ASYMPTOTIC STABILITY OF UNSTEADY INVISCID STRATIFIED FLOWS

S. CARMI*, and S. C. SINHA**

[Manuscript received: 15 March, 1978]

The stability of modulated atmospheric flows is analyzed. The equations governing the disturbance motion are solved by Galerkin expansions with time-dependent coefficients. Asymptotic stability bounds are then established by constructing Liapunov functions for the resulting differential systems.

1. Introduction

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1. Introduction

The stability of inviscid stably stratified flows in a gravitational field is a subject of intense interest to researchers of atmospheric phenomena. Models of continuous velocity and temperature profiles in infinite or semi-infinite domains and ones of two semi-infinite layers with constant but different velocities were analyzed by many authors (for reviews see for example DRAZIN and HOWARD [5], LINDZEN [9] and EINAUDI and LALAS [6]). In the above studies linear theory was used and the basic velocity and density were varying only with height.

In the current work the asymptotic stability of modulated atmospheric flows will be investigated. The basic flow variables will be time dependent in addition to their spatial variation and the modulation will be generated through the boundary (YIH [12] and ROSENBLAT and TANAKA [10]). Nonlinear stability analyses of thermoconvective and other modulated viscous flows were recently presented by DAVIS [4], HOMSY [7] and CARMI [2], but no such investigations were attempted yet for inviscid stratified fluids.

2. Basic flow and perturbation equations

The governing equations of motion for an inviscid, nonconducting, continuously stratified fluid are

\[
\frac{d \varrho}{dt} + \varrho \nabla \cdot \mathbf{V} = 0, \quad (1)
\]

\[
\varrho \frac{d \mathbf{V}}{dt} = \nabla p - \varrho g \mathbf{k}, \quad (2)
\]

* S. CARMI, Ph. D., Mechanical Engineering Department, Wayne State University, Detroit, Michigan 48202, USA

** S. C. SINHA, Ph. D.: Mechanical Engineering Department, Kansas State University, Manhattan, Kansas 66506, USA

with $V = (U, V, W)$, $\rho$, $p$ are the velocity, density and pressure, respectively, $g =$ gravitational constant, $k =$ normal unit vector and where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V \cdot \nabla .$$

To system (1), (2) we apply the Boussinesq approximation yielding the solenoidity and incompressibility conditions in addition to the momentum equation

$$\nabla \cdot V = 0 ,$$

$$\frac{d \rho}{d t} = - \frac{\chi^2}{\rho} \text{w} V = 0 ,$$

$$\rho \frac{d V}{d t} = - \nabla p - g \rho k ,$$

with $\chi^2 = - g \left( \frac{1}{\rho_s} \frac{d \rho}{dz} + \frac{g}{c_0^2} \right) =$ Brunt–Väisälä frequency, $c_0 =$ speed of sound and where

$$V = V_s(z) + V_w(x, t) ,$$

$$p = p_s(z) + p_w(x, t) ,$$

$$\rho = \rho_s(z) + \rho_w(x, t) .$$

The subscripts $s$, $w$ designate the steady and modulated parts, respectively. In the Boussinesq approximation we used $\rho_w = 0$ ($\sigma$) with

$$\sigma = \frac{1}{\rho_m} \max_\zeta (\Delta \rho_s) \ll 1 \text{ and } \rho_m = \text{Ave}_z \rho_s .$$

By subtracting two flows ("starred" and "unstarred") which satisfy the same equations and boundary conditions but vary in their initial conditions we get the system governing the disturbance flow

$$\nabla \cdot u = 0 ,$$

$$\frac{d \rho'}{d t} + u \cdot \nabla \rho' + u \cdot \nabla \rho_w - \frac{\chi^2}{\rho_s} \rho_w V = 0 ,$$

$$\rho_s \left( \frac{d u}{d t} + u \cdot \nabla u + u \cdot \nabla p' \right) = - \nabla p' - g \rho' k ,$$

where $u = (u, v, w) = V^* - V, p' = p^* - p, \rho' = \rho^* - \rho$ are the disturbance variables. System (4) can serve as a starting point for both a linear and nonlinear theory analysis.
Here, we will consider infinitesimal disturbances and will restrict our attention to two-dimensional systems (assuming that Squire’s theorem holds) obtaining from (4)

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,
\]

\[
\frac{d\varphi}{dt} - \frac{\eta^2}{g} \varphi = 0,
\]

\[
\varphi_s \frac{du}{dt} + \varphi_s \frac{\partial U}{\partial z} w + \frac{\partial p'}{\partial x} = 0,
\]

\[
\varphi_s \frac{dw}{dt} + \frac{\partial p'}{\partial z} + g\varphi = 0.
\]

In the above, the atmospheric velocity is a two dimensional parallel flow of a boundary layer type

\[ V = (U, 0, 0, ) \text{ with } U = U(z, t) = U_s(z) + \varepsilon U_w(z, t), \]

where \( \varepsilon \) is not necessarily small. This model is dynamically admissible since it is obtained by solving (1) with periodic upper boundary conditions.

