of the system and asymptotic stability of the closed-loop system can be confirmed as the states converge to zero in about 2.5 periods. Figure 2(C) shows the control torque required for the linear controller to achieve asymptotic stability. In Figure 3(A), the response of uncontrolled nonlinear system is shown. Figures 3(B) and 3(C) show the controlled states and the control effort, respectively. In this case, the control effort is smooth indicating that no switching of control law is involved. For parameter set II, the uncontrolled system is marginally stable and Figure 4(A) shows the uncontrolled trajectories of the linear equations of motion. Controlled states and the linear control torque are plotted in Figures 4(B) and 4(C), respectively. The results from nonlinear simulations are summarized in Figures 5(A), 5(B), and 5(C) depicting the uncontrolled states, the controlled states, and the control effort, respectively. The discontinuity in the control effort is attributed to the fact that a switching occurs in the control law (*cf.* equation (39)) at that point.

6. DISCUSSION AND CONCLUSIONS

We have presented a practical control scheme for guaranteeing asymptotic stability of multi-input multi-output (MIMO) linear and nonlinear stabilizable dynamical systems with time-periodic coefficients. The methodology incorporates backstepping, L–F transformation, and

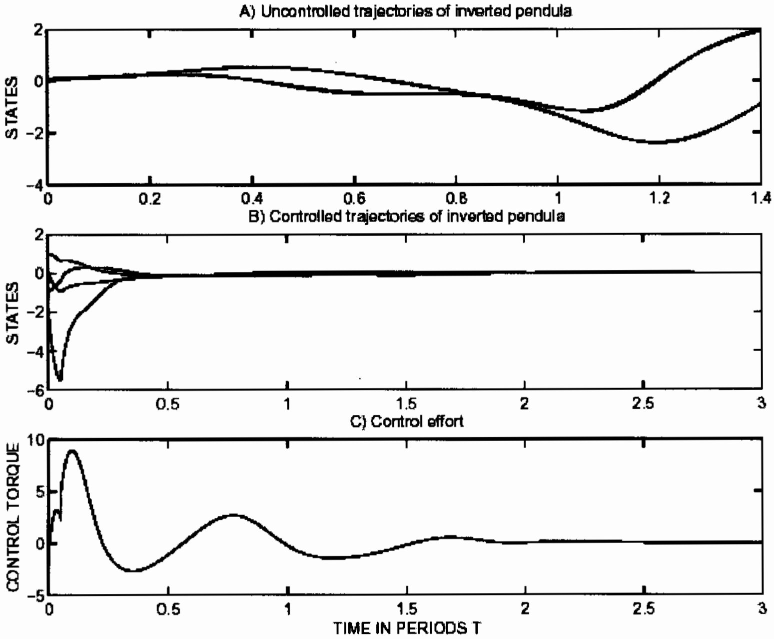

Figure 3. Simulation results for the nonlinear system (case I).

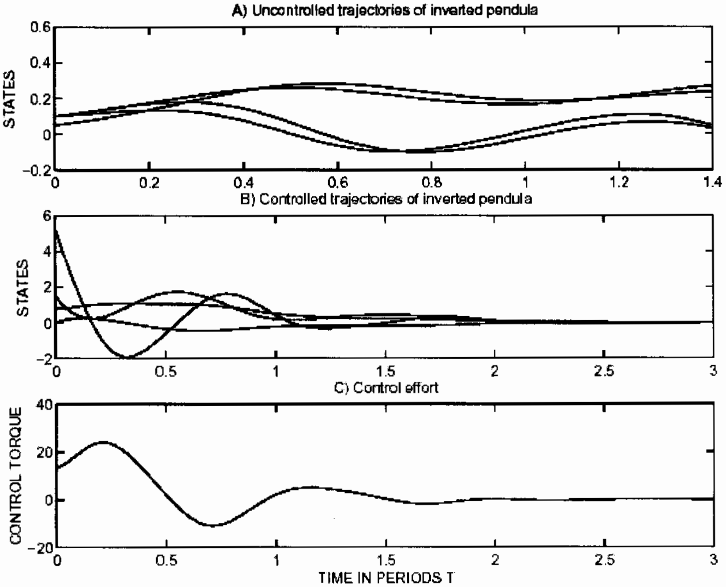

Figure 4. Simulation results for the linear system (case II).

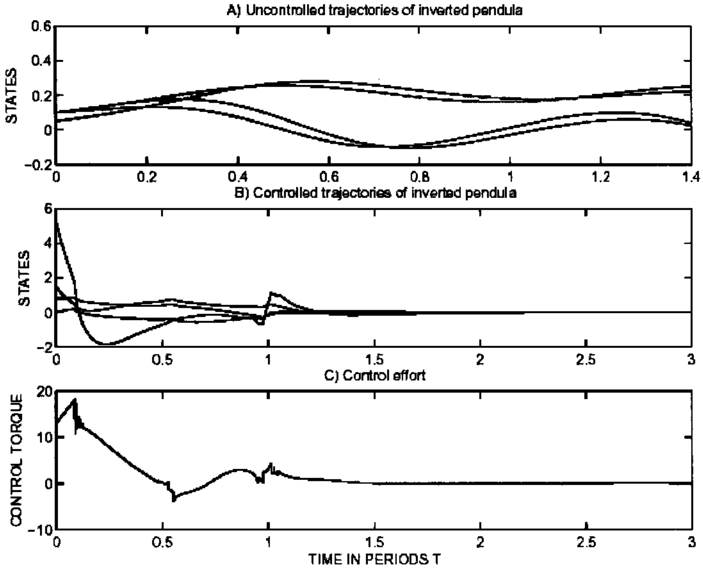

Figure 5. Simulation results for the nonlinear system (case II).

Floquet theory in order to design controllers for linear systems. The linear design process also yields a time-periodic Lyapunov function that can be used to design a nonlinear controller. The effectiveness of the proposed technique is demonstrated by successfully designing controllers for a nonlinear system comprising a statically coupled pendula subjected to axial periodic forces. It should be observed that the success of the scheme depends on the structure of the matrix, $B^{k+1}(t)$. Since it is required that the number of rows must be less than the number of columns, in certain situations it may not be possible to realize this property. However, it appears that for mechanical or structural systems this situation may not arise in most cases unless we are dealing with degenerate systems. With duality and principle of separation, the scheme can be extended to the design of observer-based controller for a time-periodic dynamic output feedback problem provided the system is stabilizable and detectable. It is also possible extend this idea to design adaptive controllers for linear and nonlinear time-periodic systems. Results from this study will be reported in a subsequent paper.

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