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Bifurcation Control of Nonlinear Systems With Time-Periodic Coefficients

In this study, a method for the nonlinear bifurcation control of systems with periodic coefficients is presented. The aim of bifurcation control is to stabilize post bifurcation limit sets or modify other nonlinear characteristics such as stability, amplitude or rate of growth by employing purely nonlinear feedback controllers. The method is based on an application of the Lyapunov-Floquet transformation that converts periodic systems into equivalent forms with time-invariant linear parts. Then, through applications of time-periodic center manifold reduction and time-dependent normal form theory completely time-invariant nonlinear equations are obtained for codimension one bifurcations. The appropriate control gains are chosen in the time-invariant domain and transformed back to the original variables. The control strategy is illustrated through the examples of a parametrically excited simple pendulum undergoing symmetry-breaking bifurcation and a double inverted pendulum subjected to a periodic load in the case of a secondary Hopf bifurcation. [DOI: 10.1115/1.1636194]

1 Introduction

Nonlinear dynamic systems with time-periodic coefficients arise in the mathematical modeling of many engineering problems such as helicopter blades, asymmetric rotor-bearing systems, structures with periodic loads and even in fluid flows under micro-gravity environment, just to mention a few. Such problems often lead to very complex, even unpredictable behavior. However, in engineering predictability and stability of systems are rather important. This explains the growing interest in nonlinear control, especially concerning feedback stabilization and bifurcation control of nonlinear systems. Nonlinear systems often can be controlled even when their linearization is uncontrollable. There may be reasons to choose nonlinear control even if the system is linearly controllable. These reasons include better system performance and lower energy requirements. It is known that sometimes performance can be significantly improved if a system is operated near a stability boundary. In order to allow that one needs to guarantee that when the boundary is crossed the stability is not lost in a catastrophic manner, but instead there are only tolerably small vibrations within a slowly growing attractive domain. This can be achieved by nonlinear bifurcation control. The size and rate of growth of the post bifurcation attractors can also be controlled and made as small as desired. Also, this type of control may require less control effort due to its purely nonlinear nature.

Nonlinear bifurcation control has been studied by several authors for time-invariant systems. Aeyels [1] studied a class of critical nonlinear autonomous systems using center manifold reduction, and applied it to stabilize the equilibrium point in the case of a Hopf bifurcation. Abed and Fu [2,3] have shown that local feedback stabilization of equilibrium positions and stabilization of bifurcated attractors (bifurcation control) are essentially the same tasks. They applied nonlinear feedback control to stabilize stationary and Hopf bifurcations of autonomous systems. Control of the period-doubling (flip) bifurcation has been studied by Iooss and Joseph [4] for continuous-time systems, and in discrete time systems by Abed, Wang and Chen [5]. Several authors have applied bifurcation control to practical problems, for example, Liaw and Abed [6] have used it for compressor stall inception and achieved significant improvement of performance, while Emad and Abdelfatah [7] employed the idea in magnetic bearing systems. A more detailed review of bifurcation control can be found in Kliemann and Namachchivaya [8].

All these studies are restricted to autonomous systems and no attempt has been made to address the problem of bifurcation control of general nonlinear systems with time-periodic coefficients. In an isolated study Ouieni and Nayfeh [9] applied the method of multiple scales to suppress vibrations of a cantilevered beam under principal parametric resonance. However, the amplitude of the excitation had to be kept small. Also, the perturbation equations obtained for the amplitude and phase were too complex and the authors had to resort to numerical analysis.

There is a recently developed analytical simplification method, that is not restricted to small parameters and yields very accurate predictions of the bifurcation point as well as the post-bifurcation dynamics. Studies by Pandiyan and Sinha, Sinha et al. and Dávid and Sinha [10–12] show that it is possible to construct dynamically equivalent time-invariant forms of periodic equations such that the local stability and bifurcation characteristics are completely preserved. In the following this analysis method is applied to feedback control systems to design nonlinear bifurcation controllers.

2 Statement of the Problem

Consider the nonlinear control system with time-periodic coefficients given by

$$\dot{x}=F(x,u(x),\alpha,t) \tag{1}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}$, $F(x,u,\alpha,t+T)=F(x,u,\alpha,t)$ The vector $\alpha\in\mathbb{R}^m$, $(m\leqslant n)$ contains the parameters of the system. It is assumed that $F(0,0,\alpha,t)=0$, that is $x=0$ and $u=0$ is an equilibrium point. For the purpose of local analysis we expand equation (1) about this equilibrium into Taylor series as

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$$\dot{x}=L_x(\alpha,t)x+L_u(\alpha,t)u+Q_x(x,\alpha,t)+uL_{2x}(\alpha,t)x+L_{2u}(\alpha,t)u^2 + C_x(x,\alpha,t)+uQ_{2x}(x,\alpha,t)+u^2L_{3x}(\alpha,t)x+L_{3u}(\alpha,t)u^3 + h.o.t. \quad (2)$$

where L_{2x}, L_{2u}, L_{3x} and L_{3u} are coefficient vectors and matrices, Q_x, Q_{2x} and C_x are symmetric quadratic and cubic forms of the variables x, respectively. Further, it is assumed that there exists a critical value of the system parameters α_c, for which the linear system matrix L_x has n_1 Floquet multipliers on the unit circle of the complex plane and n_2 multipliers with magnitude less than one. This implies that the system is undergoing a bifurcation at that point. The local feedback stabilization problem for system (1) is to find a smooth feedback control input u, such that the origin is locally asymptotically stable for $\alpha=\alpha_c$, whereas the bifurcation control problem is to find an input u, such that the post bifurcation limit set is locally asymptotically stable for $\alpha\neq\alpha_c$. It will be shown that these two goals can be achieved simultaneously by employing a nonlinear controller. It is well known that in the case of codimension one bifurcations it is sufficient to keep terms only up to cubic order because it provides very good qualitative and quantitative approximation of the bifurcation phenomena. We note, however, that the method is not restricted to cubic approximations, it is for shortness' sake only. The control input is assumed in the form