Since the basic flow is a function of \( z \) and \( t \) only, we can take the \( x \)-Fourier Transform of (5) yielding in dimensionless form after using (6) and eliminating \( u, p' \)

\[
\left( \frac{\partial}{\partial t} + ikU \right) \left( \varphi + \eta^2 \varphi \right) - k^2 \Phi - ik \left( \varphi + \sigma \frac{\partial U}{\partial z} \right) \varphi = 0,
\]

\[
\left( \frac{\partial}{\partial t} + ikU \right) \varphi - \hat{f} \varphi = 0,
\]

where we introduced the new variables

\[ \varphi = \varphi_s^{1/2}, \quad \varphi = g\varphi' \varphi_s^{-1/2}, \]

and where \( \sigma = -1 \frac{d\varphi_s}{dt} \) = ratio of velocity and density scale heights, \( k = \text{horizontal wave number}, \eta = k^2 + 1/4 \sigma^2 \) and with \( \hat{f} = x^2 h^2 / U_0^2 \) which becomes the Richardson Number in the usual sense when

\[ \max_{z,t} \frac{\partial U}{\partial z} = \frac{U_0}{h}. \]

The characteristic velocity and length are \( U_0 \) and \( h \), respectively. For the
steady case, equations (7) reduce to the Taylor–Goldstein system which was previously solved for various sets of boundary conditions.

System (7) is completed by specifying initial and boundary conditions. Here we will take

$$\psi = 0 \text{ at } z = 0 \text{ and } z \to \infty,$$

with a necessarily similar requirement for \( \varphi \) as seen from (7).

3. Stability problem formulation

Since the coefficients in (7) depend on both \( z \) and \( t \) we cannot apply the Laplace Transform method as can be done in the steady case. To obtain stability bounds from system (7), (8) we first utilize the Galerkin method. Thus we assume that \( \psi, \varphi \) can be written as

$$\psi = \sum_{n=1}^{\infty} A_n(t) \psi_n(z),$$

$$\varphi = \sum_{n=1}^{\infty} B_n(t) \phi_n(z),$$

where \( \psi_n(t), \phi_n(z) \) are complete orthogonal sets in \( z : (0, \infty) \) satisfying the boundary conditions in (8) and with \( A_n(t), B_n(t) \) unknown functions of time. Substituting (9) in (7) yields

$$\sum_{n=1}^{\infty} \dot{A}_n(t)(\psi'_n - \eta^2 \psi_n) + ikU \sum_{n=1}^{\infty} A_n(t)(\psi'_n - \eta^2 \psi_n) -$$

$$- ik \sum_{n=1}^{\infty} (U - \sigma U') A_n(t) \psi_n - k^2 \sum_{n=1}^{\infty} B_n(t) \phi_n = 0,$$

$$\sum_{n=1}^{\infty} \dot{B}_n(t) \phi_n + ikU \sum_{n=1}^{\infty} B_n(t) \phi_n - J \sum_{n=1}^{\infty} A_n(t) \psi_n = 0.$$  

Since the set \( \psi_n(z), \phi_n(z) \) are known the \( z \)-dependence can be eliminated by multiplying them into (10a, b) and integrating over the \( z \)-range. After performing the integration, truncating the series at \( N \) and rearranging we get (for details see CARMI and SHULZE [3])

$$\frac{dA_m}{dt} = - ik \sum_{n=1}^{N} A_n(t) \chi_{1mn} + k^2 \sum_{n=1}^{N} B_n(t) \chi_{2mn},$$

$$\frac{dB_m}{dt} = - ik \sum_{n=1}^{N} B_n(t) \chi_{3mn} + J \sum_{n=1}^{N} A_n(t) \chi_{4mn},$$

where \( \chi_{imn}, i = 1 - 4 \) are known complex functions of \( t \).
We can reduce (11) to a single real vector equation
\[ \dot{x}_i = G_{ij}(t) x_j, \]
using summation convention with \( j = 1, \ldots, 4N \) and subject to initial conditions
\[ x_i(0) = \delta_i n. \]

The vectors \( x_i(t) \) incorporate the real and imaginary parts of \( A_m(t), B_m(t) \)
\[ x_1(t) = R_e(A_1(t)), x_2(t) = R_e(B_1(t)), \text{ etc.} \]
and the matrix \( G_{ij}(t) \) is \( T \)-periodic.