$$u(x)=x^TG(t)x+H(x,t) \quad (3)$$

where $G(t)$ is an $n\times n$ matrix of the unknown gains of the quadratic terms and $H(x,t)$ is a cubic form of the states x, containing the unknown cubic control gains. Observe that equation (3) does not contain any linear terms. We do not assume the linearized equation to be controllable, since a linear feedback may not exist. Also, our purpose is to control the system at and in the neighborhood of the bifurcation point. A linear controller would push the system away from the stability boundary. It is also unnecessary to assume nonlinear controllability since all we need is stabilizability of the origin and other, nontrivial limit sets such as limit cycles. From the following analysis it will be clear how our goals can be achieved. After substituting (3) into (2) the closed loop dynamic system becomes

$$\dot{x}=L^*(\alpha,t)_x+Q^*(x,\alpha,t)+C^*(x,\alpha,t) \quad (4)$$

where

$$L^*(\alpha,t)=L_x(\alpha,t),$$
$$Q^*(x,\alpha,t)=Q_x(x,\alpha,t)+L_u(\alpha,t)x^TG(t)x, \quad (5)$$
$$C^*(x,\alpha,t)=C_x(x,\alpha,t)+L_u(\alpha,t)H(x,t)+(x^TG(t)x)L_{2x}(\alpha,t)x$$

This system can be analyzed at the critical point through an application of a sequence of mathematical tools, such as the Lyapunov-Floquet transformation, the time-periodic center manifold reduction and the time-dependent normal form theory. The versal deformation theory may be used to study the dynamics in the neighborhood of the bifurcation point.

3 Analysis of the Closed-Loop System and Controller Design

First, we restrict the analysis to the study of the dynamics at the bifurcation point by substituting the critical values of the system parameters, α_c, in equation (4). For the resulting parameter independent system the real Lyapunov-Floquet (L-F) transformation, $Q(t)$ can be computed using an efficient technique suggested by Sinha et al. [13]. The change of coordinates $x=Q(t)y$ transforms equation (4) into the form

$$\dot{x}=Rx+Q^{-1}(t)(Q^*(x,t)+C^*(x,t)) \quad (6)$$

where R is an $n\times n$ real constant matrix. This equation is dynamically equivalent to equation (4), in the sense that all the stability and bifurcation characteristics are preserved. In order to study the dynamics at the critical point, it is sufficient to consider the reduced equation on the center manifold. Using the time-periodic center manifold theory as suggested by Malkin [14] and Pandiyan and Sinha [10], we can separate the n_1 critical states from the stable ones (also see Perko [15]). For codimension one bifurcation, where R matrix has a zero eigenvalue or a pair of complex eigenvalues with zero real parts, center manifold relations and stability properties can be found in Chow and Hale [16]. The linear part of equation (6) can be brought into a Jordan canonical form by the modal transformation $y=Mz$ as:

$$\begin{Bmatrix}\dot{z}_c\\ \dot{z}_s\end{Bmatrix}=\begin{bmatrix}J_c & 0\\ 0 & J_s\end{bmatrix}\begin{Bmatrix}z_c\\ z_s\end{Bmatrix}+\begin{Bmatrix}\tilde{Q}_c^*(z_c,z_s,t)+\tilde{C}_c^*(z_c,z_s,t)\\ \tilde{Q}_s^*(z_c,z_s,t)+\tilde{C}_s^*(z_c,z_s,t)\end{Bmatrix} \quad (7)$$

where $\tilde{Q}^*=M^{-1}Q^{-1}(t)Q^*$, $\tilde{C}^*=M^{-1}Q^{-1}(t)C^*$ and the subscripts c and s denote the critical and the stable states, respectively. According to the center manifold theory, there exists a nonlinear relation with time-periodic coefficients of the form:

$$z_s=h_{2c}(z_c,t)+h_{3c}(z_c,t) \quad (8)$$

which, upon substitution into equation (7), decouples the critical states from the stable ones in the nonlinear terms. In equation (8), the indices 2 and 3 denote the order of the nonlinearities.

Up to this point the procedure is valid for the general case, regardless of the type of the bifurcation. It should be observed that the nonlinear terms of equation (7) contain unknown control gains, therefore we cannot proceed with the computation of the center manifold relations, in general. However, we can derive formal expressions of the normal forms for each bifurcation without actually computing the relations (see Sinha et al. [11]). In the following, we consider the four different codimension one bifurcations separately.

3.1 Flip (or Period Doubling) Bifurcation. Let us assume that for some critical value, α_c, of the system parameters the Floquet transition matrix associated with the linear part, L_x, of equation (2) has an eigenvalue (Floquet multiplier) that equals to -1 and all the other eigenvalues have magnitudes less than one. Under this assumption equation (7) takes the form:

$$\begin{Bmatrix}\dot{z}_1\\ \dot{z}_s\end{Bmatrix}=\begin{bmatrix}0 & 0\\ 0 & J_s\end{bmatrix}\begin{Bmatrix}z_1\\ z_s\end{Bmatrix}+\begin{Bmatrix}\tilde{Q}_1^*(z_1,z_s,t)+\tilde{C}_1^*(z_1,z_s,t)\\ \tilde{Q}_s^*(z_1,z_s,t)+\tilde{C}_s^*(z_1,z_s,t)\end{Bmatrix} \quad (9)$$

In this case, because all the Floquet multipliers lie in the left half of the complex plane, the real L-F transformation and its inverse are $2T$-periodic and have the symmetry property $Q(t+T)=-Q(t)$. Therefore, the nonlinear part of equation (9) is also $2T$-periodic and due to the symmetry property the coefficients of the quadratic terms have zero averages in time over the period $2T$. If the center manifold relations are assumed as:

$$z_i=h_{2,i}(t)z_1^2+h_{3,i}(t)z_1^3, \quad i=2,\ldots,n \quad (10)$$

then the reduced equation on the one dimensional center manifold becomes:

$$\dot{z}_1=\tilde{Q}^*_{1,(2,0,\ldots,0)}(t)z_1^2+\left(\sum_{i=2}^{n}\tilde{Q}^*_{1,(1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i}(t) + \tilde{C}^*_{1,(3,0,\ldots,0)}(t)\right)z_1^3 \quad (11)$$

where $\tilde{Q}^*_{1,(2,0,\ldots,0)}$ and $\tilde{C}^*_{1,(3,0,\ldots,0)}$ are the $2T$-periodic coefficient functions and the subscripts in parentheses indicate the powers of the nonlinear terms (for example $z_1^2 z_2^0 \ldots z_n^0$). To further simplify this equation, we employ the time-dependent normal form theory as described by Arnold [17]. We assume a near-identity transformation in the form:

$$\dot{z}_1 = v + h_{n2}(v,t) + h_{n3}(v,t) \tag{12}$$

which can eliminate most of the nonlinear terms of equation (11). It has been shown by Sinha et al. [11] that the only terms that cannot be removed by this transformation are the time-averages of the periodic functions. Therefore, the normal form becomes:

$$\dot{v} = \left(\sum_{i=2}^{n} \overline{\tilde{Q}^*_{1,(1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i}(t)} + \overline{\tilde{C}^*_{1,(3,0,\ldots,0)}(t)} \right) v^3 = av^3 \tag{13}$$

where the bar denotes the average of the quantity over the period and a is a constant. We cannot compute the value of a because of the presence of the unknown control gains, but this form provides the necessary information about stability in order to design the controller. Equation (13) indicates that the origin of the original system (4) at the bifurcation point is asymptotically stable if and only if a is negative. Now, to study the dynamics in the neighborhood of the bifurcation, versal deformation of the normal form is constructed as (see Dávid and Sinha [12]):

$$\begin{aligned}\dot{v} &= \mu(\alpha)v + \left(\sum_{i=2}^{n} \overline{\tilde{Q}^*_{1,(1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i}(t)} + \overline{\tilde{C}^*_{1,(3,0,\ldots,0)}(t)} \right) v^3 \\ &= \mu(\alpha)v + av^3\end{aligned} \tag{14}$$

where the versal deformation parameter μ is a function of the bifurcation parameters of the original system. This equation together with the $2T$-periodic L-F transformation describes a period doubling bifurcation. A stable $2T$-periodic limit cycle exists for $\mu \geq 0$ if and only if $a<0$. So we can see that stabilizing the origin at the critical point and the limit cycle after the bifurcation requires fulfilment of the same condition implying that feedback stabilization and bifurcation control can be achieved by the same controller. We observe that $\tilde{Q}^*$ contains only quadratic terms of the control input, while $\tilde{C}^*$ has unknowns from both quadratic and cubic control terms. If we choose the control input to be a purely cubic function, then $\tilde{Q}^*$ will not contain unknowns at all and the computation of the quadratic center manifold relations, $h_{2,i}(t)$, becomes possible. We can also see, that the only effective term of the control input the cubic term that is a function of the critical state only. So if in the transformed domain (after the application L-F and modal transformations) we choose the control input as:

$$G(t) := 0, \quad M^{-1}Q^{-1}(t)L_u(t)H(z,t) := \{\tilde{u}_c \;\; 0 \ldots 0\}^T \qquad \tilde{u}_c := \beta z_1^3 \tag{15}$$

then the normal form becomes:

$$\dot{v} = \mu(\alpha)v + \left(\left(\sum_{i=2}^{n} \overline{\tilde{Q}_{1,(1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i}(t)} + \overline{\tilde{C}^*_{1,(3,0,\ldots,0)}} \right) + \beta \right) v^3 = \mu(\alpha)v + av^3 \tag{16}$$

and β can be chosen such that $a<0$. This equation can be solved in a closed form. From the solution it is easy to see how β effects the size and the rate of growth of the limit cycle, and it can be chosen to adjust these characteristics to any desired value. Figure 1 shows the uncontrolled and controlled dynamics of equation (16) (and equation (1)) around a flip bifurcation point for both the sub- and supercritical cases. These diagrams clearly illustrate the goals of bifurcation control. When the uncontrolled bifurcation is subcritical (catastrophic loss of stability) the controller makes it supercritical. If the bifurcation was originally supercritical, by controlling it we can reduce the size and rate of growth of the post-bifurcation limit cycle to make the loss of stability as soft and slow as desired. Although robustness of the method is not treated in this work, some observations can be made from these figures. In the case of the subcritical bifurcation, a locally unbounded region of instability (where solutions go to infinity) has been changed by the controller into a locally unbounded region of attraction where solutions converge to a small amplitude limit cycle. In the second case the existing domain of attraction has been made larger. Also, the parameter sensitivity of the system is greatly reduced. This is done by delaying the occurrence of secondary bifurcations and possible chaotic behavior. These factors seem to indicate an increased robustness of the controlled system. Similar observations can be made for the other codimension one bifurcations also but they are omitted for brevity. A complete study of robustness is currently being pursued and will be reported elsewhere.

3.2 Transcritical and Symmetry Breaking Bifurcations

This time it is assumed that for some critical value, α_c, the Floquet transition matrix associated with L_x in equation (4) has an eigenvalue equal to $+1$ while all the other eigenvalues have magnitudes less than 1. Since both the $+1$ and the -1 multipliers correspond to a zero eigenvalue of the transformed system, after the L-F transformation equation (4) takes the same form as in the flip bifurcation case (equation (9)). In this case, however, since at least one of the multipliers is in the right half of the complex plane, the real L-F transformation is T-periodic and it does not have a symmetry property. Therefore, the averages of the nonlinear terms do not disappear. After going through the same simplification procedure as in Section 3.1 we obtain the versal deformation equation

$$\begin{aligned}\dot{v} &= \mu(\alpha)v + \overline{Q^*_{1,(2,0,\ldots,0)}(t)}v^2 \\ &\quad + \left(\sum_{i=2}^{n} \overline{Q^*_{1,(1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i}(t)} + \overline{C^*_{1,(3,0,\ldots,0)}(t)} \right) v^3 \\ &= \mu(\alpha)v + bv^2 + av^3\end{aligned} \tag{17}$$

where a and b are real constants. Therefore, the simplest way to choose the control input is

$$\begin{aligned} &M^{-1}Q^{-1}(t)L_u(t)z^T G(t)z := \{\tilde{u}_{c2} \;\; 0 \ldots 0\}^T, \quad \tilde{u}_{c2} := \beta_2 z_1^2 \\ &M^{-1}Q^{-1}(t)L_u(t)H(z,t) := \{\tilde{u}_{c3} \;\; 0 \ldots 0\}^T, \quad \tilde{u}_{c3} := \beta_3 z_1^3 \end{aligned} \tag{18}$$