4. Stability of periodically modulated flows

We will now obtain general stability bounds for periodically modulated
flows by applying Itapunov's second method and making use of Flauquet's
theory and Sylvester's test as described in [11]. In periodic modulation of a
steady atmospheric flow the velocity field (6) can be assumed in the form
\[ U(z, t) = a(z) + \varepsilon f(z) g(t), \]
where \( g(t) \) is \( T \)-periodic. The basic velocity (14) renders (12) in the form
\[ \dot{x}_i = (C_{ij} + \varepsilon B_{ij}(t)) x_j, \]
with \( C_{ij} \) a constant matrix and \( B_{ij}(t) \) \( T \)-periodic.

Taking \( \varepsilon = 0 \) we get from (15) the steady case (using direct notation)
\[ \dot{x} = C x \]
which is stable assuming all the eigenvalues of \( C \) lie in the left half-plane.
In this case, there exists a positive definite quadratic form with constant
coefficients such
\[ V = (H x, x); \quad (H^T = H > 0), \]
whose derivative along the trajectory of the system
\[ \dot{V} = \frac{dV}{dt} = (H C x, x) + (H x, C x) = ([H C + C^T H] x, x) = - (G_0 x, x), \]
is negative definite. The matrix \( H \) can be found from (18) where \( G_0 \) is any
positive matrix (e.g. \( G_0 = \beta I, \beta > 0 \)).
For the modulated case (15) we have (see YAKUBOVICH and STARZHIN-
skii [11])

\[ \dot{V} = - ( [G_0 + \varepsilon G_1(t)] x, x) , \]  

(19)

where \( G_1(t) \) is a T-periodic matrix function given by

\[ G_1(t) = - (HB(t) + B(t)H) . \]

We now let \( \Delta_i(t, \varepsilon) \) denote the principal minors of the matrix \( G_0 + \varepsilon G_1(t) \).

As \( G_0 \) is positive definite, it follows from SYLVESTER's text (see [11, p. 30]) that \( \Delta_i(t, 0) > 0 \) \((i = 1, 2, \ldots , n)\). Since minors \( \Delta_i(t, \varepsilon) \) are continuous of both \( t \) and \( \varepsilon \) we have for sufficiently small \( \varepsilon_0 > 0 \)

\[ \Delta_i(t, \varepsilon) > 0 \] \((i = 1, 2, \ldots , n)\) for \( 0 < t < T, \ |\varepsilon| \leq \varepsilon_0 . \)  

(20)

From (20) it follows that \( G_0 + \varepsilon G_1(t) \) \( > 0 \) for \( 0 \leq t \leq T, \ |\varepsilon| \leq \varepsilon_0 \) hence by continuity there exists \( \delta > 0 \) such that all the principal minors of \( G_0 + \varepsilon G_1(t) - \delta I \) are positive for \( 0 < t < T, \ |\varepsilon| \leq \varepsilon_0 \). Therefore \( \dot{V} \) in (19) satisfies the inequality insuring the asymptotic stability of the trivial solution of (15) for \( |\varepsilon| \leq \varepsilon_0 \). The value of \( \varepsilon_0 \) can be determined from (20).

5. Stability of non-periodic modulated flows

For a non-periodic modulation of the basic steady flow equation (15) can be written in the form

\[ \dot{x} = (C + \varepsilon D(t))x \]  

(22)

Assuming as before that the steady state case \( \varepsilon = 0 \) is asymptotically stable, we will have (22) also asymptotically stable provided (see BELLMAN [1])

\[ ||D(t)|| \leq C_1 \]  

(23)

where \( C_1 \) is a constant which depends on matrix \( C \).

It should be observed that the above criterion is independent of the size of \( \varepsilon \). For example, the basic steady flow \( g(t) \) in (14) can be assumed exponentially decaying (decelerating modulation). For this case, equation (22) can be written as

\[ \dot{x} = (C + \varepsilon Ee^{-\delta t})x ; \ \delta > 0 \]  

(24)

where \( E \) is a constant matrix.
Since the steady state case $\varepsilon = 0$ is asymptotically stable, it follows that there exists a positive constant $a$ such that $\| x \| \leq C_2 e^{-at}$ for $t \geq 0$. Then if $\| \varepsilon e^{-at} \| < C_1$, one can easily show that [1] equation (24) yields $\| x \| \to 0$ as $t \to \infty$ if $C_1 C_2 < a$.

Actual stability bounds like $\varepsilon_0$ in (20) are now being established for model atmospheric flows (e.g. boundary layer type) and results will be reported in a subsequent paper.

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