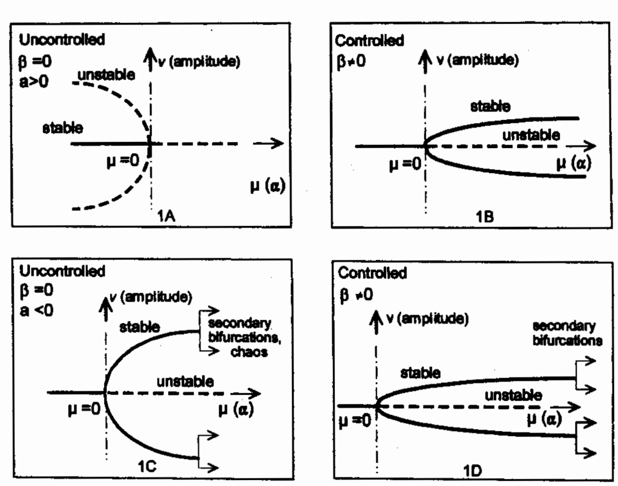

Fig. 1 Bifurcation diagrams of uncontrolled and controlled flip bifurcations

where the β's are real unknown control gains. From equation (17) we can conclude the following. If the coefficient of the quadratic term is not zero, this equation describes a transcritical bifurcation: if the equilibrium is stable for $\mu<0$ then there exists an unstable orbit around it and when for $\mu>0$ the equilibrium loses its stability, the orbit becomes stable. In this case the origin is always unstable at the bifurcation point (at $\mu=0$), therefore, in practical problems this type of bifurcation is to be avoided. However, if b is zero then equation (17) describes a symmetry breaking bifurcation, when the limit cycle exists on only one side of the bifurcation point and the coefficient of the cubic term determines the stability of this limit cycle as well as the stability of the origin at the critical point. Hence, in order to assure that the bifurcation is symmetry breaking instead of transcritical, we need to make b zero. Note, that b contains unknowns only from the quadratic control gain $G(t)$. Then, to make the symmetry breaking bifurcation supercritical, we need $a<0$. Once $G(t)$ is chosen, the quadratic part of the center manifold relation can be computed and used in the second step to choose a.

3.3 Secondary Hopf Bifurcation. In this case for some critical value, α_c, of the system parameter the Floquet transition matrix associated with L_x in equation (4) has one pair of complex eigenvalues on the unit circle while all the other eigenvalues have magnitudes less than 1. This complex pair corresponds to a pair of purely imaginary eigenvalues of matrix R of the form $0\pm\omega_c i$. If ω_c is *not* an integer multiple of $2\pi/T$, where T is the period, then the normal form becomes time-invariant, otherwise resonant periodic terms remain. We consider the non-resonant case only for brevity. Note that the resonant case can be dealt with very similarly. Under the assumptions, equation (6) takes the form:

$$\begin{Bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_s\end{Bmatrix}=\begin{bmatrix}i\omega_c & 0 & 0\\ 0 & -i\omega_c & 0\\ 0 & 0 & J_s\end{bmatrix}\begin{Bmatrix}z_1\\ z_2\\ z_s\end{Bmatrix}+\begin{Bmatrix}\tilde{Q}_1^*(z_1,z_2,z_s,t)+\tilde{C}_1^*(z_1,z_2,z_s,t)\\ \tilde{Q}_2^*(z_1,z_2,z_s,t)+\tilde{C}_2^*(z_1,z_2,z_s,t)\\ \tilde{Q}_s^*(z_1,z_2,z_s,t)+\tilde{C}_s^*(z_1,z_2,z_s,t)\end{Bmatrix} \tag{19}$$

We assume the center manifold relations to be of the form $z_i=h_{2,i}(z_1,z_2,t)+h_{3,i}(z_1,z_2,t)$, $i=3,4,\ldots,n$, where $h_{2,i}(z_1,z_2,t)=h_{2,i,(2,0)}(t)z_1^2+h_{2,i,(1,1)}(t)z_1z_2+h_{2,i,(0,2)}(t)z_2^2$ and the cubic terms can be defined in a similar fashion. The near-identity transformation is also sought in a similar form. The two dimensional versal deformation equation can be shown to be (see Dávid and Sinha [12]):

$$\begin{Bmatrix}\dot{v}_1\\ \dot{v}_2\end{Bmatrix}=\begin{bmatrix}\mu(\alpha)+\omega_c i & 0\\ 0 & \mu(\alpha)-\omega_c i\end{bmatrix}\begin{Bmatrix}v_1\\ v_2\end{Bmatrix}+\begin{Bmatrix}\left(\sum_{i=3}^{n}\overline{(Q^*_{1,(1,0,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(1,1)}(t)}+\overline{Q^*_{1,(0,1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(2,0)}(t))}+\overline{C^*_{1,(1,2,0,\ldots,0)}(t)}\right)v_1^2v_2\\ \left(\sum_{i=3}^{n}\overline{(Q^*_{2,(1,0,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(1,1)}(t)}+\overline{Q^*_{2,(0,1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(2,0)}(t))}+\overline{C^*_{2,(1,2,0,\ldots,0)}(t)}\right)v_1v_2^2\end{Bmatrix} \tag{20}$$

where the two equations are complex conjugate of each other. The versal deformation parameter $\mu(\alpha)$ can be complex, in general. However, it can be observed that around the critical value, even for very small changes of α, the real part of the eigenvalue changes several orders in magnitude, while the imaginary part remains almost constant. Therefore, it is reasonable to assume that $\mu(\alpha)$ is real. Equation (20) can be transformed into a real form in terms of polar coordinates as:

$$\begin{aligned}\dot{R}&=\mu(\alpha)R+\mathrm{Re}\left(\sum_{i=3}^{n}\overline{(Q^*_{1,(1,0,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(1,1)}(t)}\right.\\&\left.\quad+\overline{Q^*_{1,(0,1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(2,0)}(t))}+\overline{C^*_{1,(1,2,0,\ldots,0)}(t)}\right)R^3\\ \dot{\theta}R&=-\omega_cR-\mathrm{Im}\left(\sum_{i=3}^{n}\overline{(Q^*_{1,(1,0,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(1,1)}(t)}\right.\\&\left.\quad+\overline{Q^*_{1,(0,1,0,\ldots,\underset{i}{1},\ldots,0)}(t)h_{2,i,(2,0)}(t))}+\overline{C^*_{1,(1,2,0,\ldots,0)}(t)}\right)R^3\end{aligned} \tag{21}$$

Now it is easy to see that this simplified system undergoes a Hopf bifurcation at $\mu=0$, and the stability of the limit cycle depends on the sign of the constant real cubic coefficient of the amplitude equation. Therefore, there is a simplest possible controller that can stabilize this limit cycle, and in the transformed domain it has the form:

$$G(t):=0,\quad M^{-1}Q^{-1}(t)L_u(t)H(z,t):=\{\tilde{u}_c\ 0\ldots 0\}^T$$
$$\tilde{u}_c:=\begin{Bmatrix}\beta_3z_1^2z_2\\ \beta_3z_1z_2^2\end{Bmatrix} \tag{22}$$

where β_3 is a real unknown control gain. Together with the L-F transformation in the original coordinates equation (20) describes a secondary Hopf bifurcation and the limit cycle transforms into a quasi-periodic limit set, in general.

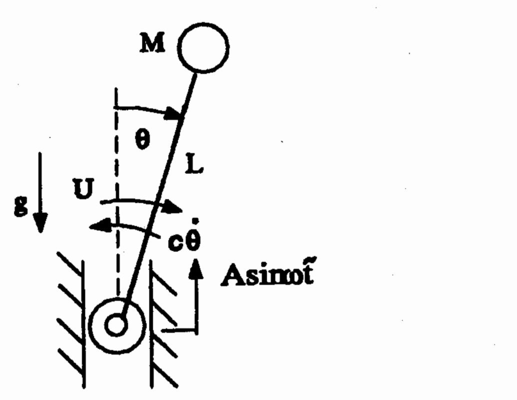

Fig. 2 Parametrically excited simple inverted pendulum

4 Illustrative Examples:

4.1 Parametrically Excited Simple Pendulum Undergoing Symmetry Breaking Bifurcation. Consider a parametrically excited simple pendulum shown in Fig. 2. The control torque U is applied at the suspension point. The nonlinear equation of motion in the state space expanded up to cubic order is obtained in the form:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(a+b\sin\omega t) & -d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{6}(a+b\sin\omega t)x_1^3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ u \end{Bmatrix} \tag{23}$$

where $a=-4g/(\omega^2 L)$, $b=A/L$, $d=4c/(ML^2\omega^2)$, $u=4U/(ML^2\omega^2)$ and $T=2\pi/\omega$ is the principal period of the system (23). For the parameter set $a=-0.2603337$, $b=1.5$, $d=0.31623$ and $\omega=2$ one of the Floquet multipliers of equation (29) is $+1$ and the system undergoes a subcritical symmetry breaking bifurcation. (Note that it is essential to obtain the bifurcation point as accurately as possible, this explains the need for seven digits in the parameter value. Luckily, due to the superconvergent nature of the computation procedure, this is very easy to do.) First, a nonlinear feedback controller is designed to make the bifurcation supercritical. Note that since the equation does not contain quadratic terms, the controller can also be purely cubic. Following the procedure described in section 3, the Lyapunov-Floquet transformation is computed and applied to equation (23). The resulting equation has a time-invariant linear part and can be brought into a Jordan canonical form as:

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_s \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} \tilde{C}_{13}(z_1,z_2,t) \\ \tilde{C}_{23}(z_1,z_2,t) \end{Bmatrix} + M^{-1}L^{-1}(t) \begin{Bmatrix} 0 \\ Tu \end{Bmatrix} \tag{24}$$

Since the normal form is time-invariant in this case, we choose the control input such that:

$$M^{-1}L^{-1}(t) \begin{Bmatrix} 0 \\ Tu(L(t)Mz,t) \end{Bmatrix} = \begin{Bmatrix} k_1 z_1^3 \\ 0 \end{Bmatrix} \tag{25}$$

where k_1 is the unknown constant control gain. After the center manifold and normal form reductions, the versal deformation is constructed to yield a scalar equation:

$$\dot{v} = \mu(a)v + (1.02106 + k_1)v^3 \tag{26}$$

where, μ is a function of the bifurcation parameter a. An approximate relationship between the original bifurcation parameter a and the versal deformation parameter μ can be obtained using a curve fitting technique (see Dávid and Sinha [12]). Let $a=a_c+\eta$, then a quadratic relationship is computed as $\mu=-151.683\eta^2 - 12.6875\eta$. If the coefficient of the cubic term in equation (26) is negative, the bifurcation is supercritical. Equation (26) can be easily solved in a closed form and for $\mu>0$ it gives a stable nontrivial equilibrium point at $v_{ss}=(-\mu/(1.02106+k_1))^{1/2}$. After applying all the transformations in the reverse order, this equilibrium point becomes a π-periodic limit cycle in the original coordinates. For illustration of the effect of the gain on the post bifurcation dynamics, three different values of k_1 are chosen. Actually, we choose the cubic coefficient to be -25, -50 and -75 and compute the corresponding gains as $k_1=-26.02106$, $k_1=-51.02106$ and $k_1=-76.02106$. Figure 3 shows the uncontrolled motion, the controlled motion for all three choices of control gain and also the effort required by each controller. Figure 4 compares the work done by the three controllers. From these figures it can be concluded, that larger gains yield smaller amplitude limit cycles. However, the amplitude of the limit cycle depends on the cubic coefficient of the normal form in an inverse quadratic fashion, as it can be seen from the solution of equation (26). Also, larger gains produce larger transient peaks in the control torque, although in the steady-state phase a larger gain corresponds to a smaller amplitude periodic torque. Comparing the work done by these controllers, we can see even clearer, how the effects of the transient peaks of torque compete with the steady-state amplitude. These observations imply that based on all three requirements (small steady-state amplitude, not too high peak force, and not too

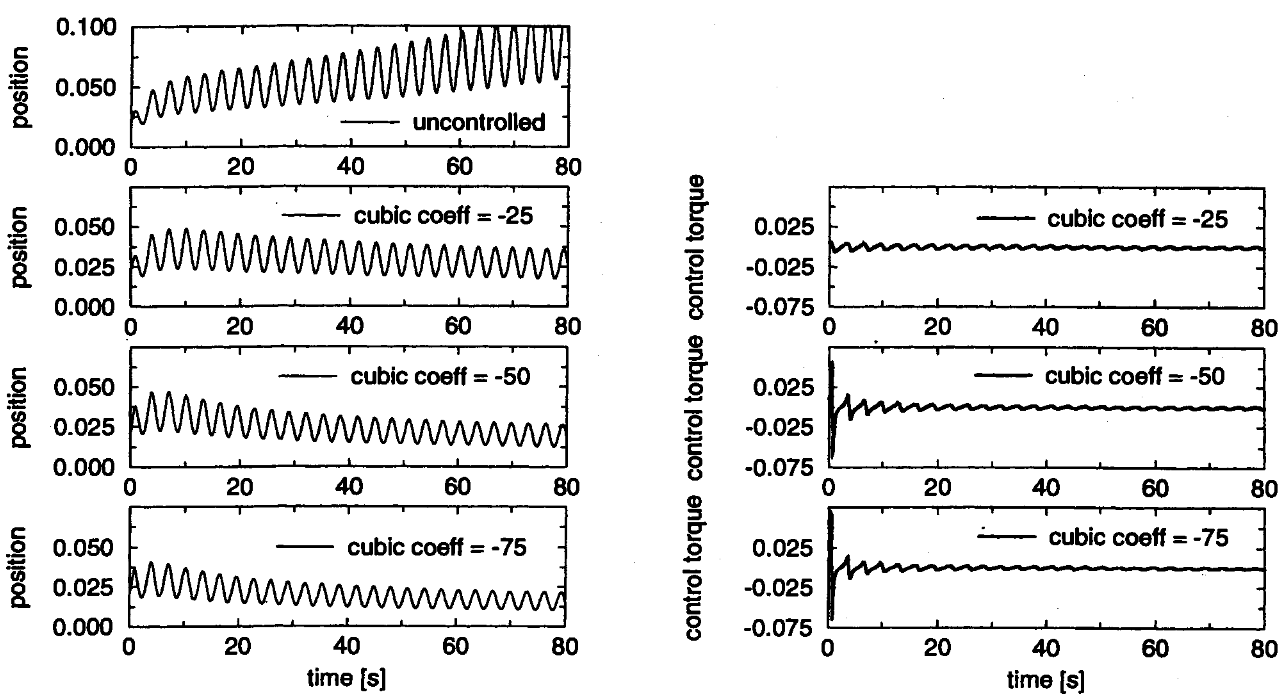

Fig. 3 Bifurcation control of the simple pendulum, comparison of different control gains

much work) there is an optimum value of the control gain. From our three data points, it seems that this optimum should be around $k_1 = -50$. In the following, this value will be used.

For comparison, two different linear feedback controllers are also designed for this system. The first one is a linear constant coefficient state-feedback controller, the design of which is based on a symbolic computation of the state transition matrix (Sinha and Butcher [18]) associated with the linear part of equation (23). The symbolic computation procedure is based on Chebyshev polynomial expansion and Picard iteration and it yields the state transition matrix as a function of the system parameters. We design a time-invariant linear feedback controller in the form:

$$u = Kx, \quad K = \{k_1, k_2\} \tag{27}$$

Following the method described in Dávid and Sinha [19] the Floquet multipliers of the closed loop system can be placed as desired by choosing the appropriate values of the parameters k_1 and k_2. For the controller shown in Fig. 4, $k_1 = 0.3477$ and $k_2 = 3.8239$ have been chosen. The other approach is the well-known technique of feedback (or exact) linearization (Khalil, [20]). The nonlinear and periodic terms of the system equations are fed back to be canceled out, and for the remaining linear autonomous equation a classical linear feedback control is designed by pole-placement. We consider equation (23) and assume the controller in the form:

$$u = u_n + u_l = -(a + b \sin \omega t)\sin\theta + u_l, \quad u_l = Kx, \quad K = \{k_1, k_2\} \tag{28}$$

After substituting this, equation (23) becomes linear and autonomous, and the values of K can be chosen by pole-placement. For the controller shown in Fig. 5, $k_1 = -0.0230$ and $k_2 = -0.0789$. Note, that while for the nonlinear control we did not need to assume controllability in the linear sense, for the application of these two methods the system has to be controllable. In the second case it is rather easy to check, while in the first one the controllability condition for periodic systems has to be satisfied. The purpose is to compare the control torque needed in the different methods to see if the nonlinear controller requires less effort as anticipated. Figure 5 shows the controlled motions for the bifurcation control and the two linear controllers. The linear controllers, of course, drive the system to zero, in the nonlinear case the steady state solution is a small periodic motion. Figure 6 compares the control torque needed in each case. We observe that the most effort is required by the feedback linearization, and the effort needed by the nonlinear controller is far the smallest, as expected.

5.2 Double Inverted Pendulum Undergoing Subcritical Secondary Hopf Bifurcation. Consider a double inverted pendulum with a periodic load as shown in Fig. 7. The nonlinear equations of motion in the state space are given as:

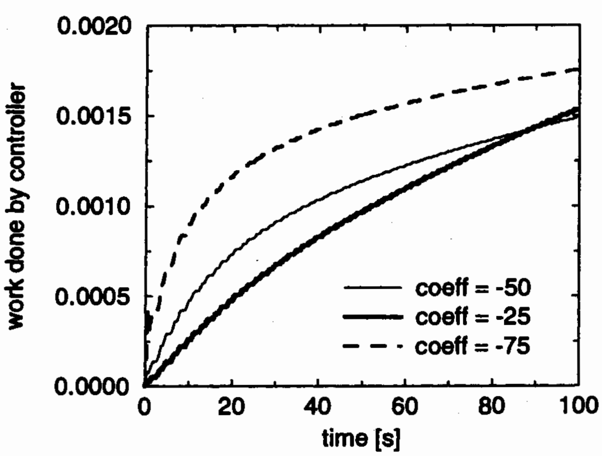

Fig. 4 Comparison of the control effort for different control gains for the simple pendulum

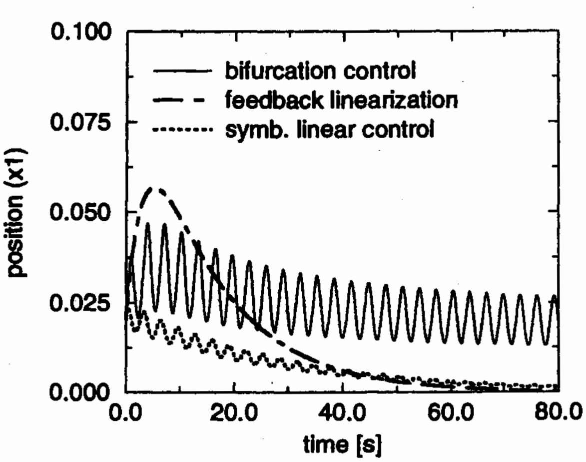

Fig. 5 Comparison of different controllers for the simple pendulum

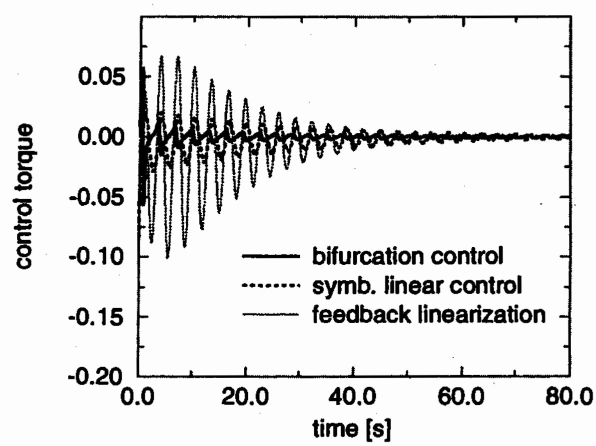

Fig. 6 Comparison of the torque required by the different controllers for the simple pendulum

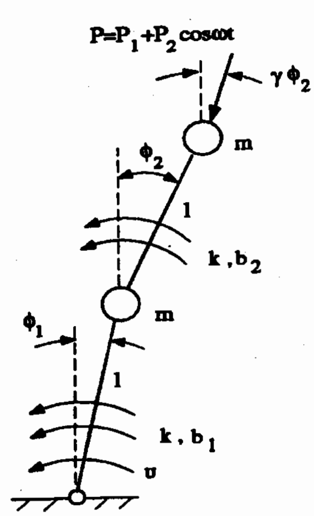

Fig. 7 Double inverted pendulum with a periodic follower load

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{Bmatrix}=T\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0.5\bar{k}(\bar{p}-3) & 0.5\bar{k}(2-\bar{p}) & -0.5(b_1+2b_2) & b_2\\ 0.5\bar{k}(5-\bar{p}) & \bar{k}(\bar{p}(1.5-\gamma)-2) & 0.5(b_1+4b_2) & -2b_2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{Bmatrix}$$

$$+T\begin{Bmatrix}0\\ 0\\ [-0.5(x_3^2+x_4^2)(x_1-x_2)-(\bar{p}\bar{k})[(x_1-\gamma x_2)^3-(1-\gamma)^3x_2^3]/12-\\ 0.25(x_1-x_2)^2[\bar{k}(\bar{p}-4)x_1+\bar{k}(3+\bar{p}(\gamma-2))x_2-(b_1+3b_2)x_3+3b_2x_4]]+u\\ [0.5(x_1-x_2)(3x_3^2+x_4^2)+\bar{p}\bar{k}[(x_1-\gamma x_2)^3-3(1-\gamma)^3x_2^3]/12+\\ 0.25(x_1-x_2)^2[\bar{k}(2\bar{p}-7)x_1+\bar{k}(5+\bar{p}(\gamma-3))x_2-(2b_1+5b_2)x_3+5b_2x_4]]\end{Bmatrix} \quad (29)$$

where $\bar{k}=k/ml^2$, $B_i=b_i/ml^2$, $i=1,2$, $\bar{p}=(P_1+P_2\cos\omega\tilde{t})/ml=p_1+p_2\cos\omega\tilde{t}$ and $u=U/ml^2$. Further, $\bar{p}=p_1+p_2\cos 2\pi t$ and $T=2\pi/\omega$. For the parameter set $\bar{k}=1$, $b_1=b_2=0.01$, $p_1=-0.149899$, $p_2=2$, $\omega=2$ and $\gamma=0$, the linear system matrix in equation (29) has a pair of complex multipliers on the unit circle. This implies that the system is undergoing a secondary Hopf bifurcation. After the application of the L-F and modal transformations, the eigenvalues of the constant matrix J_c (corresponding to the critical multipliers) are $0\pm 1.5046i$. Following the procedure described earlier we assume the control input in the form:

$$M^{-1}Q^{-1}(t)T\begin{Bmatrix}0\\ 0\\ u(Q(t)Mz,t)\\ 0\end{Bmatrix}=\begin{Bmatrix}k_{12}z_1^2z_2\\ k_{12}z_1z_2^2\\ 0\\ 0\end{Bmatrix} \quad (30)$$

The versal deformation of the normal form on the center manifold is obtained as:

$$\begin{Bmatrix}\dot{v}_1\\ \dot{v}_2\end{Bmatrix}=\begin{bmatrix}\mu(p_1)+1.5046i & 0\\ 0 & \mu(p_1)-1.5046i\end{bmatrix}\begin{Bmatrix}v_1\\ v_2\end{Bmatrix}+\begin{Bmatrix}(0.1786+0.6704i+k_{12})v_1^2v_2\\ (0.1786-0.6704i+k_{12})v_1v_2^2\end{Bmatrix} \quad (31)$$

The quadratic relationship between the bifurcation parameter p_1 and the real part of the eigenvalues, μ, if we let $p_1=p_{1crit}+\eta$, is obtained by a curve fitting technique (Dávid and Sinha, [12]) as $\eta=0.2794\mu^2+0.0329\mu$. The controller gain k_{12} is chosen to be -50.1786 and the system behavior is studied at $\eta=0.01$, for which value the origin of uncontrolled system is unstable. Figure 8 shows the uncontrolled and controlled angles vs. time. In Fig. 9, Poincaré maps of the uncontrolled and controlled motions are shown for the pairs of the states, (x_1-x_3) and (x_2-x_4). The Poincaré maps of the controlled system clearly show that the solution converges to a small quasi-periodic attractor around the origin.

6 Concluding Remarks

Local nonlinear bifurcation control of general nonlinear systems with time-periodic coefficients has been presented for the first time. The presented technique is general, applicable to periodic systems without restrictions on the size of the parametric excitation. The control method is computationally simple and can be implemented in real time. The technique is based on the application of the Lyapunov-Floquet transformation, which transforms the linear part of a periodic quasi-linear equation into a time-invariant form, while preserving the original stability and bifurcation characteristics of the system. The L-F transformation also makes the application of the time-periodic center manifold reduction and time- dependent normal form theory possible. Using these simplification techniques, time-invariant forms of the periodic equations can be constructed for most of the codimension one bifurcations, which are suitable for the application of time-invariant control methods. It has also been suggested that a time-invariant controller design is possible even in the one case when the normal form is not autonomous (resonant secondary Hopfbifurcation). Based on the normal forms it has been observed that in each case there is an ideal control input, the simplest which can

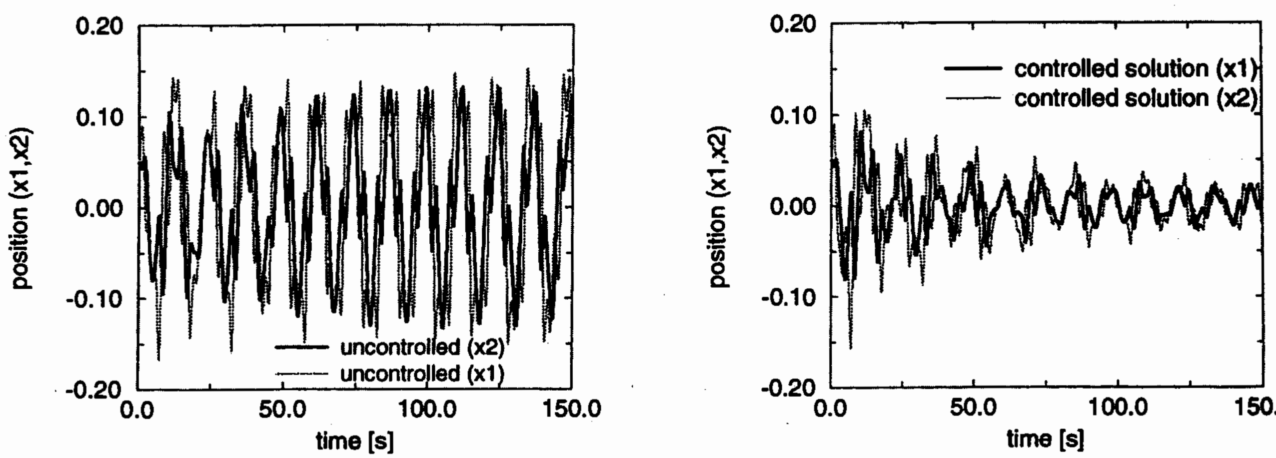

Fig. 8 Double pendulum, uncontrolled and controlled motions

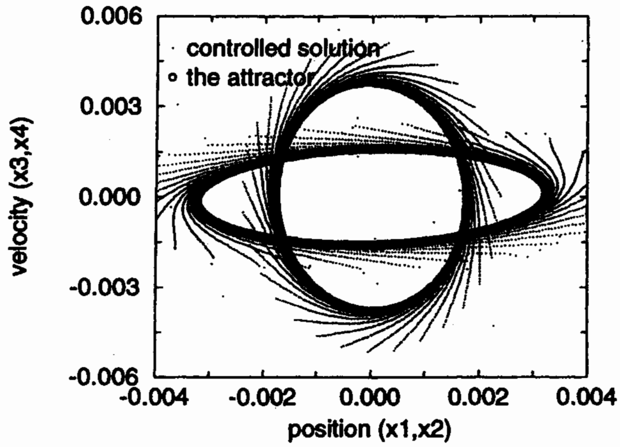

Fig. 9 Double pendulum, the uncontrolled and controlled motions in a Poincaré map representation

achieve our desired goals. In the transformed domain the controller consists of just a single nonlinear term, which is a function of the critical state with a constant control gain. The controller design has been illustrated by two examples, a parametrically excited pendulum undergoing symmetry breaking bifurcation and the secondary Hopf bifurcation of a double inverted pendulum with periodic load. The examples show that the subcritical bifurcations of the original systems can be stabilized and the controlled systems exhibit bounded dynamics. For the simple pendulum, in order to compare the effort needed to control the system, three different choices of the control gain are computed. The comparison indicates that there is an optimum choice, in the sense that controller should provide as small a steady-state amplitude with as little effort as possible. Further, two different linear control methods, a symbolically computed linear controller and feedback linearization, were also applied to the simple pendulum. The comparison of the control torques clearly showed that the purely nonlinear bifurcation control required the least effort, while the feedback linearization technique resulted in the largest control torque.

The idea of nonlinear bifurcation control can be extended to include linear control terms, also. The effect of a linear controller on the nonlinear characteristic of bifurcation is currently being studied. Further, bifurcation control might be a great tool in controlling chaos in time-periodic systems. This topic is also under investigation and will be reported later.

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